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HANDLE ATTACHING ON GENERIC MAPS II

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1. Introduction

Let V^n and M^m be smooth manifolds of dimension n and m, respectively, with $3n+3 \leq 2m$ throughout the paper unless it is said otherwise. We also assume that V is closed. In [6], Haefliger reduces the problem of homotoping a map from V into M to a smooth embedding to the construction of an equivariant map between two proper spaces, where the construction of such a map can be considered as a homotopy theoretic problem. There are a few applications of Haefliger's result, especially when $M = \mathbf{R}^m$ ([1], [3], [4], [10]).

In this paper, we study embedding and immersion problems when V and M are $([(2n-m+1)/2]+1)$-connected and $6 \leq 2n-m$. The main tools are the handle subtraction (Proposition 2 of [5]) and the handle attaching (Lemma 1 of this paper) on generic maps (see [5] for the definition of generic map).

Following the notation of [5], if $f : V \to M$ is a generic map, let $\Delta(f)$ denote the closure of the double point set, $S'(f) = \partial\Delta(f)$, the set of singular values of f, $D(f) = f^{-1}(\Delta(f))$ and $S(f) = f^{-1}(S'(f))$. All these sets are smooth manifolds and if one is contained in the other it is contained as a submanifold, and the dimension of $\Delta(f)$ is equal to $2n-m$. Let H be a $(p-r)$-handle, $-1 \leq r \leq p$, in $\Delta(f)$ relative to $S'(f)$, where $p = 2n-m-1$. We use p for this number in the rest of the paper. Now H determines uniquely an element $\theta(H)$ in $\pi_{p-r+1}(f)$ up to sign (see 5.3 how this is defined). We say that H can be subtracted from f if and only if $\theta(H) = 0$. This definition is motivated from the main result (Proposition 2) of [5] which says that given a null homotopy of $\theta(H)$, one can construct a generic map f' homotopic to f such that $\Delta(f')$ is diffeomorphic to the closure of $\Delta(f) - H$. Therefore, it follows that if f is $(p+2)$-connected, then all the handles in $\Delta(f)$ can be subtracted in any handle decomposition of

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$(\Delta(f),\ S'(f))$, thus f is homotopic to an embedding. We call f' the result of a handle subtraction on f. Note that f' depends on the null homotopy. By reversing the handle subtraction, $\Delta(f)$ can be viewed as being obtained from $\Delta(f')$ by attaching an $(r+1)$-handle to $\Delta(f')$ along $S'(f')$.

This raises the following question. Given an embedding $h : S^r \times D^{p-r} \to S'(f)$, $-1 \le r \le p$, is there a generic map f' homotopic to f such that $\Delta(f')$ is diffeomorphic to $\Delta(f)$ with an $(r+1)$-handle attached using h ? The main result (Lemma 1) of this paper shows that handle attaching can be done with some restrictions on h and r. The handle attaching lemma is proved in Section 2 following the steps of handle subtraction in [5] with time (τ) inverted. But handle attaching requires more work (see 2.1-2.5) than handle subtraction. A second proof of the lemma was suggested in [9]. Using handle attaching, we show that surgeries can be done (Lemma 2) on the interior of $\Delta(f)$ under proper conditions. It follows (2.7-2.9) that if V and M are $([(p+1)/2]+1)$-connected, then it is possible to subtract or attach handles, or do surgeries on $\Delta(f)$ below the middle dimension with one exception.

We now give some result obtained by applying these operations. If $3n < 2m$, then every map from V into M can be approximated by a generic map and an immersion can be approximated by an immersion which is generic. We will call such an immersion a generic immersion and an immersion will mean a generic immersion in this paper.

DEFINITION. An immersion from V into M is orientable if its double point set is orientable.

Note that every immersion is orientable if $m-n$ is even.

DEFINITION. A generic map f is nice if $\Delta(f)$ is orientable and $S(f)$ is two-sided in $D(f)$. Nice homotopy is defined the same way.

DEFINITION. A generic map f is a pseudo-embedding if $\Delta(f) \cong D^{2n-m}$ ($\cong$ means "is diffeomorphic to").

Note that every embedding is homotopic to a pseudo-embedding by Lemma 1.

THEOREM 1. *Let V and M be $([(p+1)/2]+1)$-connected and let $f : V \to M$ be a map. Then the following are equivalent to each other.*

(a) f is homotopic to an orientable immersion.
(b) f is homotopic to a nice map.
(c) f is homotopic to a pseudo-embedding.

The theorem suggests that there are two stages of obstructions to homotoping a map to an embedding under the connectivity assumption on V and M in Theorem 1. The first is to homotoping a map to a pseudo-embedding and the second is to homotoping a pseudo-embedding to an embedding. Next theorem identifies the second obstruction.

Given a map $f : V \to M$, let $\pi_i(f) = \pi_i(Z(f), V)$, where $Z(f)$ denotes the mapping cylinder of f, and let $\pi_i(f)^+ = \pi_i(f)/\sim$, where $a \sim b$ if $a = \pm b$.

THEOREM 2. *Let V and M be $([(p+2)/2]+1)$-connected and let $f : V \to M$ be a pseudo-embedding. Then there exists an element $\Gamma(f) \in \pi_{p+2}(f)^+$ such that f is nicely homotopic to an embedding if and only if $\Gamma(f) = [0]$.*

A similar result was obtained in [3] by assuming that the immersion has a non-trivial normal vector field with a stronger connectivity condition on M.

We now describe the first obstruction in a form of a test. Under the connectivity condition on V and M, if $m-n$ is even, the obstruction is a geometric approach to the immersion classification theorem of [7]. It may also be regarded as a partial converse of the embedding theorem of [5]: the embedding theorem was proved by showing that a relative r-handle in $(\Delta(f), S'(f))$ of a generic map f can be subtracted if the element of $\pi_{r+1}(f)$ determined by the handle is trivial. If the element is not trivial, then the handle can not be subtracted by the method of [5]. But it is possible that f may still be homotopic to an embedding. Next theorem shows that if a generic map g homotopic to f is chosen properly, then the failure of a handle subtraction from $(\Delta(g), S'(g))$ implies that g can not be homotopic to a pseudo-embedding, thus f can never be homotopic to an embedding.

The test works as follows. Suppose that V and M are $([(p+2)/2]+1)$-connected and we are given a map from V to M. We first approximate the map by a generic map f. If f is nice, then the test ends. So assume that f is not nice. Then by a sequence of handle attaching and subtraction on f, we can find a generic map (not nice) g homotopic to f such that $\pi_1(S'(g)) \cong \pi_1(\Delta(g)) \cong \mathbf{Z}_2$, and $\Delta(g)$ and $S'(g)$ are $([(p+1)/2]-1)$ and $([p/2]-1)$-connected, respectively (see 5.1).

THEOREM 3. *Under the above notation, f is homotopic to a nice generic map if and only if we can subtract all the handles except for a top dimensional one in any handle decomposition of $\Delta(g)$ relative to $S'(g)$.*

The handle attaching lemma is proved in (2.1-2.5), the surgery lemma is proved in (2.6), and the possibility of killing homotopy elements of $\Delta(f)$ and $S'(f)$ is studied in (2.7-2.9). Theorem 1, 2 and 3 are proved in Section 3, 4 and 5, respectively.

2. Handle attaching lemma

Let $f : V \to M$ be a generic map and $h : S^r \times D^{p-r} \to S'(f)$, $-1 \leq r \leq p$, be an embedding. Suppose that $f' : V \to M$ is a generic map homotopic to f such that $\Delta(f') \cong \mathcal{H}(\Delta(f), h) = \Delta(f) \cup_h D^{r+1} \times D^{p-r}$. We call f' the result of a handle attaching on f using h. If such a handle attaching is possible, it follows that $f^{-1}h$ is contractible in V. So the contractibility is a necessary condition for a handle attaching. But this condition is not sufficient to attach a handle in general. We may need to modify h before we can attach a handle.

