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COMPLETE MINIMAL SURFACES AND PUNCTURED COMPACT RIEMANN SURFACES

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Introduction

A conformal map from a complex domain D into $\mathbb{R}^3$ is said to be minimal if its component functions are harmonic. Historically speaking, the theory of minimal surfaces arose in an attempt to find the surface of least area among those bounded by a fixed curve. This problem, so called the Plateau problem, was given a solution by Douglas and Rado in the thirties. (The Plateau problem for higher dimensional submanifolds of $\mathbb{R}^n$ was given a satisfactory treatment only quite recently by Federer, Fleming, Almgren, De Giorgi, and Reifenberg. See [F] or [G] for a detailed account.)

A Riemann surface (without boundary) M is an abstract surface that looks locally like a complex domain. More precisely, each point $p \in M$ has a neighborhood diffeomorphic to a domain $D_p \subset \mathbb{C}$ and on an overlap $D_p \cap D_q$ transition functions are given by biholomorphic maps. In this article we consider conformal minimal maps from Riemann surfaces into $\mathbb{R}^3$.

Consider a conformal minimal immersion $f : M \to \mathbb{R}^3$. The *normal Gauss map* of f is the map

$$\varphi : M \to S^2 = \text{ the unit 2-sphere}$$

taking $p \in M$ to the unit outward normal vector at $f(p)$. Via the stereographic projection

$$S^2 \to \mathbb{C} \cup \{\infty\}$$

the map φ can be thought of as a meromorphic function on M. A large number of research articles have been published in recent years studying various value distribution properties of the meromorphic function

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φ. The value distribution theory deals in large part with complete minimal surfaces of infinite total curvature; impressive results have been obtained through the works Chern, Osserman, Xavier, Fujimoto, and others. The reader may consult a recent article by Mo and Osserman [MO] and references cited therein for further study. Our primary focus, however, will be on complete minimal surfaces with finite total curvature.

The *tangential Gauss map* of a conformal minimal immersion $f : M \to \mathbb{R}^3$ is the map

$$\Phi : M \to G(3,2)$$

taking $p \in M$ to the oriented tangent plane $f_* T_p M \subset \mathbb{R}^3$. $G(3,2)$ is the Grassmann manifold of oriented 2-planes in $\mathbb{R}^3$, and it lies naturally in $\mathbb{C}P^2$, the space of lines in $\mathbb{C}^2$. Moreover, the minimality of f implies that the map

$$\Phi : M \to \mathbb{C}P^2$$

is holomorphic. An important theorem of Chern and Osserman [CO] states that a complete minimal surface M is of finite total Gaussian curvature if and only if M is holomorphically equivalent to a compact Riemann surface M_g punctured at a finite number of points and the tangential Gauss map extends holomorphically to all of M_g. Thus the study of complete minimal surfaces of finite total curvature is intimately linked to the theory of compact Riemann surfaces, and we wish to exhibit some of the fruitful interactions between the two subject matters. In particular, we give a discussion of the immersion problem of Osserman: Given $r \in \mathbb{Z}^+$ and a compact Riemann surface M_g of genus g, find all complete conformal minimal immersions of finite total curvature

$$M_g \backslash \sum \to \mathbb{R}^3$$

with $|\sum| = r$. In a recent work [Y3] the author has shown that there exists at least a one-parameter family of such immersions with $r \leq 4g$. (It is easy to find examples for large r.) We discuss this and other related results in 3.

1. Minimal surfaces in R^3: the Weierstrass representation

Let M be a Riemann surface, and consider a conformal immersion

$$f = (f^i) : M \to \mathbb{R}^3.$$

Conformality means that the induced metric $f^*ds_E^2$, ds_E^2 the Euclidean metric, is compatible with the complex structure in the following sense: If z is a local holomorphic coordinate, then the induced metric can be written as

$$ds^2 = h(z)dz \cdot d\bar{z}$$

for some $h(z) > 0$. Writing $z = x + iy$, where x and y are real-valued, we can rewrite the above as

$$ds^2 = h(x, y)(dx^2 + dy^2).$$

The local functions (x, y) are called isothermal coordinates.

Let $C^\infty(M)$ denote the space of smooth complex-valued functions on M. The Laplacian of M is an operator

$$\Delta : C^\infty(M) \to C^\infty(M)$$

defined by

$$\Delta = -\left(\frac{4}{h}\right) \partial^2/\partial z \partial \bar{z}.$$

Using the chain rule one sees that Δ is well-defined.

The Gaussian curvature of (M, ds^2) can be written as

$$K = \frac{1}{2}\Delta \log(h).$$

The mean curvature of the immersion f is related to the Laplacian of f by

$$2H = (\pm)\Delta f \cdot e_3,$$

where e_3 is a unit vector field normal to $f(M)$. Proofs of the above two formulae can be found in [Y1] p. 7 and p. 12.

It follows from the preceding formula (note that Δf is normal to f) that

$$H \equiv 0 \text{ if and only if } \Delta f \equiv 0.$$

If the immersion f satisfies one of the above conditions, then it is said to be minimal.

PROPOSITION 1. *There does not exist a conformal minimal immersion*

$$f: M \to \mathbb{R}^3$$

from a compact Riemann surface M.

Proof. Suppose we had such a map $f = (f^i)$. Then each f^i would be a harmonic function on M. The maximum principle for harmonic functions states that a harmonic function on a compact surface (without boundary) must reduce to a constant, and this implies that f would have to be a constant map.

Suppose we have a conformal minimal immersion

$$f: M \to \mathbb{R}^3,$$

and let z be a local coordinate. The minimality of f gives

$$\partial^2 f^i / \partial z \partial \bar{z} = 0.$$

Define local functions (η^i) by

$$(*) \qquad \eta^i = \partial f^i / \partial z.$$

Each η^i is holomorphic since its partial with respect to $\bar{z}$ vanishes. Define local holomorphic 1-forms (ζ^i) by

$$\zeta^i = \eta^i dz.$$

If $\tilde{z}$ is another local coordinate and if $\tilde{\zeta}^i = \tilde{\eta}^i d\tilde{z}$, then

$$\tilde{\eta}^i = \partial f^i / \partial \tilde{z} = (\partial f^i / \partial z)(dz/d\tilde{z}) = \eta^i dz/d\tilde{z}$$

so that the forms (ζ^i) are globally defined on M.

Since (x, y) are isothermal we have

$$h(z) = \langle f_* \frac{\partial}{\partial x}, f_* \frac{\partial}{\partial x} \rangle = \langle f_* \frac{\partial}{\partial y}, f_* \frac{\partial}{\partial y} \rangle, \ \langle f_* \frac{\partial}{\partial x}, f_* \frac{\partial}{\partial y} \rangle = 0.$$

It follows that

$$h(z) = 2 \sum |\eta^i|^2 > 0,$$

and that

$$(\dagger) \qquad \sum (\eta^i)^2 = 0.$$

The holomorphic 1-forms (ζ^i) give rise to a well-defined holomorphic map

$$\Phi_f : M \to \mathbb{C}P^2, \ z \mapsto [\eta^1(z), \eta^2(z), \eta^3(z)],$$

where $[(\eta^i(z))]$ denotes the complex line in $\mathbb{C}^3$ through $(\eta^i(z))$.

