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DEVELOPMENT OF SINGULARITIES FOR A SINGLE QUASI-LINEAR EQUATION

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1. Discontinuities in the solution of quasi-linear equations with smooth initial data

Let $f : R^1 \to R^1$ be a C^2 function. In this paper, we treat the following quasi-linear equation with initial data;

$$(IVP) \qquad \begin{cases} u_t + f(u)_x = 0, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R, \end{cases}$$

where u_0 is a given real valued function on R^1, and $u = u(x,t)$ is to be found on the upper half plane $t \geq 0$. We know that $u(x,t)$ is constant along any characteristic line $x = x(t)$ with speed $\frac{dx}{dt} = \frac{d}{du}f(u)$, and that the characteristic line is a straight one. Hence it can be given implicitly by the formula

$$u(x,t) = u_0(x(t) - t\frac{d}{du}f(u_0(\xi))),$$

if the characteristic line passing through (x,t) meets with initial line $t = 0$ at $(\xi, 0)$ (see [2]). If $u_0 \in C^1(R)$, by the implicit function theorem, we can solve **(IVP)** locally for sufficiently small $t > 0$. Indeed, if we let $F(x,t,u) = u - u_0(x - f'(u)t)$ then

$$\frac{\partial F}{\partial u} = 1 + tu_0' f''(u) \neq 0 \text{ for sufficiently small } t.$$

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Also, we have

$$(1.1) \qquad u_t = -\frac{u_0' f'(u_0(\xi))}{1 + u_0' f''(u_0(\xi))t} \text{ and } u_x = \frac{u_0'}{1 + u_0' f''(u_0(\xi))t},$$

where $u(x,t) = u_0(x(t) - f'(u_0(\xi))t)$. If there exist two characteristic lines issuing from two distinct points in the initial line $t = 0$ so that they meet at a certain positive time $t > 0$, then the solution $u(x,t)$ must be a multivalued function since u is constant along characteristics. Now, we extract the condition for global solution to exist.

THEOREM 1. *If* $\frac{d}{dx}[\frac{d}{du}f(u_0(x))] \geq 0$ *for any* $x \in R$, *then there exists a unique solution* $u(x,t) \in C^1(R \times (0,\infty))$ *for* **(IVP)**.

Proof. Assume that $\frac{d}{dx}[\frac{d}{du}f(u_0(x))] \geq 0$, for any $x \in R$. Then $\frac{d}{du}f(u_0(x))$ is an increasing function of $x \in R$. This means that for any $x_1 < x_2$,

$$\frac{d}{du}f(u_0(x_1)) \leq \frac{d}{du}f(u_0(x_2)).$$

Since $\frac{d}{du}f(u_0(x_i))$ $(i = 1,2)$ is the speed of characteristic line issuing from x_i, the two lines can not meet at any point (x,t). Note that on each characteristic line, u is determined uniquely by its initial value. This proves the theorem.

COROLLARY 1. *Assume that* $u_0 \in C^1$ *and* $f \in C^2$. *Then there is a global solution* $u(x,t) \in C^1(R \times (0,\infty))$ *if and only if*

$$\frac{d}{dx}[\frac{d}{du}f(u_0(x))] \geq 0 \text{ for any } x \in R.$$

Proof. By Theorem 1, the necessary part was proved. Conversely, assume that $u(x,t) \in C^1(R \times (0,\infty))$ is a solution for **(IVP)**. If $\frac{d}{dx}$ $[\frac{d}{du}f(u_0(x_0))] < 0$ for some x_0, by continuity, there exists an interval (a,b) containing x_0 such that it holds for any $x \in (a,b)$. Let $a < x_1 <$

$x_2 < b$. Since $\frac{d}{du}f(u_0(x_i))(i = 1,2)$ is the speed of the characteristic lines issuing from x_i, respectively, and $u(x,t) \in C^1(R \times (0,\infty))$, we have

$$\frac{d}{du}f(u_0(x_1)) \leq \frac{d}{du}f(u_0(x_2)).$$

Hence

$$\frac{d}{du}f(u_0(x_1)) - \frac{d}{du}f(u_0(x_2)) = \frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))] \cdot (x_1 - x_2) \leq 0,$$

where $x_1 < \overline{x} < x_2$. This leads to a contradiction.

