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A CLASS OF BCH–ALGEBRAS

J.W. NAM, S.M. HONG AND Y.B. JUN

In 1986, K. Iséki[6] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. In [4] and [5], Q.P. Hu and Xin Li discussed the BCH-algebra. The notion of BCH-algebras generalizes the notion of BCI-algebras in the sense that every BCI-algebra is a BCH-algebra, but not *vice versa*(see[5]). Changchang Xi[8] discussed the BCI-algebra satisfying $(s * y) * z \leq x * (y * z)$. In this paper, we investigate some properties of BCH-algebras and study the BCH-algebra satisfying $(x * y) * z \leq x * (y * z)$ for all x, y, z in the algebra, which is called a quasi-associative BCH-algebra.

Let us recall definitions.

DEFINITION 1. A BCI-algebra is an abstract algebra $(X; *, 0)$ of type (2,0) with the following conditions:

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y)) * y = 0$,

(3) $x * x = 0$,

(4) $x * y = y * x = 0$ implies $x = y$,

(5) $x * 0 = 0$ implies $x = 0$,

for all $x, y, z \in X$.

DEFINITION 2. A BCH-algebra is an algebra $(X; *, 0)$ of type (2,0) satisfying the following conditions: for every $x, y, z \in X$,

(3) $x * x = 0$,

(4) $x * y = y * x = 0$ implies $x = y$,

(6) $(x * y) * z = (x * z) * y$.

We will use the symbol " $\leq$ " defined by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$.

A BCH-algebra has the following basic properties (for the proofs, see [4] and [5]) :

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(2) $(x * (x * y)) * y = 0$,
(5) $x * 0 = 0$ implies $x = 0$,
(7) $x * 0 = x$.

First of all, we give some examples of quasi-associative BCH-algebras.

EXAMPLE 1. Every quasi-associative BCI-algebra is a quasi-associative BCH-algebra.

EXAMPLE 2. Any BCI-algebra with weak unit is a quasi-associative BCH-algebra.

EXAMPLE 3. Let $X = \{0, 1, 2, 3\}$ and the operation $*$ given as follows:

*	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

Then $(X; *, 0)$ is a (proper) BCH-algebra (see [4]), and it is quasi-associative but not associative.

EXAMPLE 4. Let $X = \{0, 1, 2, 3\}$ and the operation $*$ given as follows:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	3
3	3	0	0	0

Then X is a (proper) BCH-algebra (see [4]), and it is quasi-associative but not associative.

PROPOSITION 1. *In a BCH-algebra X, we have*

$$x * y \leq z \text{ implies } x * z \leq y,$$

for all $x, y, z \in X$.

Proof. It is obvious by (6).

THEOREM 1. *If a BCH-algebra X astisfies the condition*

$$\text{(I)}\quad x \leq y \text{ implies } z * y \leq z * x \text{ for all } x, y, z \in X,$$

then X is a partially ordered set with respect to $\leq$.

Proof. By (3) and (4), we only prove that

$$x \leq y,\ y \leq z \text{ imply } x \leq z.$$

In fact, asume that $x \leq y$ and $y \leq z$, then by (1), we have $x*z \leq x*y = 0$. It follows from (5) that $x * z = 0$, that is, $x \leq z$, which completes the proof.

LEMMA 1 ([7]). *Let X be an abstract algebra of type (2,0) with a binary operation $*$ and a constant 0. Then X is a BCI-algebra if and only if it satisfies the following conditions:*
(1) $((x * y) * (x * z)) * (z * y) = 0$,
(4) $x * y = y * x = 0$ *implies* $x = y$,
(7) $x * 0 = x$,
all $x, y, z \in X$.

THEOREM 2. *A BCH-algebra X is a BCI-algebra if and only if it satisfies*

$$\text{(II)}\quad x \leq y \text{ implies } x * z \leq y * z \text{ for all } x, y, z \in X.$$

Proof. Necessity is clear. Let X be a BCH-algebra satisfying (II). By Lemma 1, we only prove (1). In fact, we know that

$$\begin{aligned} &((x * y) * (x * z)) * (z * y) \\ =&((x * (x * z)) * y) * (z * y) && \text{by (6)} \\ \leq&(z * y) * (z * y) && \text{by (2) and (II)} \\ =&0. && \text{by (3)} \end{aligned}$$

It follows from (5) that $((x * y) * (x * z)) * (z * y) = 0$ for all $x, y, z \in X$. This completes the proof.

Following [8] we have

COROLLARY 1. *If a BCH-algebra X satisfies the condition (II), then $0 * x \leq x$ if and only if $0 * x = 0 * (0 * x)$ for all $x \in X$.*

Following [2], [3], [6] and [8], we have

COROLLARY 2. *If a BCH-algebra X satisfies the condition (II), then we have the following:*

(a) $x * y \geq 0$ implies $y * x \geq 0$,
(b) $((x * y) * z)(x * (y * z)) \leq (0 * z) * z$,
(c) $(y * x) * (z * x) \leq y * z$,
(d) $((x * y) * z) * (u * z) \leq (x * u) * y$,
(e) $((x * y) * z) * ((x * u) * y) \leq u * z$,
(f) $(x * y) * (z * u) \leq x * (z * (u * y))$,
(g) $(x * y) * (x * (z * (u * y))) \leq z * u$,
(h) $x * (x * (x * z)) = x * z$,
(i) $(a * (x * y)) * (y * x) \leq a$,
(j) $0 * (x * y) = (0 * x) * (0 * y)$,
(k) $0 * (0 * (0 * x)) = 0 * x$.