REMARK. Because of (†) the image of the Gauss map actually lies in the complex quadric $Q_1 \subset \mathbb{C}P^2$. The quadric Q_1 can be naturally identified with the Grassmann manifold $G(3,2)$ of oriented 2-planes in $\mathbb{R}^3$, and upon this identification the Gauss map takes $p \in M$ to the (negatively oriented) tangent plane $f_* T_p(M)$.

From (*) we see that the minimal immersion f can be recovered from the holomorphic 1-forms (ζ^i):

$$f^i(z) = 2\,\mathrm{Re} \int_{z_0}^{z} \zeta^i,$$

where we assume that $f(z_0) = 0 \in \mathbb{R}^3$. In particular, the forms (ζ^i) have no real periods as the above integrals are well-defined.

Reversing the above process we can manufacture minimal surfaces from holomorphic 1-forms.

PROPOSITION 2. *Let M be a Riemann surface, and suppose we have holomorphic 1-forms (ζ^i) on M satisfying*

(1) $$\sum |\eta^i|^2 > 0, \text{ where } \zeta^i = \eta^i dz \text{ locally;}$$

(2) $$\sum (\eta^i)^2 = 0;$$

(3) (ζ^j) *have no real periods.*

Then

$$f = (f^i) = 2\ \mathrm{Re} \int_{z_0}^{z} (\zeta^i)$$

defines a conformal minimal immersion of M into $\mathbb{R}^3$ with $f(z_0) = 0$.

For a proof of this well-known result see [Y1] pp.15–16.

REMARK. Note that when M is simply connected, e.g., $M = \mathbb{C}$, the condition (3) is satisfied automatically. So, even in the absence of (3) the holomorphic 1-forms (ζ^i) define a conformal minimal immersion on the universal cover of M.

Let φ be a meromorphic function on a Riemann surface M, and also let μ be a not identically zero holomorphic 1-form on M. We further require that φ has a pole of order m at $p \in M$ if and only if μ has a zero of order $2m$ at p. Put

$$(4) \qquad \begin{cases} \zeta^1 &= \frac{1}{2}(1-\varphi^2)\mu, \\ \zeta^2 &= \frac{i}{2}(1+\varphi^2)\mu, \\ \zeta^3 &= \varphi\mu. \end{cases}$$

The ζ^i's have no common zeros, hence the condition (1) is met. The condition (2) is also easily satisfied. Therefore, the forms (ζ^i) given in (4) define a conformal minimal immersion

$$f = f_\zeta : M \to \mathbb{R}^3$$

given that they have no real periods. Up to congruence every minimal surface in $\mathbb{R}^3$ arises in this manner, and $\{\mu, \varphi\}$ is called the Weierstrass pair representing f_ζ.

We record that the induced metric of f_ζ is given by, in terms of the Weierstrass pair,

$$ds^2 = |\eta|^2(1+|\varphi|^2)^2 dz \cdot d\bar{z},$$

where $\mu = \eta dz$ locally.

REMARK. The meromorphic function φ is related to the Gauss map as follows. Let

$$\Phi_f^\perp : M \to S^2$$

denote the normal Gauss map of a conformal minimal immersion f, i.e.,

$$\Phi_f^\perp(p) = \text{ the unit outward normal vector to } f(p).$$

Then

$$\varphi = \pi \circ \Phi_f^\perp : M \to S^2 \to \mathbb{C} \cup \{\infty\},$$

where π denotes the stereographic projection.

EXAMPLES.

a) The Catenoid is given by the Weierstrass pair

$$\{\varphi(z) = z,\ \mu = \frac{1}{z^2} dz\}$$

on $M = \mathbb{C}\backslash\{0\}$, where z is the usual complex coordinate. It is a surface of revolution obtained by revolving the Catenary $x^3 = \cosh(x^1)$ about the x^1-axis.

b) Take $M = \mathbb{C}$, $\mu = dz$, and $\varphi(z) = z$. The resulting minimal surface is called Enneper's surface. For $z \in \mathbb{C}$, its image $(f^i(z)) \in \mathbb{R}^3$ is given by

$$f^1(z) = \operatorname{Re}(z - \frac{1}{3}z^3),$$
$$f^2(z) = \operatorname{Re}(iz + \frac{i}{3}z^3),$$
$$f^3(z) = \operatorname{Re} z^2.$$

Enneper's surface is not an embedded surface in $\mathbb{R}^3$.

c) Let

$$\Lambda = \mathbb{Z} \oplus i\mathbb{Z} \subset \mathbb{C}$$

denote the integral lattice. Requiring the projection

$$\pi : \mathbb{C} \to \mathbb{C}/\Lambda$$

be holomorphic $M = \mathbb{C}/\Lambda$ becomes a Riemann surface, called a complex torus. Let $\mathfrak{p}(z)$ denote the Weierstrass function relative to Λ, i.e.,

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right),$$

where the sum is taken over all $w \in \Lambda\backslash\{0\}$. The function $\mathfrak{p}(z)$, meromorphic function on $\mathbb{C}$, is an elliptic function with periods in Λ. It has a double pole at each $w \in \Lambda$ with the principal part $\frac{1}{(z-w)^2}$ and is holomorphic elsewhere. The function $\mathfrak{p}(z)$ projects down to M to give a meromorphic function on M. We again use the symbol $\mathfrak{p}$ to denote this function. (At the same time we confuse z with $\pi(z)$.) Costa's surface [C] is given by the Weierstrass pair

$$\{\mu = \mathfrak{p}(z)dz,\ \varphi(z) = 2\sqrt{2\pi}\,\mathfrak{p}(\frac{1}{2})/\mathfrak{p}'(z)\}$$

on $M\backslash\{0, \frac{1}{2}, \frac{i}{2}\}$. Hoffman and Meeks [HM1] showed that Costa's surface is actually embedded in $\mathbb{R}^3$.

2. The Riemann-Roch theorem and Weierstrass points

Let M_g denote a compact Riemann surface of genus g. Topologically, M_g is a torus with g handles.

A meromorphic function on M_g is simply a holomorphic map

$$\varphi : M_g \to \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\},$$

where it is customary to assume that $\varphi(M_g) \neq \{\infty\}$. Let φ be a non-constant meromorphic function on M_g. The following equidistribution property is well-known: each value $q \in \mathbb{C}P^1$ is taken a fixed number, called the degree of φ, of times counting multiplicity. In particular, the total number of zeros is equal to that of poles.