Assume that f is purely nonlinear, i.e., $f''(u) \neq 0$ for all $u \in R$. Since $f''(u) \in C^0(R)$, $f''(u) > 0$ for all $u \in R$ or $f''(u) < 0$ for all $u \in R$. If $u(x,t)$ is a global solution, by Corollary 1, $u_0'(x) \geq 0$ for any $x \in R$ if $f'' > 0$. Physically, $u(x,t)$ usually denotes the density of a stuff, but we have

$$|\int_{-\infty}^{\infty} u_0(x)dx| = +\infty \text{ unless } u_0(x) \equiv 0,$$

which is not realistic. Therefore, by assuming $f'' \neq 0$ in any realistic physical problem, we can not expect a global C^1 solution of the initial value problem **(IVP)** because any physically acceptable system does not permit the total density of the stuff in the system to be infinite.

THEOREM 2. *If* $B = \inf_x \frac{d}{dx}[\frac{d}{du}f(u_0(x))] < 0$ *then there exists a unique solution* $u(x,t) \in C^1(R \times (0, \frac{-1}{B}))$ *for* **(IVP)**. *After the time* $T = -\frac{1}{B}$, $u(x,t)$ *can not be continued as a single valued solution.*

Proof. Assume that $-\infty < B = \inf_x \frac{d}{dx}[\frac{d}{du}f(u_0(x))] < 0$. We shall first show that any two characteristics starting from two distinct points on x-axis won't cross each other for $0 \leq t < T\frac{-1}{B}$. Let x_1 and x_2 be any two points in the x-axis such that $x_1 < x_2$. Let l_1 and l_2 be characteristic lines pathing through $(x_1, 0)$ and $(x_2, 0)$. Then

$$\begin{cases} l_1 : x &= \frac{d}{du}f(u_0(x_1))t + x_1, \\ l_2 : x &= \frac{d}{du}f(u_0(x_2))t + x_2. \end{cases} \tag{1.2}$$

If l_1 intersects l_2 somewhere in the upper half plane $t > 0$, then we must have

$$(1.3) \qquad \begin{cases} \frac{d}{du}f(u_0(x_1)) > \frac{d}{du}f(u_0(x_2)), \\ [\frac{d}{du}f(u_0(x_1)) - \frac{d}{du}f(u_0(x_2))]t = x_2 - x_1. \end{cases}$$

By the mean value theorem, $\frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))](x_1 - x_2)t = x_2 - x_1$ for some $\overline{x} \in (x_1, x_2)$, hence we have

$$t = -(\frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))])^{-1} \geq -\frac{1}{B}.$$

Note that $\frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))] < 0$. Therefore, $u(x,t)$ is determined uniquely by its initial data for $x \in R^1$ and $0 \leq t < -\frac{1}{B} = T$. Hence $u \in C^1(R^1 \times (0, -\frac{1}{B}))$ and solves **(IVP)** by implicit function theorem. Indeed, if $\frac{d}{dx}[\frac{d}{du}f(u_0(x))] < 0$, then $t\frac{d}{dx}[\frac{d}{du}f(u_0(x))] \geq Bt > 0$. Hence

$$1 + \frac{d}{dx}[\frac{d}{du}f(u_0(x))]t \neq 0 \text{ for any } x \in R^1, 0 < t < -\frac{1}{B}.$$

Now we shall show that there are two characteristics crossing either $T = -\frac{1}{B}$ or just afterward. In other words, for any $\epsilon > 0$ they cross at $t \in [-\frac{1}{B}, -\frac{1}{B} + \epsilon)$, therefore the solution u can not be continued as a single valued solution of **(IVP)** beyond $T = -\frac{1}{B}$.

Case 1. Assume that $\equiv \frac{d}{dx}[\frac{d}{du}f(u_0(x))]$ for all $x \in (a,b), a < b$. Then for any x_1 and x_2 in the interval (a,b) so that $x_1 < x_2$, let l_1 and l_2 be characteristics starting from $(x_1, 0)$ and $(x_2, 0)$, respectively, as in (1.2). If l_1 and l_2 intersect, then the t-component of intersection point is

$$t = \frac{x_2 - x_1}{\frac{d}{du}f(u_0(x_1)) - \frac{d}{du}f(u_0(x_2))} = \frac{-1}{\frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))]} = -\frac{1}{B} > 0,$$

for $x_1 < \overline{x} < x_2$, by the mean value theorem.

Case 2. Assume that $\frac{d}{dx}[\frac{d}{du}f(u_0(x))]$ has a strict minimum B at x_0. For $|\delta|$ small, let l_1 and l_2 be characteristics starting from $(x_0, 0)$ and $(x_0+\delta, 0)$ then

$$l_1 : x = \frac{d}{du}f(u_0(x_0))t + x_0 \text{ and } l_2 : x = \frac{d}{du}f(u_0(x_0+\delta))t + x_0 + \delta.$$

Hence at the point of their intersection, we have

$$t = t(\delta) = \frac{\delta}{\frac{d}{du}f(u_0(x_0)) - \frac{d}{du}f(u_0(x_0+\delta))} = \frac{-1}{\frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))]}$$

where $\overline{x}$ is a point between x_0 and $x_0+\delta$. For $|\delta|$ small enough, $B < \frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))] < 0$ and so $0 < -\frac{1}{B} < t(\delta)$. Moreover, as $|\delta|$ decreases to 0, $\overline{x} \to x_0$ and so $t(\delta) \to -\frac{1}{B}$.

Case 3. Assume that $\frac{d}{dx}[\frac{d}{du}f(u_0(x))]$ has a strict infimum, i.e., $B < \frac{d}{dx}[\frac{d}{du}f(u_0(x))]$ for any $x \in R$. Since $\frac{d}{dx}[\frac{d}{du}f(u_0(x))]$ is continuous, $\frac{d}{dx}[\frac{d}{du}f(u_0(x))] \to B$ as $x \to -\infty$ or $x \to \infty$. Assume the convergence occurs as $x \to -\infty$. Let $\epsilon > 0$ and take x_1 sufficiently large as a negative number so that $\frac{d}{dx}[\frac{d}{du}f(u_0(x_1))] < B + \epsilon < 0$. Then for sufficiently small $|\delta|$, as in case 1, any two characteristics starting from $(x_1, 0)$ and $(x_1+\delta, 0)$ will meet each other at $t(\delta)$, hence we have

$$-\frac{1}{B} < t(\delta) = \frac{-1}{\frac{d}{dx}[\frac{d}{du}f(u_0(\overline{x}))]} < -\frac{1}{B} + \eta(\epsilon),$$

where $\overline{x} \in (x_1, x_1+\delta)$ for sufficiently small δ and $\eta(\epsilon) \to 0$ as $\epsilon \to 0$. We complete the proof.

Now, suppose that we have the region S in R such that the slopes of characteristic lines isuing from $x \in S$ decreasing monotonically, then we can formulate an envelope, whose tangent lines are those characteristic lines, as follows ;

Consider arbitrary two neighbouring characteristics issuing from $(\eta, 0)$ and $(\eta+\delta, 0)$, for $|\delta|$ small enough, so that

$$l_\eta = \frac{d}{du} f(u_0(\eta))t + \eta \text{ and } l_\delta = \frac{d}{du} f(u_0(\eta+\delta))t + \eta + \delta.$$

Then they meet at $t = \frac{\delta}{\frac{d}{du}f(u_0(\eta)) - \frac{d}{du}f(u_0(\eta+\delta))}$, which converges to $\frac{-1}{\frac{d}{d\eta}[\frac{d}{du}f(u_0(\eta))]}$ as $|\delta| \to 0$. Hence

$$E(\eta) = \left(\frac{-\frac{d}{du}f(u_0(\eta))}{\frac{d}{d\eta}[\frac{d}{du}f(u_0(\eta))]} + \eta, \frac{-1}{\frac{d}{d\eta}[\frac{d}{du}f(u_0(\eta))]}\right) \tag{1.4}$$

is the curve of the envelope inside of which any two characteristic lines meet.

Note that the envelope has a corner or cusp at time $-\frac{1}{B}$ (see figures 1.1, 2.1, and 3.1). For example, consider $u_t + uu_x = 0$ with $u_0 = \sin(s)$, then $E(s) = (-\tan(s) + s, -\sec(s))$. We can easily show that each $E(s)$ has two curves meeting at $(n\pi, 1)(n = \pm1, \pm3, \pm5, \dots)$. At these points, the envelope has a cusp.

The envelope has a corner only if the region $\frac{d}{dx}[\frac{d}{du}f(u_0(x))] < 0$ has an interval in which $\frac{d}{du}f(u_0(x))$ is linear with respect to x because if l_1 and l_2 are tangent lines of the envelope and these lines meet as $(\eta_1, 0)$, $(\eta_2, 0)$ in the line $t = 0$, respectively, then any characteristic line issuing from a point $(x, 0)$ where x lies between η_1 and η_2 must pass the point at which the two curves which consists of the envelope meet. Hence the slope of these lines $\frac{d}{du}f(u_0(\eta))$ is a linear function with respect to η between η_1 and η_2.

2. Computer Simulations

In section 1, we showed that although the initial data is smooth, we cannot generally obtain global smooth solution for **(IVP)**. If

$$-\infty < B = \inf_x \frac{d}{dx}[\frac{d}{du}f(u_0(x))] < 0, \tag{2.1}$$

then there is a smooth solution $u(x,t)$ only for $0 \le t \le T = -\frac{1}{B}$. For $t \ge T$, we must consider the solution for **(IVP)** in the distribution sense. Peter D. Lax (see [2]) proved the existence of the generalized solution for **(IVP)** with $u_0 \in L_\infty(R)$ by formulating a difference scheme as follows :

$$u_k^{n+1} = \frac{u_{k+1}^n + u_{k-1}^n}{2} + \frac{\Delta t}{2\Delta x}(f_{k-1}^n - f_{k+1}^n). \tag{2.2}$$

Here u_k^n abbreviates an approximation to u at $t = n\Delta t,\ x = k\Delta x$, and f_k^n abbreviates $f(u_k^n)$. As a stability condition, this scheme requires that

$$\frac{\Delta t}{\Delta x} \le \frac{1}{\max_{|u|\le M}|f'(u)|}. \tag{2.3}$$

where $M = \|u_0\|_{L_\infty}$. Note that this scheme is valid for solutions with shock discontinuity (see [2]). We shall use this scheme to show that for smooth data satisfying (2.1), how the solution for **(IVP)** varies before $T = -\dfrac{1}{B}$, and how the smoothness of the solution is broken after T. The surface simulated by the above scheme is shown by the field of characteristic lines in the figures. First, consider Burger's equation

$$u_t + uu_x = 0. \tag{2.4}$$

In this case,

$$\frac{d}{du}f(u_0(x)) = u_0(x) \text{ or } \frac{d}{dx}\left[\frac{d}{du}f(u_0(x))\right] = u_0'(x).$$

If $u_0(x) = x^2$, then $B = \inf_x u_0'(x) = \inf_x 2x = -\infty$. Hence we cannot obtain any strip in which the solution for (2.1) is smooth. Let $u_0(x) = \sin(x)$ for (2.4). Then discontinuity starts at time $T = 1$. By the stability condition, we must have $\frac{\Delta t}{\Delta x} \le 1$. The characteristics and integral surface are displayed in figures 1.1 and 1.2.
Let $u_0(x) = 1 + \cos(x)$ with $f(u) = \frac{1}{2}u^2$. In this case, the initial data valued on the interval $(0, \frac{\pi}{2})$ will develope a shock discontinuity since $u_0'(x) < 0$ and $f'' > 0$. We can easily show that the shock starts at $(\frac{1+\pi}{2}, \frac{1}{2})$ by the equation (1.4). (see figures 2.1, 2.2).

Consider $f(u) = u(1-u)$ for **(IVP)** with initial data as follows :

$$u_0(x) = \begin{cases} 0 & \text{if } s < -3; \\ 3+s & \text{if } -3 < s < -1; \\ 2 & \text{if } -1 < s < 0; \\ 1+\cos(s) & \text{if } 0 < s < \pi; \\ 0 & \text{if } s > \pi. \end{cases} \tag{2.5}$$

Note that because $f'' < 0$, any shock forms by the initial data whose derivatives are positive. Since $f''(u) = -2 < 0$, $\frac{d}{dx}\{f'(u)\} = f''(u)u'(x) < 0$ iff $u_0'(x) > 0$. We restrict the domain of $u_0(x)$ to $[-5, \pi]$. Then $u_0'(x) > 0$ iff $x \in [-3, -1]$. By the characteristic method, the characteristic curves are as follows :

$$x(t,s) = \begin{cases} t+s & \text{if } s < -3; \\ (-2s-5)t+s & \text{if } -3 < s < -1; \\ -3t+s & \text{if } -1 < s < 0; \\ (-1-2\cos(s))t+s & \text{if } 0 < s < \pi; \\ t+s & \text{if } s > \pi, \end{cases} \tag{2.6}$$

and the shock forms with speed $\frac{dx}{dt} = \frac{[f]}{[u]} = -1$ at $(-\frac{5}{2}, \frac{1}{2})$. In fact, the discontinuity line is $x = t - 2$ for $\frac{1}{2} < t < 1$ (see figures 3.1, 3.2). The subsequent pages are results of simulations for various data. Each pages contain characteristic field (first figure) and integral surface (second figure) in (x,t)-space.

$$f(u) = \frac{1}{2}u^2 \quad u_0(x) = \sin(x)$$

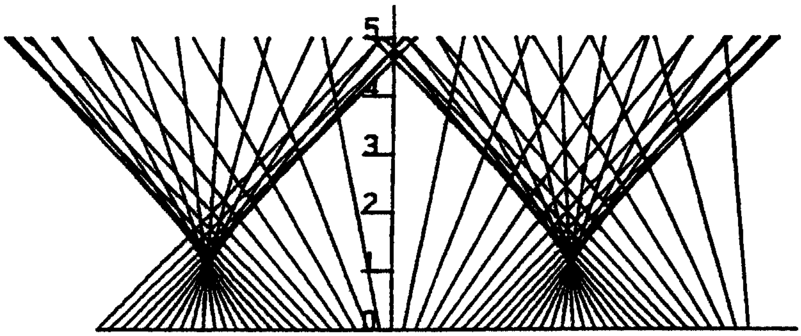

Figure 1.1

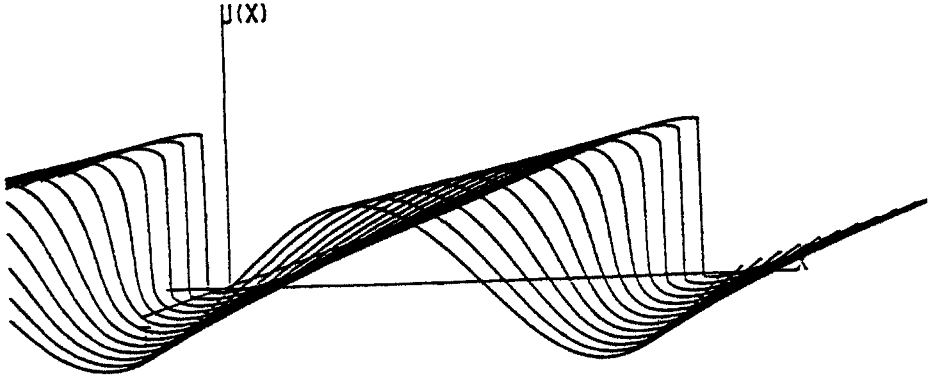

Figure 1.2

$$f(u) = \frac{1}{2}u^2 \quad u_0(x) = 1 + \cos(x)$$

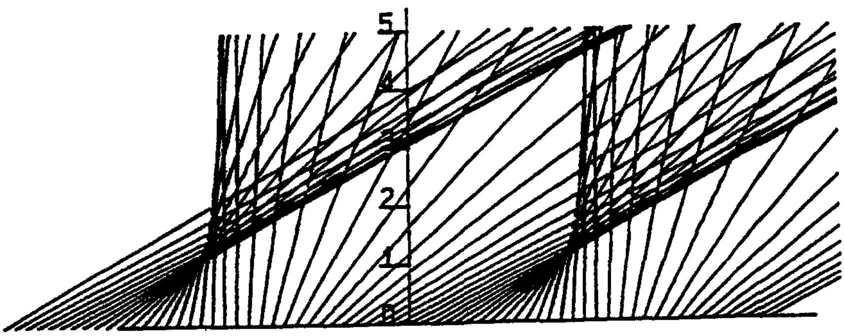

Figure 2.1

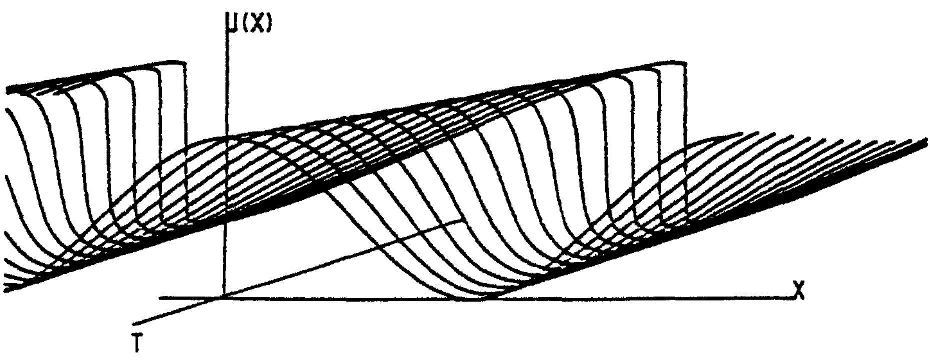

Figure 2.2

$$f(u) = u(1-u) \quad u_0(x) \ : \ (2.5)$$

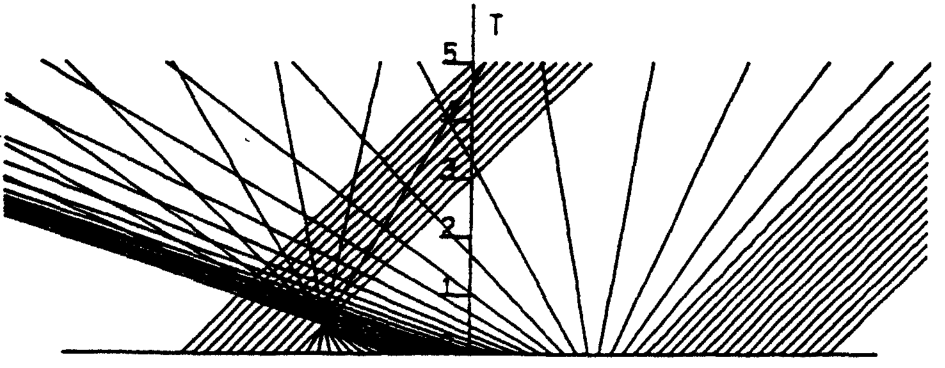

Figure 3.1

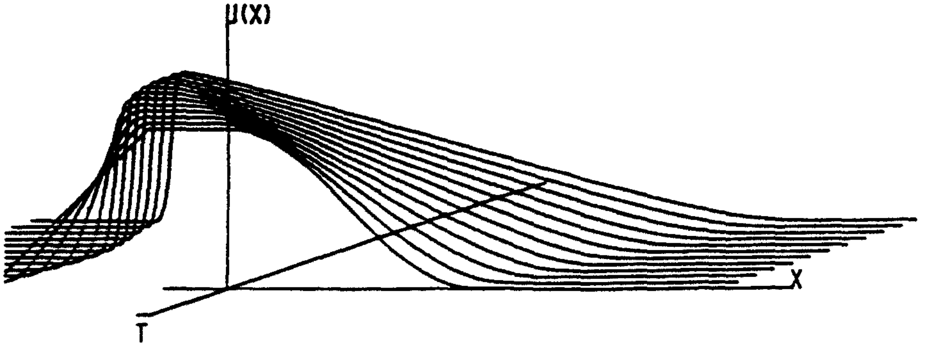

Figure 3.2

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