Let $s_* : \pi_r(SO_{p-r}) \to \pi_r(SO)$ (the stable homotopy group), $r \geq -1$, be the homomorphism induced by the inclusion map. If $r = -1$, both groups are understood to be trivial and if $r = 0$, they must be replaced by $\pi_0(O_p)$ and $\pi_0(O)$, respectively. The following discussions are made for $r \geq 1$ but with proper modification they hold true for $r = -1, 0$.

Given $\lambda \in \pi_r(SO_{p-r})$ define new embedding h_λ by $h_\lambda(x, v) = h(x, \lambda(x)v)$, $(x, v) \in S^r \times D^{p-r}$.

LEMMA 1. *Let $f : V \to M$ be a generic map, and let $h : S^r \times D^{p-r} \to S'(f)$ be an embedding such that $f^{-1}(h|S^r \times \{0\})$ is null homotopic in V. Assume that $S(f)$ is two-sided in $D(f)$ over the image of $f^{-1}h$ if $r = 1$. If $s_* : \pi_r(SO_{p-r}) \to \pi_r(SO)$ is onto, then there exists an element $\lambda \in \pi_r(SO_{p-r})$ and a generic map g homotopic to f such that $\Delta(g)$ is diffeomorphic to $\mathcal{H}(\Delta(f), h_\lambda)$. Furthermore, λ can be replaced with any element in the coset $\mathrm{Ker}(s_*) + \lambda$ in $\pi_r(SO_{p-r})$.*

We prove the lemma by following the steps and notation of the proof of Proposition 2 of [5].

2.1. Let g_τ, $0 \leq \tau \leq 1$, be the family of maps from

$$\mathbf{R}^n(x_1, x_2, \cdots, x_{m-n}, u_1, \cdots, u_{p-r}, v_1, \cdots, v_{r+1})$$

to

$$\mathbf{R}^m(X_1, \cdots, X_{m-n}, Y_1, \cdots, Y_{m-n}, U_1, \cdots, U_{p-r}, V_1, \cdots, V_{r+1})$$

defined in 4.4 of [5]. We repeat the definition for the convenience of readers.

Let $u^2 = \sum u_j^2$, $v^2 = \sum v_j^2$, $w^2 = u^2 + v^2$, $R^2 = U^2 + V^2$ and $\rho^2 = \sum_{1<i\leq m-n} x_i^2$. Let γ be an even function with variable x so that $\gamma(x) = 1$ for $|x| \leq 1$, $\gamma(x) = 0$ for $|x| \geq 2$, and it increases for $x < 0$. Define $\theta(w^2) = \theta_0^2\gamma(w^2/\theta_0^2)$, where θ_0 is a real number, $0 < \theta_0 \leq 1$. Let α be an even function in ρ such that $0 \leq \alpha(\rho) \leq 1$, $\alpha(0) = 1$ and $\alpha(\rho) = 0$ for $\rho > \varepsilon$, where ε is a positive real number. Then g_τ is defined as follows.

$$g_\tau : \begin{cases} X_1 &= x_1(1 - 2Y_1) \\ Y_1 &= A(u^2, v^2, \rho^2, \tau)\gamma(x_1)/(1 + x_1^2) \\ X_i &= x_i,\ Y_i = x_1 x_i,\ 1 < i \leq m - n \\ U_j &= u_j,\ V_j = v_j, \end{cases}$$

where $A(u^2, v^2, \rho^2, \tau) = [1 + \theta(w^2)(1 - 2\alpha(\rho)\tau) + 2v^2)]/(1 + w^2)$. It can be checked that g_τ is a generic map for $\tau \neq 1/2$.

Let K_ε be the subset of $\mathbf{R}^n$ defined by $|x_1| \leq 2$, $\rho^2 \leq \varepsilon^2$ and $w \leq 2\theta_0$, and let K'_ε be the subset of $\mathbf{R}^m$ defined by $|X_1| \leq 2$, $\sum_{i>1} X_i^2 \leq \varepsilon^2$, $0 \leq Y_1 \leq 1$, $\sum_{i>1} Y_i^2 \leq 4\varepsilon^2$ and $R \leq 2\theta_0$. Then $g_\tau^{-1}(K'_\varepsilon) = K_\varepsilon$ for all τ and g_τ is fixed outside of K_ε in τ.

Let K denote $K_\varepsilon(x_i = 0,\ i > 1)$, i.e., K is the intersection of K_ε with the plane $x_i = 0$, $i > 1$, and $K' = K'_\varepsilon(X_i = Y_i = 0,\ i > 1)$.

Let H_+^{p-r+1} be the $(p-r+1)$-dimensional upper half disk consisted of points in $(p - r + 1)$-disk whose last coordinate is non-negative and let H_-^{p-r+1} be the lower half. Let D_0^{p-r} be the subset of points in H_+^{p-r+1} whose last coordinate is equal to 0. Regard D^{p-r+1} as the union of H_{+}^{p-r+1} and H_{-}^{p-r+1} identified along D_0^{p-r}. We use 1 for θ_0 from now on.

Let S_0^r and S_1^r be the r-spheres contained in K_ε and K'_ε, respectively, satisfying the following equations.

$$S_0^r; x_i = 0,\ u_j = 0 \quad \text{and} \quad v^2 = 1.$$

$$S_1^r; X_i = 0,\ Y_1 = 1/2,\ Y_j = 0,\ 1 < j,\ U_k = 0,\ V^2 = 1.$$

We also let $D_0^{r+1} = K_\varepsilon(x_i = 0,\ u_j = 0,\ v^2 \leq 1)$ and $D_1^{r+1} = K'_\varepsilon(X_i = 0,\ Y_1 = 1/2,\ Y_j = 0,\ 1 < j,\ U_k = 0,\ V^2 \leq 1)$.

From the description of $\Delta(g_r)$ in 4.4 of [5], $\Delta(g_1) \cap K'_\varepsilon$ can be naturally identified with $S_1^r \times H_{+}^{p-r+1}$, $S'(g_1) \cap K'_\varepsilon$ with $S_1^r \times D_0^{p-r}$, $D(g_1) \cap K_\varepsilon$ with $S_0^r \times D^{p-r+1}$ and $S(g_1)$ with $S_0^r \times D_0^{p-r}$. These identifications can be done so that $\partial/\partial U_1, \cdots, \partial/\partial U_{p-r}$ and ν'_0 are independent normal vector fields to $S_1^r \times \{0\}$ in $S_1^r \times H_{+}^{p-r+1}$ (0 denotes the center of a disk), where ν'_0, a linear combination of $\partial/\partial V_1, \cdots, \partial/\partial V_{r+1}$, is a normal vector field to $S_1^r \times D_0^{p-r}$ in $S_1^r \times H_{+}^{p-r+1}$, and $\partial/\partial u_1, \cdots, \partial/\partial u_{p-r}$ and $\partial/\partial x_1$ are independent normal vector fields to $S_0^r \times \{0\}$ in $S_0^r \times D^{p-r+1}$. We may further assume that $g_1(v, y) = (v, y) \in S_1^r \times H_{+}^{p-r+1}$ for $(v, y) \in S_0^r \times H_{+}^{p-r+1}$ and $g_1(v, y') = (v, y) \in S_1^r \times H_{+}^{p-r+1}$ for $(v, y') \in S_0^r \times H_{-}^{p-r+1}$, where the last coordinate of y is equal to $-$ (last coordinate of y') and the other coordinates of y and y' are the same.

Now $\Delta(g_0) \cap K'_\varepsilon$ is diffeomorphic to $\Delta(g_1) \cap K'_\varepsilon$ with an ambient $(r+1)$-handle attached in K'_ε using a small thickening of S_1^r in $S_1^r \times D_0^{p-r}$, where the core of the handle is D_1^{r+1}.

Given a generic map $f : V \to M$ and an embedding $h : S^r \times D^{p-r} \to S'(f)$ with $f^{-1}(h|S^r \times \{0\})$ null homotopic in V, if $s_* : \pi_r(SO_{p-r}) \to \pi_r(SO)$ is onto, then we construct diffeomorphisms $H : K_\varepsilon \to V$ and $H' : K'_\varepsilon \to M$ such that

(1) $H'g_1 = fH$ over K_ε

(2) $f^{-1}H'(K'_\varepsilon) = H(K_\varepsilon)$

(3) $H' \mid S_1^r \times D_0^{p-r} = h_\lambda$ for some $\lambda \in \pi_r(SO_{p-r})$.

If generic map g is defined by $g = f$ outside of $H(K_\varepsilon)$ and $g = H'g_0H^{-1}$ over $H(K_\varepsilon)$, then g satisfies the conclusions of Lemma 1. We construct H and H' in several steps as in 4.5 through 4.6 of [5].

2.2. Over $(D(g_1) \cap K_\varepsilon) \cup$ (a neighborhood of $S(g_1) \cap K_\varepsilon$ in K_ε) and $(\Delta(g_1) \cap K'_\varepsilon) \cup$ (a neighborhood of $S'(g_1) \cap K'_\varepsilon$ in K'_ε).

Define $H_1' : S_1^r \times H_+^{p-r+1} \to M$ such that $H_1' \mid S_1^r \times D_0^{p-r} = h$ and H_1' gives a collar neighborhood of $H_1'(S_1^r \times D_0^{p-r})$ in $\Delta(f)$. Then we can find a diffeomorphism $H_1 : S_0^r \times D^{p-r+1} \to V$ so that the conditions (1), (2) and (3) hold. Here we use the assumption that $S(f)$ is two-sided in $D(f)$ over $f^{-1}(\mathrm{Im}(h))$ when $r = 1$.

Let $e_i = \partial/\partial x_i$, $1 \le i \le m-n$ and $e_i' = \partial/\partial X_i$, $e_i'' = \partial/\partial Y_i$, $2 \le i \le m-n$. Since $2(2n-m) < n$ and $H_1 \mid (S_0^r \times D_0^{p-r})$ is null homotopic in V, there exist independent normal vector fields $\xi_1, \cdots, \xi_{m-n}, \nu$ to $H_1(S_0^r \times D_0^{p-r})$ in V, where $\xi_1 = dH_1(e_1)$ is a unit normal vector field to $H_1(S_0^r \times D_0^{p-r})$ in $H_1(S_0^r \times D^{p-r+1})$. Define an isomorphism $dH_2 : T(K_\varepsilon) \mid S_0^r \times D_0^{p-r} \to T(V) \mid H_1(S_0^r \times D_0^{p-r})$ by $dH_2(e_i) = \xi_i$, $1 \le i \le m-n$, $dH_2(\nu_0) = \nu$, where ν_0 is the unit normal vector field to $S_0^r \times D^{p-r+1}$ in K, thus ν_0 is a linear combination of $\partial/\partial v_1, \cdots, \partial/\partial v_{r+1}$. By 3.2 and 3.3 of [5], there exist diffeomorphisms H_2 from $(D(g_1) \cap K_\varepsilon) \cup$ (a neighborhood of $S(g_1) \cap K_\varepsilon$ in K_ε) into V and H_2' from $(\Delta(g_1) \cap K_\varepsilon') \cup$ (a neighborhood of $S'(g_1) \cap K_\varepsilon'$ in K_ε') into M such that H_2 and H_2' extend H_1 and H_1', respectively, and (1), (2) and (3) hold.

Before further extensions of H_2 and H_2' are considered, we study the possibility of choosing H_2 and H_2' differently, which is crucial in the proof of the lemma.

2.3. Let $\lambda_1 \in \pi_r(SO_{p-r})$, $\lambda_2 \in \pi_r(SO_{m-n-1}) \cong \pi_r(SO)$, $u = (u_1, \cdots, u_{p-r})$, $v = (v_1, \cdots, v_{r+1})$, $U = (U_1, \cdots, U_{p-r})$ and $V = (V_1, \cdots, V_{r+1})$. Define diffeomorphisms $J(\lambda_1, \lambda_2)$ of $\mathbf{R}^n(|v| \ne 0)$ onto itself and $J'(\lambda_1, \lambda_2)$ of $\mathbf{R}^m(|V| \ne 0)$ onto itself by $J(\lambda_1, \lambda_2)(x_1, x_2, \cdots, x_{m-n}, u, v) = (x_1, \lambda_2(v/|v|)(x_2, \cdots, x_{m-n}), \lambda_1(v/|v|)(u), v)$ and $J'(\lambda_1, \lambda_2)(X_1, X_2, \cdots, X_{m-n}, Y_1, Y_2, \cdots, Y_{m-n}, U, V) = (X_1, \lambda_2(V/|V|)(X_2, \cdots, X_{m-n}), Y_1, \lambda_2(V/|V|)(Y_2, \cdots, Y_{m-n}), \lambda_1(V/|V|)(U), V)$. Then $g_1 J(\lambda_1, \lambda_2) = J'(\lambda_1, \lambda_2) g_1$ over $\mathbf{R}^n(|v| \ne 0)$, $J(\lambda_1, \lambda_2)(K_\varepsilon, |v| \ne 0) = (K_\varepsilon, |v| \ne 0)$, $J(\lambda_1, \lambda_2)(D(g_1) \cap K_\varepsilon) = D(g_1) \cap K_\varepsilon$, $J(\lambda_1, \lambda_2) \mid S_0^r = id$ of S_0^r, $J'(\lambda_1, \lambda_2)(K_\varepsilon', |V| \ne 0) = (K_\varepsilon', |V| \ne 0)$, $J'(\lambda_1, \lambda_2)(\Delta(g_1) \cap K_\varepsilon') = \Delta(g_1) \cap K_\varepsilon'$ and $J'(\lambda_1, \lambda_2) \mid S_1^r = id$ of S_1^r.

By choosing proper domains, H_2 and H_2' can be replaced by $H_2 J(\lambda_1, \lambda_2)$ and $H_2' J'(\lambda_1, \lambda_2)$, respectively. Clearly $H_2' J'(\lambda_1, \lambda_2) \mid S_1^r \times D_0^{p-r} = h_{\lambda_1}$.

If $dJ(\lambda_1, \lambda_2)$ is restricted over $T(K_\varepsilon) \mid S(g_1) \cap K_\varepsilon$, then $dJ(\lambda_1, \lambda_2)(e_1) = e_1$, $dJ(\lambda_1, \lambda_2)(\nu_0) = \nu_0$ and $dJ(\lambda_1, \lambda_2)(e_i) = \lambda_2(v/|v|)(e_i)$, $2 \le i \le$

$m-n$.

If $dJ'(\lambda_1, \lambda_2)$ is restricted over $T(K'_\varepsilon) \mid S'(g_1) \cap K'_\varepsilon$, then $dJ'(\lambda_1, \lambda_2)$ $(\partial/\partial X_1,\ \partial/\partial Y_1) = (\partial/\partial X_1, \partial/\partial Y_1)$, $dJ'(\lambda_1, \lambda_2)(e'_i) = \lambda_2(V/|V|)(e'_i)$, $2 \le i \le m-n$, $dJ'(\lambda_1, \lambda_2)(e''_i) = \lambda_2(V/|V|)(e''_i)$, $2 \le i \le m-n$, and $dJ'(\lambda_1, \lambda_2)(\nu'_0) = \nu'_0$, where ν'_0 is defined in 2.1.

REMARK. Suppose that $\bar{H}_2$ and $\bar{H}'_2$ are two diffeomorphisms that can replace H_2 and H'_2, respectively, in 2.2. If we assume that $H_2 = \bar{H}_2$ over S^r_0 and $H'_2 = \bar{H}'_2$ over S^r_1, then $H_2^{-1} \bar{H}_2$ and $H'^{-1}_2 \bar{H}'_2$ are diffeomorphisms of small neighborhoods of S^r_0 and S^r_1 into themselves, respectively. It can be seen that if there exist $\lambda_1 \in \pi_r(SO_{p-r})$ and $\lambda_2 \in \pi_r(SO_{m-n-1})$ such that $d(H_2^{-1} \bar{H}_2) = dJ(\lambda_1, \lambda_2)$ on the tangent subspace generated by e_i, $1 < i \le m-n$ and $\partial/\partial u_1, \cdots, \partial/\partial u_{p-r}$ along S^r_0, then $d(H'^{-1}_2 \bar{H}'_2) = dJ'(\lambda_1, \lambda_2)$ on the tangent subspace generated by e'_i, e''_i, $1 < i \le m-n$ and $\partial/\partial U_1, \cdots, \partial/\partial U_{p-r}$ along S^r_1.

2.4. Over K and K'.

As in 4.6 of [5], we can find diffeomorphisms $H_3 : K \to V$ and $H'_3 : K' \to M$ satisfying (1), (2) and (3) since $H_2 \mid S^r_0$ is null homotopic in V. Notice that $H'_3 \mid S^r_1 \times D^{p-r}_0 = h$. Furthermore, H_3 and H'_3 agree with H_2 and H'_2 in the neighborhoods of $S(g_1) \cap K_\varepsilon$ in K_ε and $S'(g_1) \cap K'_\varepsilon$ in K'_ε, respectively. The argument for the construction of H_3 and H'_3 is easier here than that of 4.6 of [5] because $g_1(K)$ is a deformation retract of K' in our case.

2.5. Over K_ε and K'_ε.

To extend H_3 and H'_3 over K_ε and K'_ε, it suffices to construct vector bundle isomorphisms dH from $N(K_\varepsilon, K)$ (the normal bundle of K in K_ε) to $N(V, H_3(K))$ covering H_3 and dH' from $N(K'_\varepsilon, K')$ to $N(M, H'(K'))$ covering H'_3 so that $dH' dg_1 = df dH$ and they agree with dH_2 and dH'_2 over $S(g_1) \cap K_\varepsilon$ and $S'(g_1) \cap K'_\varepsilon$, respectively.

Observe that dH_2 defines a bundle isomorphism from $N(K_\varepsilon, K) \mid S^r_0$ to $N(V, H_3(K)) \mid H_3(S^r_0)$ and it gives an element $\lambda \in \pi_r(SO_{m-n-1})$ by using a fixed trivialization of $N(V, H_3(K))$.

Replace H_2 and H'_2 with $\bar{H}_2 = H_2 J(0, -\lambda)$ and $\bar{H}'_2 = H'_2 J'(0, -\lambda)$, respectively, where J and J' are defined in 2.3. Construct $\bar{H}_3$ and $\bar{H}'_3$ from $\bar{H}_2$ and $\bar{H}'_2$. Then $d\bar{H}_2 = dH_2 dJ(0, -\lambda)$ as bundle maps from $N(K_\varepsilon, K) \mid S^r_0$ to $N(V, \bar{H}_3(K)) \mid \bar{H}_3(S^r_0)$. Since we may assume that

$\bar{H}_3 = H_3$ over K, $d\bar{H}_2 = \lambda - \lambda = 0$ in $\pi_r(SO_{m-n-1})$ in terms of the above trivialization of $N(V, \bar{H}_3(K))$.

Since $S(g_1) \cap K_\varepsilon$ and K deformation retract to S_0^r and D_0^{r+1}, respectively, the bundle isomorphism $d\bar{H}_2$ from $N(K_\varepsilon, K) \mid S(g_1) \cap K_\varepsilon$ to $N(V, \bar{H}_3(K)) \mid \bar{H}_3(S(g_1) \cap K_\varepsilon)$ extends to an isomorphism dH from $N(K_\varepsilon, K)$ to $N(V, \bar{H}_3(K))$ covering $\bar{H}_3$.

REMARK. If we replace $\bar{H}_2$ and $\bar{H}_2'$ in the above with $\bar{H}_2 J(\lambda_1, \lambda_2)$ and $\bar{H}_2' J'(\lambda_1, \lambda_2)$, respectively, where $\lambda_1 \oplus \lambda_2 = 0$ in $\pi_r(SO_{p-r+m-n-1})$, we can still construct H_3, H_3' and dH.

We assume that H_2 and H_2' have been chosen so that H_3, H_3' and dH are constructed as required. Let ξ_i'', $1 < i$, be the independent vector fields of $N(M, H_3'(K'))$ defined over $H_3'(\Delta(g_1) \cap K')$ by

$$\xi_i'' = dH_2'(e_i'') \text{ over } H_3'(S'(g_1) \cap K'),\ i > 1, \text{ and}$$
$$\xi_i''(d) = [df dH(e_i(d_1)) - df dH(e_i(d_2))]/[x_1(d_1) - x_1(d_2)],$$

where $d \in H_3'(\mathring{\Delta}(g_1) \cap K')$ and $d = fH_3(d_1) = fH_3(d_2)$. If ξ_i'', $1 < i$, can be extended over $H_3'(K')$ so that $df dH(e_i)$, ξ_i'', $1 < i$, are independent at every point in $H_3'(g_1(K) - \Delta(g_1))$, then dH' can be constructed as in 4.7 of [5] with the desired properties.

Let N_0 be the subvector bundle of $N(M, H_3'(K')) \mid H_3' g_1(D_0^{r+1})$ generated by $\{df dH(e_i)\}_{1<i}$ and let N_1 be the complementary bundle of N_0. Then N_1 is generated by $\xi_i'' = dH_2'(e_i'')$, $1 < i$, over $H_3' g_1(S_0^r) = H_3'(S_1^r)$. If these vector fields can be extended to independent vector fields of N_1, then ξ_i'', $1 < i$, can be extended over $N(M, H_3'(K'))$ as required, since $H_3' g_1(D_0^{r+1}) \cup H_3'(\Delta(g_1) \cap K')$ is a deformation retract of $H_3'(K')$.

For a fixed trivialization of N_1, $dH_2'(e_i'')$, $1 < i$, determine an element $\lambda \in \pi_r(SO_{m-n-1})$. We may regard λ as an element of $\pi_r(SO_{p-r})$ by the assumption that s_* is onto. Replace H_2 and H_2' with $\bar{H}_2 = H_2 J(\lambda, -\lambda)$ and $\bar{H}_2 = H_2' J'(\lambda, -\lambda)$, respectively. By the above remark, $\bar{H}_3$, $\bar{H}_3'$ and $d\bar{H}$ can be constructed. Choose an $(n+1)$-dimensional submanifold Q of M containing $L = H_3'$ (a tubular neighborhood of $g_1(D_0^{r+1})$ in K') so that $N(Q, L) \mid H_3' g_1(D_0^{r+1}) = N_0$ and $N(M, Q) \mid H_3' g_1(D_0^{r+1}) = N_1$.

Let N_0' be subbundle of $N(M, \bar{H}_3'(K')) \,|\, \bar{H}_3' g_1(D_0^{r+1})$ generated by $\{df d\bar{H}(e_i)\}_{1<i}$, and let N_1' be its complementary bundle. Then $\bar{H}_3$, $\bar{H}_3'$ and $d\bar{H}$ can be constructed so that $N_1' = N(M,Q) \,|\, \bar{H}_3' g_1(D_0^{r+1})$ and $H_3' g_1(D_0^{r+1}) = \bar{H}_3' g_1(D_0^{r+1})$. Hence the trivialization of N_1 induces a trivialization of N_1' and with respect to this trivalization, $d\bar{H}_2'(e_i'')$, $1 < i$, represent the trivial element of $\pi_r(SO_{m-n-1})$. Therefore, we have $\bar{H}_3$, $\bar{H}_3'$, $d\bar{H}$ and $d\bar{H}'$ satisfying the requirements given at the beginning of this subsection.

Let H and H' be the diffeomorphisms covered by $d\bar{H}$ and $d\bar{H}'$. Then $H' \,|\, S_1^r \times D_0^{p-r} = h_\lambda$ for some $\lambda \in \pi_r(SO_{p-r})$. If g is constructed as in 2.1, then $\Delta(g)$ is diffeomorphic to $\mathcal{H}(\Delta(f),\ h_\lambda)$. Furthermore, we may replace λ with any element of $\mathrm{Ker}(s_*) + \lambda$. This completes the proof of Lemma 1.

REMARK. In the above construction, g depends on the null homotopy of $f^{-1}(h \,|\, S^r \times \{0\})$.

2.6. The handle attaching lemma implies the following surgery lemma. Given an embedding $h : S^r \times D^{p-r+1} \to \mathrm{Int}(\Delta(f))$, let $\chi(\Delta(f), h)$ denote the result of a surgery on $\Delta(f)$ using h.

LEMMA 2. *Let $f : V \to M$ be a generic map and let $h : S^r \times D^{p-r+1} \to \mathrm{Int}(\Delta(f))$ be an embedding, $r \neq 1$. Suppose that $\pi_0(\Delta(f), S'(f)) = 0$, $\pi_1(f) = \pi_2(f) = \pi_r(V) = \pi_{r+1}(f) = 0$, and $s_* : \pi_r(SO_{p-r}) \to \pi_r(SO)$ is onto, then there exists $\lambda \in \pi_r(SO_{p-r})$ and a generic map g homotopic to f such that $\Delta(g)$ is diffeomorphic to $\chi(\Delta(f),\ h_{s_*(\lambda)})$, where $s_*(\lambda) \in \pi_r(SO_{p-r+1})$.*

Regarding $\mathrm{Im}(h) = h(D_+^r \times D^{p-r+1}) \cup h(D_-^r \times D^{p-r+1})$, we subtract the 0-handle $h(D_+^r \times D^{p-r+1})$ by [5]. This can be done since $\pi_1(f) = 0$. Then $h(D_-^r \times D^{p-r+1})$ is an r-handle in $Cl(\Delta(f) - h(D_+^r \times D^{p-r+1}))$ and it can be subtracted since $\pi_{r+1}(f) = 0$. Therefore there exists a generic map f' homotopic to f such that $\Delta(f') = Cl(\Delta(f) - \mathrm{Im}(h))$.

Define h' as the composition: $S^r \times D^{p-r} = S^r \times D_+^{p-r} \subset S^r \times S^{p-r} = S^r \times \partial D^{p-r+1} \xrightarrow{h|} h(S^r \times \partial D^{p-r+1}) \subset \Delta(f')$. Then h' is an embedding of $S^r \times D^{p-r}$ into $S'(f')$ and $f'^{-1}(h' \,|\, S^r \times \{0\})$ is null homotopic in V. By Lemma 1, there exist $\lambda \in \pi_r(SO_{p-r})$ and a generic map f'' homotopic to f' such that $\Delta(f'')$ is diffeomorphic to $\mathcal{H}(\Delta(f'), h'_\lambda)$.

We claim that $\Delta(f'')$ is diffeomorphic to $\chi(\Delta(f),\ h_{s_*(\lambda)})$ with the interior of an embedded $(p+1)$-disk in the interior of $\chi(\Delta(f), h_{s_*(\lambda)})$ removed.

$$\begin{aligned}\Delta(f'') &= \Delta(f') \cup (D^{r+1} \times D^{p-r}) \text{ (identified by } h'_\lambda) \\ \chi(\Delta(f), h_{s_*(\lambda)}) &= \Delta(f') \cup (D^{r+1} \times S^{p-r}) \text{ (identified by } h_{s_*(\lambda)}) \\ &= \Delta(f') \cup [(D^{r+1} \times D_+^{p-r}) \cup (D^{r+1} \times D_-^{p-r})] \\ &\qquad \text{(identified by } h_{s_*(\lambda)})\end{aligned}$$

To see the claim, remove the interior of $D^{r+1} \times D_-^{p-r}$ from $\chi(\Delta(f), h_{s_*(\lambda)})$.

Now $S'(f'') = S^p \cup S'(f)$. Find an embedded 1-handle $D^1 \times D^p$ in $\Delta(f'')$ such that $(D^1 \times D^p) \cap S'(f'') = S^0 \times D^p$, $\{1\} \times D^p \subset S^p$ and $\{-1\} \times D^p \subset S'(f)$. Since $\pi_2(f) = 0$, this handle can be subtracted, which implies that there exists a generic map g homotopic to f such that $\Delta(g)$ is diffeomorphic to $\chi(\Delta(f), h_{s_*(\lambda)})$, thus completing the proof of Lemma 2.

In the following subsections, we study the possibility of killing the homotopy groups of $\Delta(f)$ and $S'(f)$.

2.7. Let $f : V \to M$ be a generic map and let $i : S^r \to S'(f)$ be an embedding such that $f^{-1}i$ is null homotopic in V. We show that if $S(f)$ is two-sided in $D(f)$ over $f^{-1}i(S^r)$, then the normal bundle of i is stably trivial. Therefore, if $2r < p$, then the normal bundle of i is trivial. Note that $S(f)$ is always two-sided over $f^{-1}i(S^r)$ if $r \neq 1$.

Denote the normal bundle of i by $N(S'(f), i)$ or $N(S'(f), i(S^r))$. Push the embedding i into the interior of $\Delta(f)$ using a collar structure of $S'(f)$ in $\Delta(f)$. Let i_1 be the resulting embedding. Now $f^{-1}i_1$ can be regarded as two embeddings i' and i'' of S^r into V and both are null homotopic. The extra assumption is used here when $r = 1$. Clearly, $N(\Delta(f), i_1)$ is isomorphic to $N(S'(f), i) \oplus \varepsilon^1$, where ε^1 is the trivial line bundle, and $N(M, \Delta(f)) \,|\, i_1(S^r)$ is isomorphic to $N(V, D(f)) \,|\, i'(S^r) \oplus N(V, D(f)) \,|\, i''(S^r)$. Observe that the following bundles are stably trivial.

$$N(M,i_1) = N(\Delta(f),i_1) \oplus N(M,\Delta(f))\,|\,i_1(S^r),$$
$$N(D(f),i') \oplus N(V,D(f))\,|\,i'(S^r) \quad \text{and}$$
$$N(D(f),i'') \oplus N(V,D(f))\,|\,i''(S^r).$$

Therefore, $N(\Delta(f),i_1) \oplus N(V,D(f))\,|\,i'(S^r) \oplus N(V,D(f))\,|\,i''(S^r)$ is stably trivial and $N(\Delta(f),i_1)$ is stably equivalent to $N(D(f),i') \oplus N(D(f),i'')$. But $N(\Delta(f),i_1)$, $N(D(f),i')$ and $N(D(f),i'')$ are isomorphic bundles. This implies that $N(\Delta(f),i_1)$ is stably tribial and so is $N(S'(f),i)$. The above argument also shows that the normal bundle of any embedding i of S^r into the interior of $\Delta(f)$ is stably trivial if $r \geq 2$ and $\pi_r(V) = 0$.

2.8. By [2] and [8], it can be shown that the inclusion homomorphism $s_* : \pi_r(SO_{r+j}) \to \pi_r(SO)$, $1 \leq r$ and $-1 \leq j$, is onto in any one of the following cases.

(1) $j \geq 1$

(2) $j = 0$ or -1 and $r \neq 1, 3, 7$.

2.9. The discussions above imply that if V and M are $[(p+1)/2]$-connected and $f : V \to M$ is generic, then there is no difficulty in killing the homotopy groups of $\Delta(f)$ and $S'(f)$ below the middle dimension except for the fundamental group. If f is nice, then the fundamental group can also be killed. Finally, if f' is the result of a handle subtraction or attaching on a nice map f, then the construction can be done such that f' is again nice.

3. Proof of Theorem 1

Clearly, (a) implies (b) by the definition, and (c) implies (a) by Theorem 1 of [9].

It suffices to show that (b) implies (c).

Suppose that f is homotopic to a nice map g. If $S'(g)$ is empty, subtract a 0-handle from g to find a nice map g' homotopic to g such that $S'(g') \cong S^p$. To save notation, we will use g for g'. By subtracting or attaching 1-handles, make $\Delta(g)$ and $S'(g)$ connected and $(\Delta(g), S'(g))$ 1-connected. We may assume that g is a nice map. Represent the generators of $\pi_1(S'(g))$ by disjointly embedded circles. They have trivial normal bundle by 2.7. Attach 2-handles to kill the generators by Lemma 1 and 2.8, thus making $\Delta(g)$ and $S'(g)$ simply connected. By repeating the corresponding steps in higher dimensions, and by doing surgery on $\Delta(g)$, we can make $\Delta(g)$ and $S'(g)$ $([(p+1)/2]-1)$ and $([p/2]-1)$-connected, respectively. We now consider two cases.

(1) $p+1=2k$.

We have $H_i(\Delta(g),\ S'(g)) \cong 0$ if $i \neq k,\ 2k$, and $H_k(\Delta(g), S'(g)) \cong H^k(\Delta(g)) \cong \mathrm{Hom}(H_k(\Delta(g)), \mathbf{Z})$ is a finitely generated free abelian group since $\Delta(g)$ and $S'(g)$ are $(k-1)$ and $(k-2)$-connected, respectively, where the homology groups are taken over $\mathbf{Z}$. Now $\pi_k(\Delta(g), S'(g)) \cong H_k(\Delta(g), S'(g))$ by Hurewicz isomorphism theorem. Represent elements of a basis of $H_k(\Delta(g), S'(g))$ by disjointly embedded k-handles relative to $S'(g)$ by the method of §4 of [12] and subtract them using that $\pi_{k+1}(g) = 0$. Then $\Delta(g)$ is contractible and $S'(g)$ is a homotopy sphere. It follows that $\Delta(g)$ is diffeomorphic to the $(2n-m)$-disk by the h-cobordism theorem ([11]).

(2) $p+1=2k+1$

In this case, $\Delta(g)$ and $S'(g)$ are $(k-1)$-connected. Subtract k-handles relative to $S'(g)$ from $\Delta(g)$ to make $(\Delta(g), S'(g))$ k-connected. Then $\Delta(g)$ and $S'(g)$ are still $(k-1)$-connected. Hence we have a short exact sequence of free abelian groups.

$$0 \to H_{k+1}(\Delta(g), S'(g)) \xrightarrow{\partial} H_k(S'(g)) \to H_k(\Delta(g)) \to 0.$$

We again use the technique in §4 of [12] to represent elements of a basis of $\mathrm{Im}(\partial)$ by disjoint embeddings of $S^k \times D^k$ into $S'(g)$, and attach $(k+1)$-handles by Lemma 1 and 2.8 if $k \neq 3, 7$. If $k = 3, 7$, we use Lemma 3 (mirror handle attaching lemma) of [9] to attach $(k+1)$-handles. Now $S'(g)$ is a homotopy sphere, and $H_i(\Delta(g)) \cong 0$ for $i \neq k,\ k+1,\ 0$, and $H_k(\Delta(g)) \cong H_{k+1}(\Delta(g))$ is free abelian. We do surgery to eliminate $H_k(\Delta(g))$ using Lemma 2, 2.7 and 2.8 to make $\Delta(g)$ contractible. Finally, apply the h-cobordism theorem to finish the proof.

4. Proof of Theorem 2

4.1. Let f be an orientable immersion. By Theorem 1 and 2.9, f is nicely homotopic to a pseudo-embedding g. Define an element $\theta(g) \in \pi_{p+2}(g)$ as follows. Let $x_0 \in V$ be a base point and regard $D^{p+2} = D_+^{p+1} \times I/(x,t) = (x,0)$ for $x \in \partial D_+^{p+1}$, where D_+^{p+1} is the upper hemisphere of $\partial D^{p+2} = S^{p+1}$. There exists a commutative diagram unique up to orientation and isotopy,

$$\begin{array}{ccc} S^{p+1} & \xrightarrow{e} & D^{p+2} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ V & \xrightarrow{g} & M \end{array}$$

where e is an inclusion, $j|D_+^{p+1}$ is a diffeomorphism onto $\Delta(g)$, $j(x,t) = j(x,0)$ for each t, and i is a diffeomorphism onto $D(g)$. Let $\theta(g)$ be the above diagram. Then $\theta(g)$ represents an element of $\pi_{p+2}(g)$ up to sign.

Let F be a nice homotopy from f to g, i.e., $F|V \times \{0\} = f$ and $F|V \times \{1\} = g$. Let $u : Z(f) \to Z(F)$ be the inclusion map. We regard $(\{x_0\} \times I) * \theta(g)$ as an element of $\pi_{p+2}(F)$, where $*$ denotes the action of a path on a homotopy element. Define $\Gamma(f) = [u_*^{-1}((\{x_0\} \times I) * \theta(g))] \in \pi_{p+2}(f)^+$.

4.2. $\Gamma(f)$ is well defined. Let F' be a nice homotopy from f to another pseudo-embedding g'. Define nice homotopy $G = F \cup (-F')$ from g' to g by

$$G(x,t) = F'(x, 1-2t), \quad 0 \le t \le 1/2$$
$$G(x,t) = F(x, 2t-1), \quad 1/2 \le t \le 1.$$

As in the proof of Theorem 1, we can find a generic map G' homotopic to G relative to $V \times \{0\} \cup V \times \{1\}$ such that $\Delta(G')$ is a h-cobordism. Here we use $3n + 3 < 2m$ and $([(p+2)/2] + 1)$-connectedness of V and M. By the h-cobordism theorem, $\Delta(G') \cong \Delta(g') \times I$. Hence $\theta(g)$ and $\theta(g')$ are freely homotopic in $(Z(G'), V \times I)$. This implies that $\Gamma(f)$ is well defined since V is simply connected.

4.3. Suppose that f is nicely homotopic to an embedding by a nice homotopy F'. Let F be a nice homotopy from f to a pseudo-embedding g. We show that $\theta(g) = 0$ in $\pi_{p+2}(g)$, which implies that $\Gamma(f) = [0]$.

Let $G = F' \cup (-F)$. By attaching and subtracting handles on G as in the proof of Theorem 1, we can find a generic map G' homotopic to G relative to $V \times \{0\} \cup V \times \{1\}$ such that there exists a diffeomorphism h from $\Delta(G')$ to D^{p+2} with corners along S^p, where $S^{p+1} = \partial D^{p+2} = D_+^{p+1} \cup_{S^p} D_-^{p+1}$, and $h|\Delta(g)$ is a diffeomorphism onto D_-^{p+1}.

Let $\theta(g)$ be represented by a diagram as in 4.1. Then $(G')^{-1}h^{-1}(D^{p+2})$ is diffeomorphic to D^{p+2}, and its boundary is equal to $\mathrm{Im}(i)$. Hence i is null homotopic in V. Furthermore, there exists a map $h' : D^{p+3} \to V \times I$ such that $h'(D_-^{p+2}) \subset V \times \{0\}$, $h' \,|\, S^{p+1} = i$ and $h' \,|\, D_+^{p+2}$ is a diffeomorphism onto $(G')^{-1}h^{-1}(D^{p+2})$, where $S^{p+2} = \partial D^{p+3} = D_+^{p+2} \cup_{S^{p+1}} D_-^{p+2}$. Now $G'(h' \,|\, D_-^{p+2})$ can be regarded as a map of S^{p+2} into $M \times \{0\}$ and $PG'h'$ is a null homotopy of this map, where $P : M \times I \to M \times \{0\}$ is the projection. This shows that $\theta(g)$ is trivial in $\pi_{p+2}(g)$.

To show the converse, suppose that $\Gamma(f) = [0]$, then f is nicely homotopic to an embedding by a handle subtraction on a pseudo-embedding which is nicely homotopic to f.

5. Proof of Theorem 3

5.1. Suppose that f is a generic map that is not nice. Without loss of generality, we assume that $\Delta(f)$ and $S'(f)$ are connected and $(\Delta(f), S'(f))$ is 1-connected.

Define a homomorphism $\phi : \pi_1(S'(f)) \to \mathbf{Z}_2 \oplus \mathbf{Z}_2$ as follows. For $\alpha \in \pi_1(S'(f))$, represent α by an embedded circle S in $S'(f)$. Then $\pi(\alpha) = (a, b) \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$, where $a = 1$ if $S(f)$ is not two-sided in $D(f)$ over $f^{-1}(S)$ and $a = 0$ otherwise, and $b = 1$ if the normal bundle of S in $S'(f)$ is non-trivial and $b = 0$ otherwise. Since f is not nice, ϕ is a non-trivial homomorphism. Furthermore, $\mathrm{Im}(\phi) = \mathbf{Z}_2 \oplus \{0\}$ if $m - n$ is even, and $\mathrm{Im}(\phi)$ is the subgroup generated by (1,1) in $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ if $m - n$ is odd by 4.11 of [5].

By Lemma 1 and 2.8, kill the kernel of ϕ thus making the inclusion homomorphism $\pi_1(S'(f)) \to \pi_1(\Delta(f))$ an isomorphism with both groups isomorphic to $\mathbf{Z}_2$. We can also kill the homotopy groups below the middle dimension as in the simply connected case. Hence after finite steps we can find a generic map g homotopic to f such that $\pi_1(S'(g)) \cong \pi_1(\Delta(g)) \cong \mathbf{Z}_2$, $\tilde{\Delta}(g)$ and $\tilde{S}'(g)$ are $([(p+1)/2]-1)$ and $([p/2]-1)$-connected, respectively, where $\tilde{\Delta}(g)$ denotes the universal covering space of $\Delta(g)$.

5.2. Suppose that f is homotopic to a pseudo-embedding g'. To prove the theorem we must produce a sequence of null homotopies for the handles in any given handle decomposition of $\Delta(g)$ except for a top dimensional handle.

Let $F : V \times I \to M \times I$ be a generic homotopy (not necessarily level preserving) from g to g'. We will homotopy F relative to $V \times \{0\} \cup V \times \{1\}$ so that $\Delta(F)$ becomes a h-cobordism between $\Delta(g)$ and $S'(F) \cup \Delta(g')$.

By attaching and subtracting handles on F away from 0 and 1-level, make $\Delta(F)$ and $S'(F)$ connected and $(\Delta(F), S'(F))$ 1-connected. Attach 2-handles along $S'(F)$ to make $\phi : \pi_1(S'(F)) \to \mathbf{Z}_2 \oplus \mathbf{Z}_2$ a monomorphism. The inclusion of $\pi_1(S'(F))$ into $\pi_1(\Delta(F))$ is an isomorphism. This implies that the inclusion homomorphism of $\pi_1(S'(g))$ into $\pi_1(S'(F))$ is also an isomorphism since the following diagram commutes.

$$\begin{array}{ccc} \pi_1(S'(g)) & \overset{\phi}{\searrow} & \\ \text{inclusion homomorphism} \downarrow & & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\ \pi_1(S'(F)) & \underset{\phi}{\nearrow} & \end{array}$$

We now divide the argument into two cases.

(1) $p = 2k$.

As in 5.1, make $\Delta(F)$ k-connected and $S'(F)$ $(k-1)$-connected. Now $\partial\Delta(F) = D^{p+1} \cup S'(F) \cup \Delta(g)$. Let $N = D^{p+1} \cup S'(F) = Cl(\partial\Delta(F) - \Delta(g))$. In this section, the homology of a manifold or a pair is understood to be the homology of the universal covering space with coefficients $\mathbf{Z}$, i.e., $H_i(N)$ denotes actually $H_i(\tilde{N})$, where $\tilde{N}$ is the universal cover of N. This group can be considered as a $\mathbf{Z}\pi_1(N)$-module. Similarly, the cohomology of a manifold N (or a pair) is the cohomology induced from the cellular chain complex of $\mathbf{Z}\pi_1(N)$-modules of the universal covering space of N with coefficients $\mathbf{Z}\pi_1(N)$ unless it is said otherwise. (See [12] for more information.)

From the connectivities of $\Delta(g)$, $\Delta(F)$ and $S'(F)$, $H_i(\Delta(F), N) \cong 0$ for all i except for $i = k+1$ by duality. Also $H^{k+2}(\Delta(F), N) = H_k(\Delta(F), \Delta(g)) = 0$. (This is true for every $\mathbf{Z}\mathbf{Z}_2$-module as coefficients.) By Lemma 2.3 of [12], $H_{k+1}(\Delta(F), N)$ is a finitely generated stably free $\mathbf{Z}\mathbf{Z}_2$-module. By attaching trivial $(k+1)$-handles along $S'(F)$, make $H_{k+1}(\Delta(F), N)$ a free $\mathbf{Z}\mathbf{Z}_2$-module. Now represent, a set of basis elements of $H_{k+1}(\Delta(F), N)$ by disjoint $(k+1)$-handles relative to $S'(F)$ as in §4 of [12], and subtract the handles.

It follows that $\Delta(F)$ is an h-cobordism between $\Delta(g)$ and $S'(F) \cup \Delta(g')$. Hence $\Delta(F)$ is diffeomorphic to $\Delta(g) \times I$ since $Wh(\mathbf{Z}_2) = 0$.

We now need a lemma whose proof is postponed until 5.3.

DEFINITION. Let C be a compact manifold with a non-empty boundary, and let A be a compact codimension 0 submanifold of ∂C or empty. Define the wedge $W(C, A)$ of C relative to A as the manifold with corners obtained by rotating C by 90° about A. If $A = \emptyset$, then $W(C, A)$ is defined to be $C \times I$. Let $W_0(C, A)$ be the copy of C at 0° in $W(C, A)$ and $W_1(C, A)$ the copy of C at 90°. Note that $W(C, A)$ contains naturally a copy of A and we denote it by A again. Observe that $W(C, A)$ can be regarded as $C \times I$ with $A \times I$ collapsed to $A \times \{0\}$.

REMARK. $S^{r+1} (r \geq 0)$ may be regarded as the union of four copies of $W(D^r, S^{r-1})$.

LEMMA 3. *Let $F : V \times I \to M \times I$ be a generic map such that $F(V \times \{0\}) \subset M \times \{0\}$ and $F(V \times \{1\}) \subset M \times \{1\}$. Let g be $F|V \times \{0\}$. Let C be a compact connected manifold with non-empty boundary, and let A be a compact submanifold of ∂C of codimension 0 or empty. Suppose that there is an embedding $h : W(C, A) \to \Delta(F)$ such that $h(W(C, A)_0) \subset \Delta(g)$, $h(A) = S'(g) \cap h(W(C, A)_0)$, $h(W(C, A)_1) \subset S'(F)$ and $h(W(C, A)) \cap M \times \{1\} = \emptyset$. Then there exists a generic map $F' : V \times I \to M \times I$ such that $\Delta(F') = Cl(\Delta(F) - Im(h)), g' = F' \mid V \times \{0\}$ is a generic map obtained from g by a sequence of handle subtractions, $\Delta(g) = Cl(\Delta(g) - h(W(C, A)_0))$ and $F' \mid V \times \{1\} = F|V \times \{1\}$.*

Using the lemma, we finish the proof of the first case. Given a handle decomposition of $\Delta(g)$ relative to $S'(g)$ with a top dimensional handle H (this is a $(2n-m)$-disk and there is one in any handle decomposition), define a trivialization $T : W(\Delta(g), S'(g)) \to \Delta(F)$ by the h-cobordism theorem such that $T|W_0(\Delta(g), S'(g))$ is the identity and $T(W_1(H,\emptyset)) = \Delta(g') \times \{1-\varepsilon\} \subset M \times \{1-\varepsilon\}$ for some small $\varepsilon > 0$, where we assume that F is a product for $t \geq 1-2\varepsilon$, $t \in I$ and $W(H,\emptyset)$ is regarded as a subset of $W(\Delta(g), S'(g))$.

Let $C = Cl(\Delta(g) - H)$, $A = S'(g) \subset \partial C$ and $h : W(C,A) \to \Delta(F)$ be an embedding defined by $h = T|W(C,A)$, where $W(C,A)$ is regarded as a subset of $W(\Delta(g), S'(g))$. By Lemma 3 we can subtract all the handles except for H.

(2) $p = 2k+1$.

After the initial modification $S'(g)$ is $(k-1)$-connected, and $\Delta(g)$ k-connected. We may also assume that $S'(F)$ and $\Delta(F)$ are k-connected. Let $N = Cl(\partial\Delta(F) - \Delta(g))$. By subtracting $(k+1)$-handles in $\Delta(F)$ relative to $S'(F)$, make $(\Delta(F), N)$ $(k+1)$-connected with the same connectivity conditions on $\Delta(F)$ and $(\Delta(F), \Delta(g))$.

Since $H_i(\Delta(F), \Delta(g)) = 0$, $i \leq k$, $H_i(\Delta(F), \Delta(g)) \cong H^{2k+3-i}(\Delta(F), N) \cong 0$, $i \geq k+2$, and $H^{k+2}(\Delta(F), \Delta(g)) \cong H_{k+1}(\Delta(F), N) \cong 0$ (with any $\mathbf{Z}\,\mathbf{Z}_2$-module as coefficients), $H_{k+1}(\Delta(F), \Delta(g))$ is a finitely generated stably free $\mathbf{Z}\,\mathbf{Z}_2$-module. By attaching trivial $(k+1)$-handles along $S'(F)$, make $H_{k+1}(\Delta(F), \Delta(g))$ a free $\mathbf{Z}\,\mathbf{Z}_2$-module. Represent the basis elements by a disjoint union H of $(k+1)$-handles in $\Delta(F)$ relative to $\Delta(g)$. Let $W = Cl(\Delta(F) - H)$ and $N' = Cl(\partial W - N)$. From the long exact sequence of the triple $(\Delta(F),\ H \cup \Delta(g),\ \Delta(g))$, it can be seen that $(W; N', N)$ is an h-cobordism. Since $Wh(\mathbf{Z}_2) = 0$, there exists a diffeomorphism $T : W(N', \partial N') \to W$ such that $T|W_0(N', \partial N')$ is the identity and $(T|W_1(N', \partial N'))^{-1}(\Delta(g')) \subset W_1(N' - \Delta(g), \phi)$. Furthermore, we may assume that $H \cap \Delta(g)$ is contained in a top dimensional handle of any handle decomposition of $\Delta(g)$ relative to $S'(g)$ since $\Delta(g)$ is k-connected.

Let $C = Cl(\Delta(g)$-the above top dimensional handle). Then we have $T(W(C, S'(g))) \subset \Delta(F)$, $T|W_0(C, S'(g))$ is the identity, and $T(W_1(C, S'(g))) \subset S'(F)$. As in the first case apply Lemma 3 to subtract all the handles in C. This completes the proof.

5.3. Proof of lemma 3.

It suffices to prove the lemma when C is an r-handle. Suppose that $H = D^r \times D^{2n-m-r}$ is an r-handle of $\Delta(g)$ relative to $S'(g)$ with $S^{r-1} \times D^{2n-m-r}$ as A. Put $q = 2n - m$. Let $h : W(D^r \times D^{q-r}, S^{r-1} \times$

$D^{q-r}) \to \Delta(F)$ be an embedding satisfying the assumptions of the lemma. Let $W = W(D^r \times D^{q-r}, S^{r-1} \times D^{q-r})$.

Define reflection $\alpha : M \times [-1,1] \to M \times [-1,1]$ by $\alpha(x,t) = (x,-t)$, $(x,t) \in M \times [-1,1]$. Now construct generic map $G : V \times [-1,1] \to M \times [-1,1]$ by $G|V \times [0,1] = F$ and $(G|V \times [-1,0])(x,t) = \alpha(F(x,-t))$.

There is a natural $(r+1)$-handle in $\Delta(G)$ relative to $S'(G)$ obtained from h as follows. Regard $D^{r+1} \times D^{q-r}$ as the union of two copies W^1 and W^2 of W with W_0^1 identified with W_0^2 (W_0 is the 0-level of W.) Define $(r+1)$-handle $h' : (D^{r+1} \times D^{q-r}, S^r \times D^{q-r}) \to (\Delta(G), S'(G))$ by $h' \,|\, W^1 = h$ and $h' \,|\, W^2 = \alpha h$.

Then h' determines a commutative diagram $\theta(h')$ as in 4.1.

$$\begin{array}{ccc} S^{r+1} & \xrightarrow{e} & D^{r+2} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ V \times [-1,1] & \xrightarrow{g} & M \times [-1,1] \end{array}$$

In the above diagram, e is an inclusion, $j\,|D_+^{r+1}$ is a diffeomorphism onto $h'(D^{r+1} \times \{0\})$ and i is a diffeomorphism onto $G^{-1}(h'(D^{r+1} \times \{0\}))$.

Using the symmetry of G in $V \times \{0\}$, it is easy to produce a null homotopy of $\theta(h')$ which restricts to a null homotopy of $\theta(H)$ in 0-level. We subtract the handle h' from G to get a generic map G'. The construction of G_τ, $0 \le \tau \le 1$ in §4 of [5] can be done so that G_τ preserves 0-level for all time (see §5 of [5]). Let $F' = G'|V \times [0,1]$. Then F' has the desired properties of the lemma.

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