A meromorphic 1-form (also called an Abelian differential) μ on M_g is locally given by

$$\eta(z)dz,$$

where z is a local coordinate and $\eta(z)$ is a meromorphic function. Let φ be a meromorphic function on M. Then the total differential $d\varphi$ is a meromorphic 1-form. Locally

$$d\varphi = \varphi'(z)dz.$$

Let μ be a meromorphic 1-form on M_g given locally by $\eta(z)dz$. Then the residue of μ at a point $p \in M_g$ is defined to be

$$\mathrm{Res}_p\mu = \mathrm{Res}_p\eta.$$

To see that the residue is well-defined just observe that

$$\mathrm{Res}_p\mu = \frac{1}{2\pi i}\int_\gamma \mu,$$

where γ is a small path around p of index 1.

PROPOSITION 3. *Let μ be a meromorphic 1-form on M_g. Then the total residue must vanish, i.e.,*

$$\sum_{p \in M} Res_p\mu = 0.$$

Proof. Triangulate M_g so that each singularity of μ lies in the interior of a triangle. Let $\Delta_1, \cdots, \Delta_k$ be the triangles in this trianglulation. Then

$$\sum \mathrm{Res}_p \mu = \frac{1}{2\pi i} \sum \int_{\gamma_i} \mu,$$

where γ_i is the boundary of Δ_i. Since each edge appears exactly twice with opposite signs the integral vanishes.

A *divisor* D on M_g is a finite formal sum

$$D = \sum a_i p_i,\ a_i \in \mathbb{Z}\backslash\{0\},\ p_i \in M_g.$$

If $a_i \geq 0$ for every i, then D is called an integral divisor and we write $D \geq 0$. The set of all divisors on M_g, denoted by $\mathrm{Div}(M_g)$, forms an Abelian group under addition: it is isomorphic to the free Abelian group on the points of M_g.

There is a group homomorphism

$$\deg : \mathrm{Div}(M_g) \to \mathbb{Z},\ \deg D = \sum a_i.$$

By way of notation we put

$$\mathrm{Ker}(\deg) = \mathrm{Div}^0(M_g).$$

Let φ be a not identically zero meromorphic function on M_g. It is convenient to use the sheaf notation and write

$$\varphi \in H^0(M_g, \mathcal{M}^*),$$

where $\mathcal{M}^*$ denotes the sheaf of germs of not identically zero meromorphic functions on M_g. The *divisor* of φ, denoted by (φ), is

$$(\varphi) = \sum a_i p_i - \sum b_j q_j,$$

where the p_i's are the zeros (p_1 with multiplicity a_i) and the q_j's are the poles (q_j with multiplicity b_j) of φ. We also write

$$(\varphi)_0 = \sum a_i p_i,\ (\varphi)_\infty = \sum b_j q_j.$$

By the equidistribution property we then have

$$(\varphi) \in \mathrm{Div}^0(M_g).$$

A divisor is called a principal divisor if it is (φ) for some $\varphi \in H^0(M_g, \mathcal{M}^*)$.

REMARK. The set of principal divisors is exactly $\mathrm{Div}^0(M_g)$ if and only if the genus is zero, i.e., M_g is biholomorphic to $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$.

Let $\mu \neq 0$ be a meromorphic 1-form on M_g. Take a (finite) open cover (U_a) of M_g and write

$$\mu|_{U_a} = \eta_a(z)dz.$$

The divisor of μ, denoted by (μ), is defined to be the divisor D such that

$$D|_{U_a} = (\eta_a).$$

Define the order of μ at $p \in M_g$ to be

$$\mathrm{ord}_p\mu = \mathrm{ord}_p\eta,$$

where $\mu = \eta dz$ locally. A divisor is called a canonical divisor if it is of the form (μ) for some meromorphic 1-form μ.

PROPOSITION 4. *Let $\varphi \in H^0(M_g, \mathcal{M}^*)$. Then*

$$deg(d\varphi) = 2g - 2.$$

In fact, the degree of an arbitrary canonical divisor on M_g is $2g-2$.

To prove Proposition 4 we first need to establish the

Riemann-Hurwitz Formula. Consider a nonconstant holomorphic map

$$f : M_{g_1} \to M_{g_2}.$$

Let m denote the degree of f, i.e., every value $q \in M_{g_2}$ is assumed m times taking into account multiplicity. We know that about any point $p \in M_{g_1}$ there is the local normal form

$$f(z) = z^n,\ n \in \mathbb{Z}^+.$$

The number $n-1$ is called the branch number at p, and is denoted by $b_f(p)$. Let

$$B_f = \sum b_f(p)$$

be the total branching number of f. We then have

$$2(g_1 - 1) - 2m(g_2 - 1) = B_f.$$

Proof. Let $S = \{f(p) \in M_{g_2} : b_f(p) > 0\}$. S is a finite set and we can triangulate M_{g_2} so that every point of S occurs as a vertex. Put

$$\begin{aligned} F_2 &= \text{ the number of triangles,} \\ E_2 &= \text{ the number of edges,} \\ V_2 &= \text{ the number of vertices} \end{aligned}$$

of this triangulation. Lifting this triangulation to M_{g_1} via f we obtain a triangulation of M_{g_1} with $F_1 = mF_2$, $E_1 = mE_2$, $V_1 = mV_2 - B$. Now

$$2 - 2g_i = F_i - E_i + V_i$$

and the result follows.

Proof of Proposition 4. Near a pole $p \in M_g$ of φ we have the Laurant series expansion

$$\varphi(z) = c_{-k}z^{-k} + \cdots + c_0 + c_1 z + \cdots (c_{-k} \neq 0).$$

Thus

$$d\varphi(z) = (-kc_{-k}z^{-k-1} + \cdots + c_{-1}z^{-2} + c_1 + 2c_2 z + \cdots)dz.$$

Near a nonpole $q \in M_g$ we have the Taylor series expansion

$$\varphi(z) = c_n z^n + c_{n+1}z^{n+1} + \cdots (c_n \neq 0),$$

and

$$d\varphi(z) = (nc_n z^{n-1} + \cdots)dz.$$

It follows that

$$\deg(d\varphi) = \sum b_\varphi(q) - \sum(k(p) + 1),$$

where p runs over all poles with multiplicity $k(p)$ and q runs over all branch points with branch number $n-1 = b_\varphi(q)$ with the proviso that q is not a pole. Now

$$\deg\varphi = m = \sum k(p) = \text{ the total number of poles,}$$

and

$$B = \sum b_\varphi(q) + \sum (k(p) - 1).$$

The Riemann-Hurwitz formula applied to φ now gives

$$\begin{aligned} 2(g-1) &= -2m + B \\ &= -2\sum k(p) + \sum b_\varphi(q) + \sum (k(p) - 1) \\ &= \sum b_\varphi(q) = \sum (k(p) - 1) = \deg(d\varphi). \end{aligned}$$

For $D \in \mathrm{Div}(M_g)$ we put

$$L(D) = \{\varphi \in H^0(M_g, \mathcal{M}^*) : (\varphi) + D \geq 0\} \cup \{0\}.$$

For some $a_i, b_j \in \mathbb{Z}^+$, $p_i, q_j \in M_g$ distinct points we can write

$$D = \sum a_i p_i - \sum b_j q_j.$$

We then see that $\varphi \in L(D)\backslash\{0\}$ if and only if φ is holomorphic outside $\cup p_i$ and

$$\mathrm{ord}_{q_j}\varphi \geq b_j;\ \mathrm{ord}_{p_i}\varphi \geq -a_i.$$

The following properties concerning $L(D)$ are easily verified:

a) $L(D)$ is a complex vector space;

b) $L(D) = 0$ if $\deg D < 0$;

c) $L(0) =$ constant functions $\cong \mathbb{C}$.

PROPOSITION 5. *Let $D \geq 0$ be an integral divisor on M_g. Then*

$$\dim L(D) \leq \deg D + 1.$$

Proof. Write $D = \sum a_i p_i$, $a_i > 0$, the p_i's distinct. (If $D = 0$, then $\dim L(D) = 1$.) Suppose $\varphi \in L(D)$. Then about each p_i we have the Laurant expansion

$$\varphi = \sum_{k=-a_i}^{\infty} c_{ik} z_i^k,$$

where z_i is a local coordinate about p_i. Map

$$\Phi : L(D) \to \mathbb{C}^{\deg(D)}, \quad \varphi \mapsto (c_{ik}),\ -a_i \le k \le -1.$$

This map is linear with

$$\mathrm{Ker}(\Phi) = \{\text{constant functions}\}.$$

In fact we have the famous

Riemann-Roch Theorem. For any divisor $D \in \mathrm{Div}(M_g)$

$$\dim L(D) = \deg D - g + 1 + \dim L(Z - D),$$

where Z is any canonical divisor.

For a very readable proof of the Riemann-Roch theorem we refer the reader to [Ke] pp.291–293.

DEFINITION. Let $p \in M_g$ be an arbitrary point. A positive integer m is called a gap at p if there does not exist a meromorphic function φ on M_g with

$$(\varphi)_\infty = mp.$$

The point p is called a Weierstrass point if the set of gaps at p is not

$$\{1, 2, \cdots, g\}.$$

EXAMPLES.

a) Consider $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. If $p \in \mathbb{C}$, then put

$$\varphi(z) = \frac{1}{(z-p)^m}, \quad \infty \mapsto 0.$$

If $p = \infty$, then we put $\varphi(z) = z^m$. Either way $(\varphi)_\infty = mp$, and there are no gaps anywhere.

b) Consider a complex torus $M = \mathbb{C}/\Lambda$, and $p \in M$ be arbitrary. Then there does not exist a meromorphic function φ with $(\varphi)_\infty = p$: if there were such a φ, then φ would give a homeomorphism between M and $\mathbb{C}P^1$. So 1 is a gap at p. Now by the Riemann-Roch theorem

$$\dim L(mp) = m, \ m \geq 2.$$

Consequently, there exists a meromorphic function in $L(mp)\backslash L((m-1)p)$, and m is not a gap value. So at an arbitrary point of a complex torus the set of gaps is $\{1\}$, and there are no Weierstrass points.

PROPOSITION 6. *Let $p \in M_g$ be arbitrary. Then there are exactly g gaps $\{m_1, \cdots, m_g\}$ at p with*

$$m_1 = 1 < \cdots < m_g \leq 2g-1.$$

Proof. We first show that there are no gaps $\geq 2g$. The preceding examples take care of the cases $g = 0, 1$. We assume that $g \geq 2$. For D with $\deg D \geq 2g-1$ we have $L(Z-D) = 0$, where Z is a canonical divisor. It follows that

$$\dim L(Z-mp) = \dim L(Z-(m-1)p) = 0.$$

On the other hand

$$\dim L(Z-mp) = (2g-2-m)+1-g+\dim L(mp),$$
$$\dim L(Z-(m-1)p) = (2g-2-m+1)+1-g+\dim L((m-1)p).$$

Therefore

$$\dim L(mp) - \dim L((m-1)p) = 1, \quad m \geq 2g,$$

and there are no gaps $\geq 2g$ at p. We now show that for any $m \geq 1$,

$$\dim L(mp) - \dim L((m-1)p) = 0 \text{ or } 1.$$

Suppose that $\dim L(mp) - \dim L((m-1)p) \neq 0$. Given a meromorphic function φ in $L(mp)$ we have the Laurant series expansion

$$\varphi(z) = a_{-m}z^{-m} + \cdots + a_{-1}z^{-1} + a_0 + \cdots ,$$

where z is a local coordinate centered at p. Note that

$$a_{-m} \neq 0 \text{ if and only if } \varphi \in L(mp)\backslash L((m-1)p).$$

Recall the linear map

$$\Phi : L(mp) \to \mathbb{C}^m, \quad \varphi \mapsto (a_{-m}, \cdots , a_{-1}).$$

Suppose $\varphi_1, \varphi_2 \in L(mp)\backslash L((m-1)p)$. Then we can find c_1, c_2 such that

$$\Phi(c_1\varphi_1 + c_2\varphi_2) = (0, \cdots),$$

i.e., $c_1\varphi_1 + c_2\varphi_2 \in L((m-1)p)$. It follows that one of the φ_i's is in the span of the other, and consequently

$$\dim L(mp) = \dim L((m-1)p) = 1.$$

We have shown that a positive integer m is a gap at p if and only if

$$\dim L(mp) - \dim L((m-1)p) = 0;$$

m is not a gap at p if and only if

$$\dim L(mp) = \dim L((m-1)p) = 1.$$

Now for any point p we have

$$\dim L(0) = 1, \ \dim L((2g-1)p) = g, \ \dim L(2gp) = g + 1.$$

The rest follows.

Proposition 7. *Let W denote the number of Weierstrass points on M_g. Then*

a) $2g+2 \leq W \leq (g-1)g(g+1)$;

b) $W = 2g+2$ *if and only if at every Weierstrass point the gaps are given by* $\{1, 3, \cdots, 2g-1\}$;

c) $W = (g-1)g(g+1)$ *if and only if at every Weierstrass point the gaps are given by* $\{1, 2, \cdots, g-1, g+1\}$.

For a proof of Proposition 7 see [FK] pp.85–86.

Recall that a function element (or a power series) determines upon analytic continuation a multivalued holomorphic function on $\mathbb{C}$. Let $c_1, \cdots, c_{2g+2}$ $(g \geq 2)$ be distinct points in $\mathbb{C}$. Consider the Riemann surface M of the multivalued function

$$w(z) = \sqrt{\Pi(z-c_i)}, \ \infty \mapsto \infty.$$

We think of w as a multivalued function $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. The Riemann surface M is a two-sheeted cover of $\mathbb{C} \cup \{\infty\}$ branched over $c_1, \cdots, c_{2g+2}$. Note that w is a single-valued holomorphic function $M \to \mathbb{C} \cup \{\infty\}$, i.e., w is a meromorphic function on M. Moreover, the degree of w is 2; the Weierstrass points of M are precisely at $c_1, \cdots, c_{2g+2}$.

Compact Riemann surfaces of genus at least 2 arising in the above manner are known as *hyperelliptic* surfaces. More precisely, a compact Riemann surface of genus $g \geq 2$ is said to be hyperelliptic if there exists a meromorphic function of degree 2 on it. It is not difficult to show that every Riemann surface of genus 2 is hyperelliptic. A generic compact Riemann surface of genus at least 3 is not hyperelliptic, however.

Let M_g be a Riemann surface of genus at least 2, and also let W denote the number of Weierstrass points on it. If $W = 2g+2$, then at a Weierstrass point p, 2 is a nongap. Thus there exists a meromorphic function φ whose polar divisor is given by $2p$. In particular, the degree of φ is 2, showing that M_g is hyperelliptic. In fact, it can be shown that M_g is hyperelliptic if and only if the number of Weierstrass points is $2g+2$.

3. Complete minimal surfaces of finite total curvature

Let $f : M \to \mathbb{R}^3$ be a conformal minimal immersion from a Riemann surface, and also let $\Phi_f : M \to \mathbb{C}P^2$ be its Gauss map. We say that the Gauss map is algebraic if

a) M is biholomorphic to a compact Riemann surface M_g punctured at a finite number of points $\{p_1, \cdots, p_r\}$;

b) Φ_f extends to a holomorphic map $\Phi : M_g \to \mathbb{C}P^2$.

Suppose Φ_f is algebraic. Then the image $\Phi(M_g)$ is an algebraic curve: $\Phi(M_g)$ can be realized as the zero locus of a complex homogeneous polynomial in 3 variables. The degree of Φ_f can be defined as the degree of a polynomial defining $\Phi(M_g)$. The following result is a variant of so called the Wirtinger theorem from Algebraic Geometry, and a proof can be found in [Y1] pp. 24–25.

PROPOSITION 8. *Let τ_f denote the total curvature of f, i.e.,*

$$\tau_f = \int_M K\, dA,$$

where $K \leq 0$ is the Gaussian curvature and dA is the area element of the induced metric. Then

$$-\tau_f = 2\pi \deg(\Phi_f).$$

In particular, the total curvature is an integral multiple of 2π.

A Riemannian manifold (N, ds_N^2) is said to be complete if it is a complete metric space. It is a well-known result that N is complete if and only if every geodesic can be extended for arbitrary large values of the arclength parameter.

A minimal surface $f : M \to \mathbb{R}^3$ is said to be complete if it is complete with respect to the induced metric. We have the following fundamental result.

The Chern-Osserman Theorem [CO]. Suppose $f : M \to \mathbb{R}^3$ is a complete minimal surface. Then the total curvature is finite if and only if the Gauss map is algebraic.

Let $f : M \to \mathbb{R}^3$ be a conformal minimal immersion, and suppose that the Gauss map is algebraic. An end of the minimal surface f is, by definition, $f(\Delta_i)$, where Δ_i is a sufficiently small punctured disc in M centered at a puncture p_i. Note that any path approaching the puncture p_i has to have an infinite arclength.

PROPOSITION 9. *Let $f : M \to \mathbb{R}^3$ be a complete conformal minimal immersion of finite total curvature. Also let r denote the number of ends or punctures, and g the genus of the underlying compact Riemann surface. Then*

$$\tau_f \leq 4\pi(1 - g - r).$$

Proof. Identify M with $M_g \backslash \{p_1, \cdots, p_r\}$ and note that each $\zeta^i = \frac{\partial f^i}{\partial z} dz$ gives a meromorphic 1-form on M_g. Let m_j denote the maximum order of the poles of (ζ^j) at p_j. Picking suitable constants (c^i) the meromorphic 1-form

$$\zeta = \sum c^i \zeta^i$$

has a pole of order exactly m_j at each p_j, $1 \leq j \leq r$. Since (ζ) is a canonical divisor on M_g we have

$$2g - 2 = \deg(\zeta)_0 - \deg(\zeta)_\infty.$$

But $\deg(\zeta)_0$ is just the number of zeros of ζ counted according to multiplicity, and

$$\deg(\zeta)_0 = \sum m_j + (2g - 2) \geq 2g - 2 + 2r,$$

since each $m_j \geq 2$. (We leave the verification of this as an exercise to the reader.) But

$$-\tau_f = 2\pi. \text{ (the number of zeros of } \zeta)$$

since the number of zeros of ζ is also the number of intersections between the algebraic curve $\Phi(M_g)$ and the hyperplane $\{(z^i) : \sum c^i z^i = 0\}$.

Jorge and Meeks [JM] showed that the inequality of Proposition 9 must in fact be an equality for a complete embedded minimal surface.

PROPOSITION 10 (OSSERMAN). *Let $f : M \to \mathbb{R}^3$ be a complete conformal minimal immersion of finite total curvature. Consider the meromorphic Gauss map*

$$\varphi = \pi \circ \Phi_f^{\perp} : M \to \mathbb{C} \cup \{\infty\}.$$

If φ misses more than 3 points, then $f(M)$ is a plane.

Proof. Identify M with $M_g \backslash \{p_1, \cdots, p_r\}$. We have $\{\mu, \varphi\}$, the Weierstrass pair of f. The meromorphic Gauss map φ extends to M_g giving a holomorphic map

$$\hat{\varphi} : M_g \to \mathbb{C}P^1, \quad \hat{\varphi}\,|\,_M = \varphi.$$

Applying a rotation to $f(M)$ if necessary we may (and do) assume:
a) $\mathrm{support}(\hat{\varphi})_\infty \cap \{p_1, \cdots, p_r\} = \emptyset$;
b) $(\hat{\varphi})_\infty$ consists only of simple poles.
Put

$$m = \deg \hat{\varphi}, \quad B = \text{the total branching number of } \hat{\varphi}.$$

Applying the Riemann-Hurwitz formula to $\hat{\varphi}$ we obtain

$$g = -m + 1 - B/2, \text{ or } B = 2(g + m - 1).$$

We now look at the differential μ and see how it extends to all of M_g. μ has double zeros at the poles of φ and no other zeros. Near p_i, one of the punctures, we have

$$|\eta|^2(1 + |\varphi|^2)^2 = \frac{c}{|z|^{2m_i}} + \text{ higher order terms}$$

with $2 \le m_i \le \infty$, where z is a local coordinate centered at p_i. Thus μ extends to a meromorphic 1-form $\hat{\mu}$ on M_g with a pole of order m_i at each p_i (and no other poles). So

$$\mathrm{support}(\hat{\mu})_\infty = \{p_1, \cdots, p_r\}, \ \mathrm{ord}_{pi} = m_i \ge 2,$$
$$\mathrm{support}(\hat{\mu})_\infty \cap \mathrm{support}(\hat{\varphi})_\infty = \emptyset.$$

The degree of the divisor of μ, (μ), is $2g - 2$ since (μ) is a canonical divisor. Hence

$$2g - 2 = 2m - \sum_{i=1}^{r} m_i.$$

Since $m_i \geq 2$ we must have

$$(*) \qquad g - 1 + d \leq m.$$

Suppose φ misses the points $q_1, \cdots, q_k$ of $\mathbb{C}P^1$. Then

$$\varphi^{-1}(\{q_1, \cdots, q_k\}) \subset \{p_1, \cdots, p_r\}.$$

Each q_i has m preimages counting multiplicity. So

$$km \leq \sum_{i=1}^{r}(1 + n_i) = r + \sum n_i,$$

where $1 + n_i$ ($n_i \geq 0$) is the multiplicity of $\hat{\varphi}$ at p_i. Now $\sum n_i$ is the sum of branching numbers at $\{p_1, \cdots, p_r\}$, hence it does not exceed the total branching number B. It follows that

$$(\dagger) \qquad km \leq r + B = r + 2(g + m - 1).$$

Adding the inequalities in $(*)$ and $(\dagger)$ and rewriting we obtain

$$1 - g \leq (3 - k)m.$$

The inequality in $(*)$ says that $r - m \leq 1 - g$. So $r - m \leq (3 - k)m$, or

$$r \leq (4 - k)m.$$

But since M is not compact $r \geq 1$, hence $k < 4$.

It is not known whether there exists a complete conformal minimal immersion $M \to \mathbb{R}^3$ of finite total curvature whose meromorphic Gauss map misses exactly 3 points.

REMARK. In 1988 Fujimoto [F] proved that the meromorphic Gauss map of any complete minimal surface in $\mathbb{R}^3$, whether with finite total curvature or not, can not omit more than 4 points. Since it is not hard to construct a complete minimal surface whose meromorphic Gauss map misses 4 or more points Fujimoto's bound is sharp. In 1989 Osserman and Mo [MO] gave the following refinement of Fujimoto's result:

the meromorphic Gauss map of a nonplanar complete minimal surface in $\mathbb{R}^3$ of infinite total curvature takes on every value infinitely often, with the possible exception of 4 points. The following question seems to be still open: Let $f : M \to \mathbb{R}^3$ be a nonplanar complete minimal surface with infinite total curvature. Then does the meromorphic Gauss map of f take on every value of its image infinitely often?

We now state the

Immersion Problem. Given $r \in \mathbb{Z}^+$ and a compact Riemann surface M_g find all complete conformal minimal immersions of finite total curvature

$$f : M_g \backslash \sum \to \mathbb{R}^3$$

with $|\sum| = r$.

Klotz and Sario [KS] proved that there exists a complete minimal surface in $\mathbb{R}^3$ of finite total curvature of every genus. Hoffman and Meeks [HM2] later constructed a complete minimal surface in $\mathbb{R}^3$ with finite total curvature of every genus with 3 punctures that is actually embedded.

A major step toward solving the immersion problem was taken by Gackstatter and Kunert [GK].

THEOREM A (GACKSTATTER-KUNERT). *Any compact Riemann surface of genus g can be immersed as a complete minimal surface with finite total curvature in $\mathbb{R}^3$ with at most $4g + 1$ punctures.*

Later the author [Y2] proved the following result.

THEOREM B (YANG). *Given any nonconstant meromorphic function F_1 on a compact Riemann surface M_g of genus $g > 0$ there exists another meromorphic function F_2 such that $\{dF_1, F_2\}$ is the Weierstrass pair defining a complete conformal minimal immersion of finite total curvature into $\mathbb{R}^3$ defined on M_g punctured at the supports of the polar divisors of F_1 and F_2.*

Since there are always an abundant supply of meromorphic functions on a Riemann surface Theorem B implies that any compact Riemann surface can be immersed in $\mathbb{R}^3$ as a complete minimal surface with finite total curvature with finitely many punctures. Indeed in [Y2] the author using Theorem B recoved the Gackstatter-Kunert theorem and proved the following result.

THEOREM C (YANG). *Any hyperelliptic Riemann surface can be immersed in* $\mathbb{R}^3$ *as a complete minimal surface with finite total curvature with at most* $3g+4$ *punctures.*

We now give a theorem improving the results in Theorems A–C.

THEOREM D [Y3]. *Let* M_g *be any compact Riemann surface of genus* $g > 0$. *Then there exists at least a one-parameter family of nonisometric complete conformal minimal immersions of finite total curvature*

$$M_g \backslash \sum \to \mathbb{R}^3,$$

where $\sum$ *is a finite set. For* $g = 1$, *we can have* $\sum$ *with* $|\sum| \leq 5$. *For* $g \geq 2$ *and* M_g *hyperelliptic, we can have* $\sum$ *with* $|\sum| \leq 3g+2$. *For* $g \geq 2$ *and* M_g *arbitrary, we can have* $\sum$ *with* $|\sum| \leq 4g$.

Proof. Let M_g be a compact Riemann surface of genus $g > 0$, and also let F_1 be any nonconstant meromorphic function on M_g. For some $b_i \in \mathbb{Z}^+$ and distinct points $p_i \in M_g$ we have

$$(F_1)_\infty = \sum_{i=1}^{n} b_i p_i.$$

Put

$$d = \sum b_i = \deg(F_1)_\infty.$$

Consider the meromorphic 1-form dF_1. We have

$$(dF_1)_\infty = \sum (b_i + 1) p_i.$$

For some $a_j \in \mathbb{Z}^+$ and distinct points $q_j \in M_g$ we have

$$(dF_1)_0 = \sum_{j=1}^{m} a_j q_j.$$

Since

$$(dF_1) = (dF_1)_0 - (dF_1)_\infty$$

is a canonical divisor we must have

$$\deg(dF_1) = 2g - 2$$

so that

$$\sum a_j = (2g-2) + n + d.$$

Introduce a divisor $D \in \mathrm{Div}(M_g)$ by

$$D = \sum_{j=1}^{m} a_j q_j - \sum_{i=1}^{n} c_i p_i = D^+ - D^-,$$

where the c_j's are some positive integers satisfying the conditions

$$c_i \geq b_i + 1; \sum c_i = 3g - 2 + n + d.$$

The first condition means that

$$D^- \geq (dF_1)_\infty,$$

and the second condition means that

$$\deg D = -g.$$

Consider the complex vector space

$$L(-D) = \{F \in H^0(M_g, \mathcal{M}^*) : (F) \geq D\} \cup \{0\}.$$

So a not identically zero meromorphic function F is a member of $L(-D)$ if and only if it has zeros of order at least a_j at q_j and poles of order at most c_i at p_i (or no poles). Now $\deg(dF_1) = 2g - 2 \geq 0$ and since $(dF_1)_\infty > 0$ we must have

$$\mathrm{support}(dF_1)_0 \neq \emptyset.$$

Consequently nonzero constant functions can not be in $L(-D)$. By the Riemann-Roch theorem

$$\begin{aligned}\dim L(-D) &= \deg(-D) - g + 1 + \dim L((dF_1) + D) \\ &= 1 + \dim L((dF_1) + D) \geq 1.\end{aligned}$$

So we can (and do) choose a nonconstant meromorphic function $G \in L(-D)$. (We may replace G by a nonzero complex multiple, and obtain a one-parameter family of noncongruent minimal surfaces in $\mathbb{R}^3$. We will not exploit this fact in the present article, however.) We put

$$(G)_0 = \sum_{j=1}^{m} \tilde{a}_j q_j + \sum_{k=1}^{l} \tilde{a}_{m+k} q_{m+k}; (G)_\infty = \sum_{i=1}^{n} \tilde{c}_i p_i$$

with

$$\tilde{c}_i \le c_i, \ \tilde{a}_j \ge a_j;$$
$$\sum \tilde{a}_j + \sum \tilde{a}_{m+k} = \sum \tilde{c}_i.$$

The two inequalities mean that $G \in L(-D)$, and the equality comes from the fact that (G) is a principal divisor. Define a nonconstant meromorphic function F_2 on M_g by

$$F_2 = \sum_{\alpha=1}^{\lambda} c_\alpha / G^\alpha,$$

where $\lambda = 2(n+m+l-1)+(4g+1)$, and (c_α) is a nonzero vector in $\mathbb{C}^\lambda$ to be chosen suitably later. Consider meromorphic 1-forms $F_2 dF_1$ and $F_2^2 dF_1$ on M_g. Observe that

$$\{q_{m+1}, \cdots, q_{m+l}\} \subset \text{support}(F_2 dF_1)_\infty \subset \{q_1, \cdots, q_{m+l}; p_1, \cdots, p_n\},$$
$$\{q_1, \cdots, q_{m+l}\} \subset \text{support}(F_2^2 dF_1)_\infty \subset \{q_1, \cdots, q_{m+l}; p_1, \cdots, p_n\}.$$

We claim that we can choose $(c_\alpha) \in \mathbb{C}^\lambda \backslash \{0\}$ such that the forms $F_2 dF_1$ and $F_2^2 dF_1$ have neither residues nor periods on M_g. Put

$$R_{i,\alpha} = \text{Res}_{p_i} dF_1 / G^\alpha,$$
$$R_{j,\alpha} = \text{Res}_{q_j} dF_1 / G^\alpha,$$
$$R_{k,\alpha} = \text{Res}_{q_{m+k}} dF_1 / G^\alpha,$$

where Res denotes the residue. We then have

$$\text{Res}_{p_i} F_2 dF_1 = \sum_\alpha c_\alpha R_{i,\alpha},$$
$$\text{Res}_{q_j} F_2 dF_1 = \sum c_\alpha R_{j,\alpha},$$
$$\text{Res}_{q_{m+k}} F_2 dF_1 = \sum c_\alpha R_{k,\alpha}.$$

So the meromorphic 1-form $F_2 dF_1$ on M_g has no residues if and only if the c_α's satisfy the following system of linear equations:

$$(I) \qquad \sum c_\alpha R_{i,\alpha} = 0; \ \sum c_\alpha R_{j,\alpha} = 0; \ \sum c_\alpha R_{k,\alpha} = 0.$$

The system has $n + m + l$ equations. We know that on a compact Riemann surface the total residue of any meromorphic 1-form must vanish. It follows that if any $n + m + l - 1$ residues of $F_2 dF_1$ were to vanish, then the remaining residue would have to vanish also. Consequently, we may (and do) throw out an equation from the linear system (I). Let $(e_1, \cdots, e_{2g})$ be 1-cycles on M_g representing a canonical homology basis. Put

$$P_{a,\alpha} = \int_{e_a} dF_1/G^\alpha, \ 1 \le a \le 2g, \ 1 \le \alpha \le \lambda.$$

So the e_a-period of $F_2 dF_1$ is given by the sum $\sum c_\alpha P_{a,\alpha}$. It follows that the differential $F_2 dF_1$ has no periods on M_g if and only if

$$(II) \qquad \sum c_\alpha P_{a,\alpha} = 0, \quad 1 \le a \le 2g.$$

This system is linear in (c_α) and contains at most $2g$ independent equations. We now consider the differential $F_2^2 dF_1$. Put

$$R_{i,2\alpha} = \mathrm{Res}_{p_i} dF_1/G^{2\alpha}, \quad 1 \le \alpha \le \lambda.$$

Similarly define $R_{j,2\alpha}$, and $R_{k,2\alpha}$. Thus the residue of $F_2^2 dF_1$ at p_i is given by

$$R_i(c_\alpha) = \sum_{\alpha=1}^{\lambda} c_\alpha^2 R_{i,2\alpha} + 2 \sum_{1 \le \alpha < \beta \le \lambda} c_\alpha c_\beta R_{i,\alpha+\beta}.$$

We also let $R_j(c_\alpha)$ and $R_k(c_\alpha)$ denote the residues of $F_2^2 dF_1$ at q_1 and q_{m+k} respectively. Observe that $R_i(c_\alpha)$, $R_j(c_\alpha)$, and $R_k(c_\alpha)$ are all homogeneous polynomials in (c_α) of degree 2. The zero locus in $\mathbb{C}^\lambda$ of one of these polynomials is a (possibly degenerate) homogeneous quadric. Now the meromorphic differential $F_2^2 dF_1$ has no residues if and only if the c_α's satisfy

$$(III) \qquad R_i(c_\alpha) = 0; \; R_j(c_\alpha) = 0; \; R_k(c_\alpha) = 0.$$

Again using the vanishing of the total residue we can throw out one of the equations from (III). Put

$$P_{a,2\alpha} = \int_{e_a} dF_1/G^{2\alpha}, \; P_{a,\alpha+\beta} = \int_{e_a} dF_1/G^{\alpha+\beta},$$

where $1 \le \alpha, \beta \le \lambda$, $1 \le a \le 2g$. So the e_a-period of $F_2^2 dF_1$ is given by

$$P_a(c_\alpha) = \sum c_\alpha^2 P_{a,2\alpha} + 2 \sum_{1 \le \alpha < \beta \le \lambda} c_\alpha c_\beta P_{a,\alpha+\beta}.$$

The form $F_2^2 dF_1$ has no periods on M_g if and only if

$$(IV) \qquad P_a(c_\alpha) = 0, \quad 1 \le a \le 2g.$$

Each $P_a(c_\alpha)$ is a homogenous polynomial in (c_α) of degree 2. The number of equations in the system (I-IV) is $2(n+m+l-1)+4g = \lambda - 1$. We can now establish our earlier claim: First note that the solution set of an equation in (I-IV) is either a hyperplane or a (possibly degenerate) homogenous quadric in $\mathbb{C}^\lambda$. At any rate it corresponds to an algebraic hypersurface in $\mathbb{P}^{\lambda-1}$. But we know that $\lambda - 1$ algebraic hypersurfaces in $\mathbb{P}^{\lambda-1}$ must intersect: This follows at once from the codimension formula: for any two algebraic varieties $V_1, V_2 \subset \mathbb{P}^N$

$$\operatorname{codim}(V_1 \cap V_2) \le \operatorname{codim} V_1 + \operatorname{codim} V_2.$$

In fact we see that the set of c_α's solving (I-IV) is itself a homogeneous affine variety. In particular, if a vector (c_α) solves the system, then so does any complex multiple of it. We let

$$\mathrm{Sol} \subset \mathbb{C}^\lambda$$

denote this solution variety. Fix $c = (c_\alpha) \in \mathrm{Sol}\backslash\{0\}$. We define holomorphic 1-forms $\zeta^1, \zeta^2, \zeta^3$ on

$$M = M_g \backslash \{\mathrm{support}(F_1)_\infty \cup \mathrm{support}(F_2)_\infty\} = M_g \backslash \{p_i; q_j; q_{m+k}\}$$

by the formulae

$$\zeta^1 = \frac{1}{2}(1 - F_2^2)dF_1,$$
$$\zeta^2 = \frac{i}{2}(1 + F_2^2)dF_1,$$
$$\zeta^3 = F_2 dF_1.$$

The fact that the differentials F_2dF_1 and $F_2^2dF_1$ have no residues and no periods on M_g guarantees that the holomorphic differentials $\zeta^1, \zeta^2, \zeta^3$ have no real periods on M. Consequently the formula

$$f = (f^\epsilon) = 2\mathrm{Re}\int^z (\zeta^\epsilon), \quad 1 \le \epsilon \le 3,$$

defines a conformal minimal immersion $M \to \mathbb{R}^3$. At a puncture

$$p \in \{p_i;\, q_j;\, q_{m+k}\} = M_g \backslash M$$

each ζ^ϵ has at worst a pole, hence the Gauss map of f extends holomorphically to all of M_g (cf. [Y1] p.29). It is not hard to see that any path approaching one of the punctures has an infinite arclength. Take a p_i, for example. If we let z be a holomorphic coordinate centered at p_i and write

$$\zeta^\epsilon = \eta^\epsilon dz,$$

then

$$h(z) = 2\sum |\eta^\epsilon|^2 = C/|z|^{2m} + \text{higher order terms}, \; m \ge 2$$

since dF_1 has a pole of order $b_i + 1 \ge 2$ at p_i. The arclength of any path approaching p_i must be infinite since the induced metric on M is given by

$$f^* ds_E^2 = h(z) dz \cdot d\bar{z}$$

near the puncture. Thus the induced metric is complete. We have shown that each

$$c = (c_\alpha) \in \mathrm{Sol}\backslash\{0\}$$

gives rise to a complete conformal minimal immersion of finite total curvature

$$f_c : M = M_g \backslash \sum \to \mathbb{R}^3,$$

where $\sum$ denotes the finite puncture set. Let $\tilde{c} = (\tilde{c}_\alpha) \in \mathrm{Sol}\backslash\{0\}$ be given by

$$\tilde{c}_\alpha = \rho c_\alpha,$$

where ρ is any nonzero complex number. Let $\{d\tilde{F}_1 = dF_1, \tilde{F}_2\}$ be the Weierstrass pair coming from the choice $\tilde{c} \in \mathrm{Sol}\backslash\{0\}$. We see that

$$\tilde{F}_2 = \rho F_2.$$

Let

$$\tilde{h}(z) dz \cdot d\bar{z}$$

denote a local expression for the induced metric of $\tilde{f} = f_{\tilde{c}}$. We compute that

$$\tilde{h} = |\eta|^2 (1 + |\rho F_2|^2)^2,$$

where $dF_1 = \eta dz$ locally. On the other hand, the induced metric of f_c is given by $h(z) dz \cdot d\bar{z}$ with

$$h = |\eta|^2 (1 + |F_2|^2)^2.$$

It follows that the surfaces f_c and $f_{\tilde{c}}$ are not isometric for $|\rho| \neq 1$ showing that there exists at least a one-parameter family of nonisometric complete conformal minimal immersions of finite total curvature

$$M_g \backslash \sum \to \mathbb{R}^3.$$

Let r denote the number of punctures, i.e., $r = |\sum|$. So $r = n + m + l$, where

$$\begin{aligned} n =& \text{ the number of distinct poles of } F_1, \\ m =& \text{ the number of distinct zeros of } dF_1. \end{aligned}$$

Moreover,

$$m + l \leq (3g - 2) + d + n,$$

where $d = \deg(F_1)_\infty$. This is so since

$$m + l \leq \sum \tilde{c}_i \leq \sum c_i = 3g - 2 + d + n.$$

Thus

$$r \leq (3g-2) + d + 2n.$$

Suppose M_g is a complex torus, i.e., $g = 1$. Then for any point $p \in M_g$ we can find a meromorphic function F on M_g such that

$$(F)_\infty = 2p.$$

Set $F_1 = F$. Then $d = 2$ and $n = 1$. Hence

$$r \leq 5.$$

Suppose M_g is hyperelliptic, and let $p \in M_g$ be a Weierstrass point. We then know that there exists a meromorphic function F on M_g with

$$(F)_\infty = 2p.$$

Letting $F_1 = F$ we see that $d = 2$, $n = 1$, and

$$r \leq 3g + 2.$$

We now suppose that M_g is an arbitrary Riemann surface of genus $g \geq 2$. On M_g there are at least $2g+2$ Weierstrass points. Let $p \in M_g$ be a Weierstrass point. This means that the gap sequence at p is not given by $\{1, 2, \cdots, g\}$. Since there are exactly g gaps it follows that we must have a nogap $d \leq g$. (The worst possible gap sequence at p is $\{1, 2, \cdots, g-1, g+1\}$.) But this means that there is a meromorphic function F on M_g with

$$(F)_\infty = dp.$$

Letting $F_1 = F$ we have $n = 1$, and

$$r \leq (3g-2) + (d + 2n) \leq (3g-2) + (g+2) \leq 4g.$$

The stage is set for the following

CONJECTURE.

a) There exists a compact Riemann surface M_g that can not be conformally minimally and completely immersed into $\mathbb{R}^3$ with finite total curvature with less than $4g$ punctures;

b) there exists a hyperelliptic Riemann surface M_g that can not be so immersed in $\mathbb{R}^3$ with less than $3g + 2$ punctures.

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