PROPOSITION 2. *If a BCH-algebra X satisfies the condition (II), then X also satisfies the condition (I).*

Proof. Assume that $x \leq y$. Then we have

$$\begin{aligned} (z * y) * (z * x) &= (z * (z * x)) * y && \text{by (6)} \\ &\leq x * y && \text{by (2) and (II)} \\ &= 0. \end{aligned}$$

It follwos from (5) that $(z * y) * (z * x) = 0$, that is, $z * y \leq z * x$ which proves (I).

REMARKS. 1. Proposition 2 is also an immediate consequence of Theorem 2 and [3 ; Lemma 1.6].

2. It does not hold in general that $(I) \Rightarrow (II)$ because, in Example 3, X satisfies (I) but not (II).

Combining Theorem 1 and Proposition 2, we have

COROLLARY 3. *If a BCH-algebra X satisfies the condition (II), then X is a partially ordered set with respect to $\leq$.*

PROPOSITION 3. *If X is a quasi-associative BCH-algebra, then $0*x = 0*(0*x)$ for all $x \in X$.*

Proof. Assume that X is quasi-associative. Then we have

$$0*x = (0*0)*x \leq 0*(0*x)$$

for all $x \in X$. On the other hand, we also have

$$\begin{aligned}(0*(0*x))*(0*x) &\leq 0*((0*x)*(0*x)) \\ &= 0*0 = 0.\end{aligned}$$

It follows from (5) that $(0*(0*x))*(0*x) = 0$, which means $0*(0*x) \leq 0*x$. Hence we have $0*x = 0*(0*x)$ for all $x \in X$.

From Theorem 2 and [8 ; Theorem 3] we have

THEOREM 3. *Lex X be a BCH-algebbra satisfying the condition (II). Then the following are equivalent:*
(i) X is quasi-associative,
(ii) $0*x \leq x$,
(iii) $0*(x*y) = 0*(y*x)$,
(iv) $(0*x)*y = 0*(x*y)$,
(v) $(x*y)*(y*x) \geq 0$,
for all $x, y \in X$.

LEMMA 2 ([4 ; LEMMA 4]). *Let X and Y be BCH-algebras and let*

$$X \oplus Y = \{(x,y) | x \in X, y \in Y\}.$$

We define the composition $*$ on $X \oplus Y$ by

$$(x,y)*(x',y') = (x*x', y*y')$$

for all $(x,y),(x',y') \in X \oplus Y$. Then $(X \oplus Y; *, (0,0))$ is a BCH-algebra, which is called the direct sum of X and Y.

We can easily extend this construction to any family of BCH-algebras. Let $(X_j)_{j\in J}$ be a family of BCH-algebras indexed by J. We define the direct sum $\oplus_{j\in J}X_j$ of BCH-algebras $X_j, j \in J$, as follows: an element of $\oplus_{j\in J}X_j$ is a family of $(x_j)_{j\in J}$ with $x_j \in X_j$ and $x_j \neq 0$ for only a finite number of subscripts. The composition $*$ is defined by

$$(x_j)_{j\in J} * (y_j)_{j\in J} = (x_j * y_j)_{j\in J}.$$

THEOREM 4. *If X and Y are quasi-associative BCH-algebras, then the direct sum $X \oplus Y$ is also quasi-associative.*

Proof. We have that for every $(x,y),(x',y'),(x'',y'') \in X \oplus Y$,

$$\begin{aligned}&(((x,y)*(x',y'))*(x'',y''))*((x,y)*((x',y')*(x'',y'')))\\&=((x*x',y*y')*(x'',y''))*((x,y)*(x'*x'',y'*y''))\\&=((x*x')*x'',(y*y')*y'')*(x*(x'*x''),y*(y'*y''))\\&=(((x*x')*x'')*(x*(x'*x'')),((y*y')*y'')*(y*(y'*y'')))\\&=(0,0).\end{aligned}$$

This means that

$$((x,u)*(x',y'))*(x'',y'') \leq (x,y)*((x',y')*(x'',y''),$$

proving that $X \oplus Y$ is quasi-associative,

COROLLARY 4. *If $(X_j)_{j\in J}$ is a family of quasi-associative BCH-algebras, then so is $\oplus_{j\in J}X_j$.*

References

1. M.A. Chaudhry, *Weakly positive implicative and weakly implicative BCI-algebras*, Math. Japonica **35** (1990), 141– 151.
2. C.S. Hoo, *Quasi-commutative BCI-algebras*, Math. Japonica **33** (1988), 49 – 56.
3. C.S. Hoo and P.V. Ramana Murty, *Quasi-commutative p-semisimple BCI-algebras*, Math. Japonica **32** (1987), 889 - 894.
4. Q.P. Hu and Xin Li, *On BCH-algebras*, Math. Sem. Notes **11** (1983), 313 – 320.
5. ———, *On proper BCH-algebras*, Math. Japonica **30** (1985), 659 – 661.
6. K. Iséki, *An algebra related with a propositional caluclus*, Proc. Japan Acad. **42** (1966), 26 – 29.
7. Li Hui Shi, *An axiom system of BCI-algebras*, Math. Japonica **30** (1985), 351 – 352.
8. Changchang Xi, *On a class of BCI-algebras*, Math. Japonica **35** (1990), 13 –17.

Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea