J. Korean Math. Soc. 28(1991), No. 2, pp. 341–358

A NOTE ON JACOBI FORMS OF HIGHER DEGREE

JAE-HYUN YANG AND JIN WOO SON

1. Introduction

Let H_{n+j} be the Siegel upper half plane of degree $n+j$ $(n,j \in Z^+)$ and let Γ_{n+j} be the Siegel modular group of degree $n+j$. We consider a Siegel modular form $f \in [\Gamma_{n+j}, k]$ of weight k. Then $f(Z)$ has a Fourier expansion

$$(1) \qquad f(Z) = \sum_{\substack{T \geq 0 \\ \text{half integral}}} c(T) e^{2\pi i \sigma(TZ)}, \quad Z \in H_{n+j},$$

where $\sigma(M)$ denotes the trace of a matrix M and ${}^tT$ denotes the transpose of T. We write

$$Z = \begin{pmatrix} Z_1 & {}^tW \\ W & Z_2 \end{pmatrix}, \quad Z_1 \in H_n,\ Z_2 \in H_j,\ W \in C^{(j,n)}$$

and

$$T = \begin{pmatrix} T_1 & \frac{1}{2}R \\ \frac{1}{2}{}^tR & T_2 \end{pmatrix}, \quad 2T_1 \in Z^{(n,n)},\ 2T_2 \in Z^{(j,j)},\ R \in Z^{(n,j)},$$

where $F^{(k,j)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. According to (1), we have the so-called Fourier-Jacobi expansion $\Phi_{T_2}(Z_1, W)$ of a Siegel modular form f. That is,

$$(2) \qquad f(Z) = \sum_{\substack{T_2 \geq 0 \\ \text{half integeral}}} \Phi_{T_2}(Z_1, W) e^{2\pi i \sigma(T_2 Z_2)}.$$

$\Phi_{T_2}(Z_1, W)$ satisfies some functional equations and so $\Phi_{T_2}(Z_1, W)$ is a Jacobi form of index T_2 (see Definition 2.4). Eichler and Zagier ([E-Z])

Received December 5, 1990.

have developped the theory of these functions called Jacobi forms in the special case $n = j = 1$. Yamazaki ([Y]) and Klingen ([K]) deal with the case for any n and $j = 1$. Ziegler ([Z]) generalized these functions to the general case for any n, j.

In the present article, we discuss holomorphic differential forms on $H_n \times C^{(j,n)}$ which are invariant under some discrete subgroup of the Jacobi group and show that the study of such holomorphic forms is closely related to that of Jacobi forms. In section 2, we give a definition of Jacobi forms. In section 3, we characterize Jacobi forms as smooth functions on the Jacobi group satisfying a certain transformation law and compute the Lie algebra of the Jacobi group. In section 4, we study some properties of theta series. In section 5, we discuss $\Gamma_{(n,j)}$-invariant holomorphic forms on $H_n \times C^{(j,n)}$ (see section 5 for the definition of $\Gamma_{(n,j)}$).

NOTATIONS. We denote by Z, R and C the ring of integers, the field of real numbers, and the field of complex numbers respectively. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, R)$ and $Z \in H_n$, we set $M\langle Z\rangle : (AZ + B)(CZ + D)^{-1}$. $\Gamma_n := Sp(n, Z)$ denotes the Siegel modular group of degree n. $[\Gamma_n, k]$ denotes the vector space of all Siegel modular forms of weight k. The symbol ":=" means that the expression on the right is the definition of that on the left. We denotes by Z^+ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. E_n denotes the identity matrix of degree n.

2. Jacobi forms

We establish notation and review some basic properties of Jacobi forms. Let $f \in [\Gamma_{n+j}, k]$ be a Siegel modular form on H_{n+j} of weight k. Then $f(Z)$ has a Fourier expansion (1). With the same notation in the previous section, we have the so-called Fourier-Jacobi expansion $\Phi_{T_2}(Z_1, W)$ (see (2)) of a Siegel modular form f. Precisely, $\Phi_{T_2}(Z_1, W)$ is given by

$$\Phi_{T_2}(Z_1, W) = \sum_{T_1, R} c(T_1, R, T_2) e^{2\pi i \sigma(T_1 Z_1 + RW)},$$

where $T_1,\ R$ runs through the set of all $T_1,\ R$ such that $\begin{pmatrix} T_1 & \frac{1}{2}R \\ \frac{1}{2}R & T_2 \end{pmatrix} \geq 0$.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. Then

$$\tilde{M} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & E_j & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & E_j \end{pmatrix}$$

is an element of Γ_{n+j}. Then $\Phi_{T_2}(Z_1, W)$ satisfies the functional equation

$$\begin{aligned} (3) \qquad & \Phi_{T_2}(M\langle Z_1\rangle,\ W(CZ_1+D)^{-1}) \\ &= \det(CZ_1+D)^k e^{2\pi i\sigma(T_2 W(CZ_1+D)^{-1}C\,{}^tW)}\Phi_{T_2}(Z_1,W). \end{aligned}$$

Let

$$M' = \begin{pmatrix} E_n & 0 & 0 & {}^t\mu \\ \lambda & E_j & \mu & \kappa \\ 0 & 0 & E_n & -{}^t\lambda \\ 0 & 0 & 0 & E_j \end{pmatrix}, \quad \kappa + \mu\,{}^t\lambda = {}^t(\kappa + \mu\,{}^t\lambda)$$

be an element of Γ_{n+j}. Then

$$M'\langle Z\rangle = \begin{pmatrix} Z_1 & {}^t(W+\lambda Z_1+\mu) \\ W+\lambda Z_1+\mu & Z_2+\lambda Z_1\,{}^t\lambda+\lambda\,{}^tW+W\,{}^t\lambda+(\kappa+\mu\,{}^t\lambda) \end{pmatrix}.$$

Thus $\Phi_{T_2}(Z_1, W)$ satisfies the functional equation

$$\begin{aligned} (4) \qquad & \Phi_{T_2}(Z_1, W+\lambda Z_1+\mu) \\ &= e^{-2\pi i\sigma(T_2(\lambda Z_1\,{}^t\lambda+2\lambda\,{}^tW+(\kappa+\mu\,{}^t\lambda)))}\Phi_{T_2}(Z_1,W) \end{aligned}$$

for any $\lambda, \mu \in Z^{(j,n)}$ and $\kappa \in Z^{(j,j)}$ with $\kappa + \mu\,{}^t\lambda$ symmetric. We consider the *Heisenberg group*

$$H_R^{(n,j)} := \{[(\lambda,\mu),\kappa] \,|\, \lambda,\mu \in R^{(j,n)}, \kappa \in R^{(j,j)}, (\kappa+\mu\,{}^t\lambda) \text{ symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa]\circ[(\lambda',\mu'),\kappa'] := [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda^t\mu'-\mu^t\lambda'].$$

$[(0,0),0]$ is the unit element of $H_R^{(n,j)}$ and the inverse element of $[(\lambda,\mu),\kappa]$ is given by $[(-\lambda,-\mu),-{}^t\kappa]$. We note that the symplectic group $Sp(n,R)$ of degree n acts on $H_R^{(n,j)}$ as follows:

$$(5)\qquad [(\lambda,\mu),\kappa]\bullet M := [(\lambda,\mu)\bullet M,\kappa],\quad M\in Sp(n,R).$$

Now we define the semidirect product $G_R^{(n,j)} := Sp(n,R)\ltimes H_R^{(n,j)}$ endowed with the multiplication law:

$$\begin{aligned}&(M,[(\lambda,\mu),\kappa])\bullet(M',[(\lambda',\mu'),\kappa'])\\ :=\ &(MM',[(\tilde\lambda+\lambda',\tilde\mu+\mu'),\kappa+\kappa'+\tilde\lambda^t\mu-\tilde\mu^t\lambda']),\end{aligned}$$

where $M,M'\in Sp(n,R)$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)\bullet M'$. We call this group $G_R^{(n,j)}$ the *Jacobi group*. It is easy to see that the Jacobi group $G_R^{(n,j)}$ acts on $H_n\times C^{(j,n)}$ by

$$(6)\quad (M,[(\lambda,\mu),\kappa])\bullet(Z,W) := (M\langle Z\rangle,(W+\lambda Z+\mu)(CZ+D)^{-1}).$$

DEFINITION 2.1. Let $\rho: GL(n,C)\to GL(V_\rho)$ be a rational representation of $GL(n,C)$ on a finite dimensional complex vector space V_ρ. We denote by $\mathcal{O}(H_n\times C^{(j,n)},V_\rho)$ the vector space of all holomorphic functions on $H_n\times C^{(j,n)}$ with values in V_ρ. For any $\Phi\in\mathcal{O}(H_n\times C^{(j,n)},V_\rho)$, $M\in Sp(n,R)$, $\zeta=[(\lambda,\mu),\kappa]\in H_R^{(n,j)}$ and $\mathcal{M}\in R^{(j,j)}$ with $\mathcal{M}\geq 0$ symmetric and half integral, we define

$$\begin{aligned}(\Phi|_{\rho,\mathcal{M}}M)(Z,W) :=&\rho(CZ+D)^{-1}e^{-2\pi i\sigma(\mathcal{M}W(CZ+D)^{-1}C^tW)}\\ &\Phi(M\langle Z\rangle,W(CZ+D)^{-1})\end{aligned}$$

and

$$(\Phi|_{\rho,\mathcal{M}}\zeta)(Z,W) := e^{2\pi i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^tW+(\kappa+\mu^t\lambda)))}\Phi(Z,W+\lambda Z+\mu).$$

From now on, for brevity we write $\Phi|M$, $\Phi|\zeta$ instead of $\Phi|_{\rho,\mathcal{M}}M$, $\Phi|_{\rho,\mathcal{M}}\zeta$.

LEMMA 2.2. *For any $M, M' \in Sp(n,R)$ and $\zeta, \zeta' \in H_R^{(n,j)}$, we have the following*

$$\begin{aligned} \Phi|M|M' &= \Phi|(MM'), \\ \Phi|\zeta|\zeta' &= \Phi|(\zeta \circ \zeta'), \\ \Phi|\zeta|M &= \Phi|M|(\zeta M). \end{aligned}$$

COROLLARY 2.3. *$G_R^{(n,j)}$ acts on $\mathcal{O}(H_n \times C^{(j,n)}, V_\rho)$ by*

$$\Phi \to \Phi \bullet (M,\zeta) := \Phi|M|\zeta,\ (M,\zeta) \in G_R^{(n,j)},\ \Phi \in \mathcal{O}(H_n \times C^{(j,n)}, V_\rho).$$

For the proof of the above lemma, we refer the reader to [Z].

We observe that an element $(M,\zeta) \in G_R^{(n,j)}$ acts trivially on $\mathcal{O}(H_n \times C^{(j,n)}, V_\rho)$ if and only if $M = E_{2n}$ and $\zeta = [(0,0),\kappa]$ such that $\sigma(\mathcal{M}\kappa) \in Z$. Let

$$\mathcal{N}_{\mathcal{M}} := \{(E_{2n}, [(0,0),\kappa]) \in G_R^{(n,j)} \,|\, \sigma(\mathcal{M}_\kappa) \in Z\}.$$

Then $\mathcal{N}_{\mathcal{M}}$ is a normal subgroup of $G_R^{(n,j)}$ and we have a faithful representation of the quotient group $G_R^{(n,j)}/\mathcal{N}_{\mathcal{M}}$ on $\mathcal{O}(H_n \times C^{(j,n)}, V_\rho)$.

DEFINITION 2.4. Let ρ and $\mathcal{M}$ be the same in Definition 2.1. Let

$$H_Z^{(n,j)} := \{[(\lambda,\mu),\kappa] \in H_R^{(n,j)} \,|\, \lambda,\mu \in Z^{(j,n)},\ \kappa \in Z^{(j,j)}\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on a subgroup $\Gamma \subset \Gamma_n$ of finite index is a holomorphic mapping $\Phi \in \mathcal{O}(H_n \times C^{(j,n)}, V_\rho)$ satisfying the following conditions (A), (B) and (C):

(A) $\Phi|M = \Phi$ for every $M \in \Gamma$.

(B) $\Phi|\zeta = \Phi$ for every $\zeta \in H_Z^{(n,j)}$.

(C) For each $M \in \Gamma_n$, the function $\Phi|M$ has a Fourier development of the following form:

$$(\Phi|M)(Z,W) = \sum_{\substack{T \geq 0 \\ \text{half integral}}} \ \sum_{R \in Z^{(n,j)}} c(T,R) e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(TZ)} e^{2\pi i \sigma(RW)}$$

with a suitable $\lambda_\Gamma \in Z$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$. If $n \geq 2$, the condition (C) is superfluous by Koecher principle (see [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. In the special case $V_\rho = C$, $\rho(A) = (\det A)^k$ ($k \in Z, A \in GL(n,C)$), we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the weight of a Jacobi form $\Phi \in J_{k,\mathcal{M}}(\Gamma)$. We observe that $\Phi_{T_2}(Z_1, W)$ is a Jacobi form in J_{k,T_2}.

Ziegler ([Z] Theorem 1.8 and [E-Z] Theorem 1.1) proves the following.

THEOREM 2.5. *The space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional.*

3. The Jacobi group

In this section, we realize Jacobi forms as smooth functions on the Jacobi group satisfying a certain transformation law and compute the Lie algebra of the Jacobi group $G_R^{(n,j)}$. It is easy to see that the Jacobi group $G_R^{(n,j)}$ acts on $H_n \times C^{(j,n)}$ transitively (cf. (6)). The stabilizer $K_R^{(n,j)}$ of $G_R^{(n,j)}$ at $(iE_n, 0)$ is given by

$$(7) \qquad K_R^{(n,j)} \cong U(n) \ltimes \{[(0,0),\kappa] \in H_R^{(n,j)} \mid \kappa = {}^t\kappa \in R^{(j,j)}\}.$$

The quotient $G_R^{(n,j)}/K_R^{(n,j)}$ can be identified with $H_n \times C^{(n,j)}$ via

$$(8) \qquad (M, [(\lambda,\mu),\kappa]) \bullet K_R^{(n,j)} \mapsto (M, [(\lambda,\mu),\kappa]) \bullet (iE_n, 0).$$

But $K_R^{(n,j)}$ is not compact. Thus we shall consider another group G_R^J so that the stabilizer K_R^J of G_R^J at $(iE_n, 0)$ is maximally compact. The center $C_R^{(n,j)}$ of $H_R^{(n,j)}$ is given by

$$C_R^{(n,j)} = \{[(0,0),\kappa] \in H_R^{(n,j)} \mid \kappa = {}^t\kappa\}.$$

Clearly $\dim_R C_R^{(n,j)} = \frac{j(j+1)}{2}$. We set

$$C_Z^{(n,j)} = \{[(0,0),\kappa] \in C_R^{(n,j)} \mid \kappa \in Z^{(j,j)}\}.$$

We consider the quotient group G_R^J of $G_R^{(n,j)}$ by

$$G_R^J := Sp(n,R) \ltimes H_R^{(n,j)}/C_Z^{(n,j)}. \tag{9}$$

We observe that $H_R^{(n,j)}/C_Z^{(n,j)}$ is a central extension of $R^{(j,n)} \times R^{(j,n)}$ by the abelian group $(S^1)^l$, where $S^1 = \{z \in C|\ |z| = 1\}$ is the unit circel and $l = \frac{j(j+1)}{2}$. G_R^J also acts on $H_n \times C^{(j,n)}$ transitively. The stabilizer K_R^J of G_R^J at $(iE_n, 0)$ is given by

$$K_R^J = U(n) \ltimes C_R^{(n,j)}/C_Z^{(n,j)}. \tag{10}$$

We note that K_R^J is a maximal compact subgroup of G_R^J. The homogeneous space G_R^J/K_R^J is diffeomorphic to the underlying space $H_n \times C^{(j,n)}$ via

$$(M, [(\lambda,\mu),\kappa]) \bullet K_R^J \mapsto (M, [(\lambda,\mu),\kappa]) \bullet (iE_n, 0). \tag{11}$$

Let

$$\Gamma^J := Sp(n,Z) \ltimes H_Z^{(n,j)}/C_Z^{(n,j)}$$

be the discrete subgroup of G_R^J. It is easy to show that $\mathrm{vol}(\Gamma^J\backslash G_R^J)$ is finite.

Let ρ be an irreducible rational representation of $GL(n,C)$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} = (\mathcal{M}_{kl})$ be a semipositive, symmetric half integral matrix of degree j. According to Corollary 2.3, we have

(12)

$$\begin{aligned}
&(\Phi|_{\rho,\mathcal{M}}(M,\zeta))(Z,W)\\
:= {}& e^{-2\pi i\sigma(\mathcal{M}(W+\lambda Z+\mu)(CZ+D)^{-1}C^t(W+\lambda Z+\mu))}\\
& e^{2\pi i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^t W+(\kappa+\mu^t\lambda))}\\
& \rho(CZ+D)^{-1}\Phi(M\langle Z\rangle,(W+\lambda Z+\mu)(CZ+D)^{-1}),
\end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,R)$ and $\zeta = [(\lambda,\mu),\kappa] \in H_R^{(n,j)}$.

Let $N_{\mathcal{M}} := \{[(0,0),\kappa] \in H_R^{(n,j)}\,|\,\sigma(\mathcal{M}\kappa) \in Z\}$. Clearly $C_Z^{(n,j)} \subset N_{\mathcal{M}}$. Thus if $\Phi \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is a Jacobi form of index $\mathcal{M}$ with respect to ρ, then Φ is invariant under the action of Γ^J. Conversely, if Φ is invariant under Γ^J, then $\Phi \in J_{\rho,\mathcal{M}}(\Gamma_n)$. On the other hand, to each Jacobi form $\Phi \in J_{\rho,\mathcal{M}}(\Gamma_n)$ we can associate a function F on G_R^J with values in V_ρ defined by

$$(13) \qquad F(g) := (\Phi|_{\rho,\mathcal{M}}g)(iE_n, 0), \quad g \in G_R^J.$$

Then $F : G_R^J \mapsto V_\rho$ is invariant under the left translation on Γ^J by Corollary 2.3. In addition, F satisfies the following transformation behaviour

$$(14) \qquad F(g \bullet (M, [(0,0), \kappa])) = e^{2\pi i \sigma(\mathcal{M}\kappa)} \rho(iC + D)^{-1} F(g),$$

where $g \in G_R^J$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n)$ and $\kappa \in R^{(j,j)}$ is symmetric. Conversely,if $F : G_R^J \mapsto V_\rho$ is a smooth function on G_R^J which is invariant under the left translation of Γ^J and satisfies the transformation behaviour (14), then the function $\Phi : H_n \times C^{(j,n)} \mapsto V_\rho$ defined by

$$(15) \qquad \Phi(g\langle iE_n, 0\rangle) = e^{2\pi i \sigma(\mathcal{M}(i\lambda+\mu)(iC+D)^{-1}C^t(i\lambda+\mu))} \\ e^{-2\pi i \sigma(\mathcal{M}(i\lambda^t\lambda+\mu^t\lambda+\kappa))} \rho(iC+D) F(g)$$

is precisely a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_n)$, where $g = (M, [(\lambda,\mu),\kappa]) \in G_R^J$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, R)$. Here we note that G_R^J acts on $H_n \times C^{(j,n)}$ transitively.

So far we proved

THEOREM 3.1. *With the notations as above, the Jacobi functions $J_{\rho,\mathcal{M}}(\Gamma_n)$ are precisely the functions in $\mathcal{O}(H_n \times C^{(j,n)}, V_\rho)$ which are invariant under the discrete subgroup Γ^J of G_R^J. A Jacobi form $\Phi \in J_{\rho,\mathcal{M}}(\Gamma_n)$ can be identified with a function $F : G_R^J \mapsto V_\rho$ defined by (13) which is invariant under the left action of Γ^J and satisfies the transformation behaviour (14).*

Now we will compute the Lie algebra of the Jacobi group $G_R^{(n,j)}$ explicitly. For each $M \in Sp(n,R)$, we let $t_M : H_R^{(n,j)} \to H_R^{(n,j)}$ be the automorphism of $H_R^{(n,j)}$ defined by

$$(16) \qquad t_M([(\lambda,\mu),\kappa) := [(\lambda,\mu) \bullet M, \kappa], \ [(\lambda,\mu),\kappa] \in H_R^{(n,j)}.$$

The mapping

$$(17)\qquad H_R^{(n,j)} \ni [(\lambda,\mu),\kappa] \overset{i_1}{\mapsto} \begin{pmatrix} E_n & 0 & 0 & {}^t\mu \\ \lambda & E_j & \mu & \kappa \\ 0 & 0 & E_n & -{}^t\lambda \\ 0 & 0 & 0 & E_j \end{pmatrix}$$

defines an embedding of $H_R^{(n,j)}$ into $Sp(n+j,R)$. Also the mapping

$$(18)\qquad Sp(n,R) \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \overset{i_2}{\mapsto} \begin{pmatrix} A & 0 & B & 0 \\ 0 & E_j & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & E_j \end{pmatrix}$$

defines an embedding of $Sp(n,R)$ into $Sp(n+j,R)$. By the above embedding i_1, we can obtain the Lie algebra $\mathcal{H}_R^{(n,j)}$ of $H_R^{(n,j)}$ as a subalgebra of the Lie algebra $sp(n+j,R)$ of $Sp(n+j,R)$ explicitly. That is,

$$\mathcal{H}_R^{(n,j)} = \left\{ \begin{pmatrix} 0 & 0 & 0 & {}^tb \\ a & 0 & b & c \\ 0 & 0 & 0 & -{}^ta \\ 0 & 0 & 0 & 0 \end{pmatrix} \in sp(n+j,R) \,\middle|\, a,b \in R^{(j,n)}, \quad c = {}^tc \in R^{(j,j)} \right\}.$$

By the above embedding i_2, the Lie algebra $sp(n,R)$ of $Sp(n,R)$ can be regarded as a subalgebra of $sp(n+j,R)$. Precisely,

$$sp(n,R) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & -{}^ta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, b = {}^tb,\ c = {}^tc,\ a,b,c \in R^{(n,n)} \right\}.$$

The following lemma is easy to prove and so we omit it.

LEMMA 3.2. *By the embeddings i_1 and i_2, the automorphism t_M of $H_R^{(n,j)}$ may be realized as the automorphism $\tilde{t}_M$ of the group $i_1(H_R^{(n,j)})$ defined by*

$$(19)\qquad \tilde{t}_M(i_1(h)) = i_2(M)^{-1} i_1(h) i_2(M),\ M \in Sp(n,R),\ h \in H_R^{(n,j)}.$$

In other words, for any $M \in Sp(n,R)$ and $h \in H_R^{(n,j)}$,

$$i_1(t_M(h)) = i_2(M)^{-1} i_1(h) i_2(M). \tag{20}$$

Let τ_M be the differential of t_M. The set $\{\tau_M \mid M \in Sp(n,R)\}$ define a representation $\tau : Sp(n,R) \to GL(\mathcal{H}_R^{(n,j)})$ of $Sp(n,R)$ on $\mathcal{H}_R^{(n,j)}$. Let $\theta = d\tau$ be the differential of τ. We note that θ is a Lie algebra homomorphism of $sp(n,R)$ into $\mathcal{G}l(\mathcal{H}_R^{(n,j)})$. Now we will determine the maps τ_M and θ. By the relations,

$$\exp(\tau_M(Y)) = M^{-1}(\exp Y)M, \quad Y \in H_R^{(n,j)}$$

and

$$\exp\theta(X) = \tau(\exp X), \quad X \in sp(n,R),$$

we obtain

$$\tau_M(Y) = M^{-1} Y M, \quad Y \in \mathcal{H}_R^{(n,j)} \tag{21}$$

and

$$\theta(X) = \log \tau_{\exp(X)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\tau_{exp(X)} - I)^k}{k}, \; X \in sp(n,R), \tag{22}$$

where I denotes the identity mapping of $\mathcal{H}_R^{(n,j)}$ and we identified M with $i_2(M)$ by the embedding i_2. Clearly $\theta(X)$ is a derivation of $\mathcal{H}_R^{(n,j)}$ for every $X \in sp(n,R)$.

In summary, we proved

THEOREM 3.3. *With the same notations as above, the Lie algebra $\mathcal{G}_R^{(n,j)}$ of the Jacobi group $G_R^{(n,j)}$ is the semidirect product of $\mathcal{H}_R^{(n,j)}$ with $sp(n,R)$ relative to θ. The Lie bracket $[\ ,\ ]$ on $\mathcal{G}_R^{(n,j)}$ is given by*

$$[(X,Y),(X_1,Y_1)] = ([X,X_1],[Y,Y_1] + \vartheta(X_1,)Y - \vartheta(X)Y_1) \tag{23}$$

for all $X, X_1 \in sp(n,R)$ and $Y, Y_1 \in \mathcal{H}_R^{(n,j)}$. Let $\mathcal{G}_1 = sp(n,R) \times \{0\}$ and $\mathcal{G}_2 = \{0\} \times \mathcal{H}_R^{(n,j)}$. Then $\mathcal{G}_1$ is a subalgebra of $\mathcal{G}_R^{(n,j)}$ and $\mathcal{G}_2$ is an ideal of $\mathcal{G}_R^{(n,j)}$.

4. Theta series

Let S be a symmetric, positive definite integral matrix of degree j and let $a, b \in Q^{(j,n)}$. We consider

$$\vartheta_{S,a,b}(Z,W) := \sum_{\lambda \in Z^{(j,n)}} e^{\pi i \sigma(S((\lambda+a)Z^t(\lambda+a)+2(\lambda+a)^t(W+b)))} \tag{24}$$

with characteristic (a, b) converging uniformly on any compact subset of $H_n \times C^{(j,n)}$.

Let $\mathcal{M}$ be a symmetric, positive definite and half-integral matrix of degree j and let $\mathcal{N}$ be a complete system of representatives of the cosets $(2\mathcal{M})^{-1}Z^{(j,n)}/Z^{(j,n)}$. We observe that $\#(\mathcal{N}) = \{\det(2\mathcal{M})\}^n$. An easy application of the Poisson summation formula gives

LEMMA 4.1. *For $a \in \mathcal{N}$, we have*

(25)

$$\vartheta_{2\mathcal{M},a,0}(-Z^{-1}, WZ^{-1}) = \{\det(2\mathcal{M})\}^{-\frac{n}{2}}\{\det(\frac{Z}{i})\}^{\frac{j}{2}} e^{2\pi i \sigma(\mathcal{M}WZ^{-1\,t}W)} \sum_{b \in \mathcal{N}} e^{-2\pi i \sigma(2\mathcal{M}b^t a)} \vartheta_{2\mathcal{M},b,0}(Z,W).$$

COROLLARY 4.2. *Let $\mathcal{M}$ be unimodular. Then $\vartheta_{2\mathcal{M},0,0}(Z,W)$ is a Jacobi form of weight $\frac{j}{2}$ and index $\mathcal{M}$.*

LEMMA 4.3. *Let $S = mI_j$ with $m > 0$. Then $\vartheta_{S,a,b}(Z,W)$ satisfies the heat equation*

$$\sum_{k=1}^{j} \frac{\partial^2 \vartheta_{S,a,b}}{\partial W_{kp}\partial W_{kq}} = \frac{4\pi i m}{2-\delta_{pq}} \frac{\partial \vartheta_{S,a,b}}{\partial Z_{pq}}, \quad 1 \le p \le q \le n. \tag{26}$$

It is easy to prove it and so we omit its proof.

DEFINITION 4.4. Let $S \in Z^{(2k,2k)}$ be a symmetric, positive definite unimodular and even matrix of degree $2k$ and let $c \in Z^{(2k,j)}$. We define the theta series

$$\vartheta_{S,c}^{(n)}(Z,W) := \sum_{\lambda \in Z^{(2k,n)}} e^{\pi i \sigma(S(\lambda Z^t\lambda + 2\lambda^t W^t c))}, \tag{27}$$

$$Z \in H_n \text{ and } W \in C^{(j,n)}.$$

We observe that $\vartheta_{S,c}^{(n)} \in J_{k,\mathcal{M}}(\Gamma_n)$ with $\mathcal{M} = \frac{1}{2}{}^t cSc$. And the Fourier coefficients $c(T,R)$ of $\vartheta_{S,c}^{(n)}$ are given by

$$c(T,R) = \#\{\lambda \in Z^{(2k,n)} \mid {}^t\lambda S\lambda = 2T,\ {}^t\lambda Sc = R\}.$$

We may define the notion of singular Jacobi forms as in the case of Siegel modular forms (see [Z]).

DEFINITION 4.5. A Jacobi form $\Phi \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that the Fourier coefficients $c(T,R)$ are zero unless $\det(4T - R\mathcal{M}^{-1}{}^tR) = 0$. In addition, if $2k < n$ and whenever $c(T,R) \neq 0$ only if $4T - R\mathcal{M}^{-1t}R = 0$, it is called *strongly singular*. For example, if $2k < n + rank(\mathcal{M})$, then $\vartheta_{S,c}^{(n)} \in J_{k,\mathcal{M}}(\Gamma_n)$ is singular. If $2k < n$, the theta series $\vartheta_{S,c}^{(n)}$ with $c \in Z^{(2k,2k)}$ is strongly singular. As in the case of singular modular forms, Jacobi singular or strongly singular forms may be written as linear combinations of theta series $\vartheta_{S,c}^{(n)}$ (see [Z] pp.23–25).

For a fixed $Z_0 \in H_n$ and $\mathcal{M}$ as above, we denote by $T_{\mathcal{M}}(Z_0)$ the vector space of all holomorphic functions $g : C^{(j,n)} \mapsto C$ satisfying

$$g(W + \lambda Z_0 + \mu) = e^{-2\pi i\sigma(\mathcal{M}(\lambda Z_0{}^t\lambda + 2\lambda^t W))} g(W)$$

for every $\lambda, \mu \in Z^{(j,n)}$. Then it is easy to show that the functions

$$\{\vartheta_{2\mathcal{M},a,0}(Z_0, W) \mid a \in \mathcal{N}\} \tag{28}$$

form a basis of $T_{\mathcal{M}}(Z_0)$ and its dimension is clearly $\{\det(2\mathcal{M})\}^n$. Now we assume that $2\mathcal{M}$ is not unimodular. Then by a classical theory, we know that for a fixed $Z_0 \in H_n$, $\{\vartheta_{2\mathcal{M},a,0}(Z_0, W) \mid a \in \mathcal{N}\}$ have no common zeros as a function of W. Therefore if $N = \{\det(2\mathcal{M})\}^n - 1$, we can define

$$\Theta : C^{(j,n)} \mapsto P^N(C)$$

by

$$\Theta(W) = [\vartheta_{2\mathcal{M},a_1,0}(Z_0, W) : \cdots : \vartheta_{2\mathcal{M},a_{N+1},0}(Z_0, W)],\ a_i \in \mathcal{M}. \tag{29}$$

Then Θ is well defined and induces a mapping

$$\Theta : C^{(j,n)}/L \mapsto P^N(C), \tag{30}$$

where $L = \{\lambda Z_0 + \mu \mid \lambda, \mu \in Z^{(j,n)}\}$ is a lattice in $C^{(j,n)}$. It is well known that Θ is holomorphic and injective.

5. Holomorphic differential forms on $H_n \times C^{(j,n)}$

In this section we consider holomorphic differential forms on $H_n \times C^{(j,n)}$ invariant under the discrete subgroup $\Gamma_{(n,j)} = Sp(n,Z) \ltimes H_Z^{(n,j)}$. We assume that n and j are arbitrary positive integers with $n \geq 2$. We write for $Z \in H_n$ and $W \in C^{(j,n)}$

$$\begin{aligned} Z = (Z_{\mu\nu}),\ dZ = (dZ_{\mu\nu}), \quad 1 \leq \mu, \nu \leq n, \\ W = (W_{kl}),\ dW = (dW_{kl}), \quad 1 \leq k \leq j,\ 1 \leq l \leq n. \end{aligned}$$

If $g = (M, [(\lambda,\mu),\kappa]) \in G_R^{(n,j)}$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n := Sp(n,Z)$, we write $(Z^*, W^*) = g \bullet (Z,W)$. that is,

$$\begin{aligned} Z^* &= (AZ+B)(CZ+D)^{-1}, \\ W^* &= (W + \lambda Z + \mu)(CZ+D)^{-1}. \end{aligned}$$

By a simple calculation, we have

(31)
$$\begin{aligned} dZ^* &= {}^t(CZ+D)^{-1} dZ (CZ+D)^{-1}, \\ dW^* &= dW(CZ+D)^{-1} \\ &\quad + \{\lambda - (W + \lambda Z + \mu)(CZ+D)^{-1}C\} dZ (CZ+D)^{-1}. \end{aligned}$$

We denote by $\Omega^\nu(H_n \times C^{(j,n)})^{\Gamma_{(n,j)}}$ the vector space of all holomorphic ν-forms on $H_n \times C^{(j,n)}$ invariant under $\Gamma_{(n,j)}$.

Now we let

$$\alpha = \sum_{\mu \leq \nu} f_{\mu\nu} dZ_{\mu\nu} + \sum_{k,l} \phi_{kl} dW_{kl}$$

be a holomorphic 1-form on $H_n \times C^{(j,n)}$ invariant under $\Gamma_{(n,j)}$. We set

$$e_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu; \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

We set

$$f(Z,W) = (e_{\mu\nu} f_{\mu\nu}(Z,W)),\ \Phi(Z,W) = {}^t(\phi_{kl}(Z,W)).$$

Then we have

$$\alpha = \sigma(f dZ + \Phi dW).$$

For each $\tilde{\gamma} = (\gamma, [(\lambda, \mu), \kappa]) \in \Gamma_{(n,j)}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, according to (31), $f(Z, W)$ and $\Phi(Z, W)$ satisfy the functional equations

$$\Phi(\tilde{\gamma} \bullet (Z, W)) = (CZ + D)\Phi(Z, W), \tag{32}$$

and

$$\begin{aligned} f(\tilde{\gamma} \bullet (Z, W)) = &(CZ + D) f(Z, W)^t(CZ + D) \\ &- \Phi(\tilde{\gamma} \bullet (Z, W))\Lambda^t(CZ + D), \end{aligned} \tag{33}$$

where $\Lambda = \lambda - (W + \lambda Z + \mu)(CZ + D)^{-1}C$.

Let $\tilde{\rho}$ be the canonical representation of $GL(n, C)$ on $C^{(n,j)}$ and ρ the representation of $GL(n, C)$ on $\mathrm{Symm}^2(C^n)$ defined by

$$\rho(A)v = Av^tA, \ A \in GL(n, C), \ v \in \mathrm{Symm}^2(C^n).$$

By the relation (32) and the Koecher principle for Jacobi forms, we have $\Phi \in J_{\tilde{\rho},0}(\Gamma_n)$. If $\Phi \equiv 0$, then $f(Z, W)$ is a Jacobi form of index 0 with respect to the representation ρ. Therefore we obtain

$$\dim_C \Omega^1(H_n \times C^{(j,n)})^{\Gamma_{(n,j)}} \geq \dim_C J_{\rho,0}(\Gamma_n) + \dim_C J_{\tilde{\rho},0}(\Gamma_n), \tag{34}$$

where 0 denotes the $j \times j$ zero matrix.

Let

$$\omega_0 = dZ_{11} \wedge dZ_{12} \wedge \cdots \wedge dZ_{nn} \wedge dW_{11} \wedge \cdots \wedge dW_{jn}$$

be a holomorphic form on $H_n \times C^{(j,n)}$ of degree $\tilde{N}$, where $\tilde{N} = \frac{n(n+1)}{2} + jn$. If $\omega = \Phi(Z, W)\omega_0$ is a $\Gamma_{(n,j)}$-invariant holomorphic form of degree $\tilde{N}$, by a simple computation, we obtain

$$\Phi(\tilde{\gamma} \bullet (Z, W)) = \det(CZ + D)^{n+j+1}\Phi(Z, W), \tag{35}$$

where $\tilde{\gamma} = (\gamma, [(\lambda, \mu), \kappa]) \in \Gamma_{(n,j)}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. Let $\tilde{\omega} = \Psi(Z, W)\omega_0^{\otimes k}$ be a $\Gamma_{(n,j)}$-invariant holomorphic tensor on $H_n \times C^{(j,n)}$

of degree $k\tilde{N}$. Then assording to (35), $\Phi \in J_{\rho_{k(n+j+1)},0}(\Gamma_n)$, where $\rho_{k(n+j+1)} = (\det)^{k(n+j+1)}$.

We write

$$\omega_1 = dZ_{11} \wedge \cdots \wedge dZ_{nn},\ \omega_2 = dW_{11} \wedge \cdots \wedge dW_{jn}.$$

Now we define

$$\omega_{ab} = \epsilon_{ab} \bigwedge_{\substack{1\le\mu\le\nu\le n\\(\mu,\nu)\neq(a,b)}} dZ_{\mu\nu} \wedge \omega_2,\ 1 \le a \le b \le n$$

and

$$\tilde{\omega}_{cd} = \tilde{\epsilon}_{cd}\omega_1 \wedge \bigwedge_{\substack{1\le k\le j,\\1\le l\le n,\\(k,l)\neq(c,d)}} dW_{kl},\ 1 \le c \le j,\ 1 \le d \le n.$$

The signs ϵ_{ab} and $\tilde{\epsilon}_{cd}$ are determined by the relations $\epsilon_{ab}\omega_{ab}\wedge dZ_{ab} = \omega_0$ and $\tilde{\epsilon}_{cd}\tilde{\omega}_{cd} \wedge dW_{cd} = \omega_0$. We let

$$\beta = \sum_{\mu\nu} f_{\mu\nu}\omega_{\mu\nu} + \sum_{k\le l} \phi_{kl}\tilde{\omega}_{kl}$$

be a $\Gamma_{(n,j)}$-invariant holomorphic form on $H_n \times C^{(j,n)}$ of degree $\tilde{N}-1$. We define

$$f = (\epsilon_{\mu\nu})f_{\mu\nu},\ f_{\mu\nu} = f_{\nu\mu},\ \epsilon_{\mu\nu} = \epsilon_{\nu\mu},\ \phi = (\tilde{\epsilon}_{kl}\phi_{kl}).$$

If we write

$$\Phi = \begin{pmatrix} f \\ \phi \end{pmatrix},$$

then we obtain

$$\beta \wedge \begin{pmatrix} dZ \\ dW \end{pmatrix} = \Phi\omega_0.$$

Let $\tilde{\gamma} = (\gamma, [(\lambda,\mu),\kappa]) \in \Gamma_{(n,j)}$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. Since β is invariant under $\Gamma_{(n,j)}$, we have

$$\begin{aligned}
&\Phi(\gamma_R \bullet (Z,W)) \det(CZ+D)^{-(n+j+1)}\omega_0 \\
&= \beta \wedge \begin{pmatrix} {}^t(CZ+D)^{-1} & 0 \\ 0 & E_j \end{pmatrix} \begin{pmatrix} dZ \\ dW \end{pmatrix} (CZ+D)^{-1} \\
&\quad + \beta \wedge \begin{pmatrix} 0 \\ \Lambda \end{pmatrix} (E_n, 0) \begin{pmatrix} dZ \\ dW \end{pmatrix} (CZ+D)^{-1},
\end{aligned}$$

where $\Lambda = \lambda - (W + \lambda Z + \mu)(CZ + D)^{-1}C$. Therefore f and ϕ satisfy the following functional equation

(36)
$$\begin{aligned} & f(\tilde{\gamma} \bullet (Z, W)) \\ = & \det(CZ + D)^{n+j+1}\,{}^t(CZ + D)^{-1} f(Z, W)(CZ + D)^{-1} \end{aligned}$$

and

(37)
$$\begin{aligned} & \phi(\tilde{\gamma} \bullet (Z, W)) \\ = & \det(CZ + D)^{n+j+1}(\Lambda f(Z, W) + \phi(Z, W))(CZ + D)^{-1} \end{aligned}$$

for all $\tilde{\gamma} \in \Gamma_{(n,j)}$.

Let $\hat{\rho}$ be the representation of $GL(n, C)$ on $\mathrm{Symm}^2(C^n)$ defined by

$$\hat{\rho}(A)v = (\det A)^{n+j+1}\,{}^tA^{-1} v A^{-1}, \ A \in GL(n, C), \ v \in \mathrm{Symm}^2(C^n)$$

and π be the representation of $GL(n, C)$ on $C^{(j,n)}$ defined by

$$\pi(A)w = (\det A)^{n+j+1} w A^{-1}, \ A \in GL(n, C), \ w \in C^{(j,n)}.$$

Then by the relation (36) and the Koecher principle for Jacobi forms, we have $f \in J_{\hat{\rho},0}(\Gamma_n)$. In addition, if $f \equiv 0$, then $\phi \in J_{\pi,0}(\Gamma_n)$. Therefore we obtain

$$\dim_C \Omega^{\tilde{N}-1}(H_n \times C^{(j,n)})^{\Gamma_{(n,j)}} \geq \dim_C J_{\hat{\rho},0}(\Gamma_n) + \dim_C J_{\pi,0}(\Gamma_n).$$

We let
$$\eta = \sum_{i_k \leq j_k} f_{i_1 j_1 \cdots i_\nu j_\nu} dZ_{i_1 j_1} \wedge \cdots \wedge dZ_{i_\nu j_\nu}$$

be a holomorphic ν-form on $H_n \times C^{(j,n)}$ invariant under $\Gamma_{(n,j)}$, where $f(Z)$ is a holomorphic function on $H_n \times C^{(j,n)}$ depending only on Z. Then we can regard η as a holomorphic ν-form on H_n invariant under $\Gamma_n = Sp(n, Z)$. We denote by $\Omega^\nu(H_n)^{\Gamma_n}$ the vector space of all holomorphic ν-forms on H_n invariant under Γ_n. By the above consideration, we have

$$\dim_C \Omega^\nu(H_n \times C^{(j,n)})^{\Gamma_{(n,j)}} \geq \dim_C \Omega^\nu(H_n)^{\Gamma_n}, \quad 0 \leq \nu \leq \frac{n(n+1)}{2}. \tag{38}$$

Let ρ be the symmetric product of the canonical representation of $GL(n,C)$ on C^n. That is, $\rho(A)v = Av\,{}^tA$ for $A \in GL(n,C)$ and $v \in \mathrm{Symm}^2(C^n)$. Then we have

$$\Omega^\nu(H_n)^{\Gamma_n} \cong [\Gamma_n, \rho^{[\nu]}], \ \rho^{[\nu]} = \bigwedge^\nu \mathrm{Symm}^2(C^n). \tag{39}$$

For detail, we refer the reader to [W]. It follows from (38) and (39) that

$$\dim_C \Omega^\nu(H_n \times C^{(j,n)})^{\Gamma_{(n,j)}} \geq \dim_C[\Gamma_n, \rho^{[\nu]}].$$

Now we know that the study of holomorphic forms on $H_n \times C^{(j,n)}$ is closely related to that of Jacobi forms and involves the computation of the dimension of $J_{\rho,\mathcal{M}}(\Gamma)$ for the special ρ, $\mathcal{M}$ and Γ. But the computation of the dimension of $J_{\rho,\mathcal{M}}(\Gamma)$ is complicated and so except for the special case $n = j = 1$, the formula for $\dim_C J_{\rho,\mathcal{M}}(\Gamma)$ is not still known.

By the vanishing theorem, we obtain

LEMMA. *Let Γ be a subgroup of Γ_n of finite index and let $\Gamma_Z = \Gamma \ltimes H_Z^{(n,j)}$. Then we have*

$$\Omega^1(H_n \times C^{(j,n)})^{\Gamma_Z} = 0 \tag{40}$$

Proof. According to (32) and Satz 1 in [F], $\Phi(Z,0) = 0$ for all $Z \in H_n$. Therefore $\Phi(Z,W)$ vanishes identically on $H_n \times C^{(j,n)}$. Hence we obtain

$$f(M\langle Z\rangle, (W + \lambda Z + \mu)(CZ+D)^{-1}) = (CZ+D)f(Z,W)(CZ+D)^{-1}$$

for all $(M, [(\lambda,\mu),\kappa]) \in \Gamma_Z$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$. Again according to Satz 1 in [F], $f(Z,0) = 0$ for all $Z \in H_n$. Hence $f(Z,W)$ vanishes identically on $H_n \times C^{(j,n)}$. Thus we obtain the desired result.

References

[E-Z] Eichler, M. and Zagier, D., *The Theory of Jacobi forms*, Progress in Math., **55**, Birkhauser, Boston-Basel-Stuttgart (1985).

[F] Freitag, E., *Ein Verschwindingssatz für automorphe Formen zur Siegelsche Modulgruppe*, Math. Zeit. **165** (1980), 11–18.

[K] Klingen, H., *Über Kernfunktionen für Jacobiformen und Siegelsche Modulformen*, Math. Annalen, **285** (1989), 405–416.

[W] Weissauer, R., *Vektorwertige Siegelsche Modulformen kleinen Gewichtes*, Jour. reine angew. Math., **343** (1983), 184–202.

[Y] Yamazaki, T., *Jacobi forms and a Maass relation for Eisenstein series*, J. Fac. Sci. Uni. Tokyo, Sec. IA. Math., **33** (1986), 295–310.

[Z] Ziegeler, C., *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

Department of Mathematics
Inha University
Incheon 402–751, Korea

Department of Mathematics
Kyungnam University
Masan 630–701, Korea