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ON COMMUTATIVITY OF s–UNITAL RINGS

H. A. S. ABUJABAL AND M. S. KHAN

1. Introduction

Let R be any ring. There are several results dealing with conditions under which R is commutative. Generally, such conditions are placed either on the ring itself or on its commutators. Further, the ring is required to satisfy certain polynomial identities as well.

In the present paper, we study the commutativity of a left s-unital ring R satisfying the polynomial identity

$$x^t[x^n, y] = y^r[x, y^m]y^s \text{ for all } x, y \in R, \tag{1}$$

where m, n, r, s and t are fixed non-negative integers.

We shall be frequently concerned with a property of R, namely the torsion freeness of commutators in R. This property has already been exploited (see [9]) to establish several results on the commutativity of R.

The results of this paper generalize some of the well-known commutativity theorems for rings which are left s-unital.

2. Preliminary Results

Throughout this paper, R will represent an associative ring (not necessarily with unity 1), $Z(R)$ the center of R, $C(R)$ the commutator ideal of R, $N(R)$ the set of all nilpotent elements in R, $N'(R)$ the set of all zero-divisors in R, and R^+ the additive group of R. As usual, for any $x, y \in R$, we write $[x, y] = xy - yx$. By $GF(q)$ we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. Set $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, in $(GF(p))_2$ for a prime p.

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DEFINITION 1. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further, R is called s-unital if it is both left as well as right s-unital, that is $x \in xR \cap Rx$ for each $x \in R$.

DEFINITION 2. If R is an s-unital (resp. a left or right s-unital) ring, then for any finite subset F of R, there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e is called the pseudo (resp. pseudo left or pseudo right) identity of F in R.

DEFINITION 3. For any positive integer m, the ring R is said to have property $Q(m)$ if for all $x, y \in R$, $m[x,y] = 0$ implies $[x,y] = 0$.

The property $Q(m)$ is an H-property in the sense of [9]. It is obvious that every m-torsion free ring R has the property $Q(m)$, and every ring has the property $Q(1)$. Also, it is clear that if a ring R has the property $Q(m)$, then R has the property $Q(n)$ for every divisor n of m.

In the proof of our results, we shall require the following well-known results.

LEMMA 1 ([3, LEMMA 2]). *Let R be a ring with unity 1, and let x and y be elements in R. If $kx^m[x,y] = 0$ and $k(x+1)^m[x,y] = 0$ for some integers $m \geq 1$ and $k \geq 1$, then necessarily $k[x,y] = 0$.*

LEMMA 2 ([14, LEMMA 3]). *Let x and y be elements in a ring R. If $[x,[x,y]] = 0$, then $[x^k, y] = kx^{k-1}[x,y]$ for all integers $k \geq 1$.*

LEMMA 3 ([18, LEMMA 3]). *Let R be a ring with unity 1, and let x and y be elements in R. If $(1-y^k)x = 0$, then $(1-y^{km})x = 0$ for some integers $k > 0$ and $m > 0$.*

LEMMA 4. *Let x and y be elements in a ring R. Suppose that there exists relatively prime positive integers m and n such that $m[x,y] = 0$ and $n[x,y] = 0$. Then $[x,y] = 0$.*

LEMMA 5 ([4, THEOREM 4 (C)]). *Let R be a ring with unity 1. Suppose that for each $x \in R$ there exists a pair n and m of relatively prime positive integers for which $x^n \in Z(R)$ and $x^m \in Z(R)$. Then R is commutative.*

Following results play an important role in proving the main results of this paper. The first is due to T. P. Kezlan [10, Theorem] and H. E.

Bell [3, Theorem 1] (also see [9, Proposition 2]), the second and third are due to Herstein.

THEOREM KB. *Let f be a polynomial in n non-commuting indeterminates $x_1,\ldots,x_n$ with relatively prime integral coefficients. Then the following are equivalent:*

(1) *For any ring satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.*
(2) *For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*
(3) *Every semi-prime ring satisfying $f = 0$ is commutative.*

THEOREM H ([7, THEOREM 18]). *Let R be a ring, and let $n > 1$ be an integer. Suppose that $(x^n - x) \in Z(R)$ for all $x \in R$. Then R is commutative.*

THEOREM H' ([8, THEOREM]). *If for every x and y in a ring R we can find a polynomial $p_{x,y}(t)$ with integral coefficients which depends on x and y such that $[x^2 p_{x,y}(x) - x, y] = 0$, then R is commutative.*

3. Main Results

Now, we are in a position to present our results.

THEOREM 1. *Let $n > 1$, m, r, s and t be fixed non-negative integers, and let R be a left s-unital ring satisfying the polynomial identity (1). Further, if R possesses property $Q(n)$, then R is commutative.*

Following lemma shows that the ring considered in Theorem 1 is in fact an s-unital ring. According to Proposition 1 of [9] this lemma enables us to reduce the proof of Theorem 1 to a ring with unity 1.

LEMMA 6. *Let $n > 0$, m, r, s and t be fixed non-negative integers such that $(r, n, s, m, t) \neq (0, 1, 0, 1, 0)$, and let R be a left s-unital ring satisfying the polynomial identity (1). Then R is an s-unital ring.*

Proof. Let x and y be arbitrary elements in R. Suppose that R is a left s-unital ring. Then there exists an element $e \in R$ such that $ex = x$ and $ey = y$. Replace x by e in (1). Then $y = ye^n \in yR$ for all $y \in R$. Thus R is an s-unital ring.

LEMMA 7. *Let $n > 0$, m, r, s and t be fixed non-negative integers, and let R be a ring satisfying the polynomial identity (1). Then $C(R)$ is nil.*

Proof. Let $x = e_{11}$ and $y = e_{12}$. Then x and y fail to satisfy the polynomial identity (1) whenever $n > 0$ except for $r = s = 0$, $m = 1$. In this later case one can choose $x = e_{12}$ and $y = e_{21}$. Hence Theorem KB ensures that

$$C(R) \subseteq N(R). \tag{2}$$

A combination of Lemma 7 with Theorem KB yields the following commutativity theorem for semi-prime ring.

THEOREM 2. *Let $n > 0, m. r, s$ and t be fixed non-negative integers. If R is a semi-prime ring satisfying the polynomial identity (1), then R is commutative.*

LEMMA 8. *Let $n > 1$, m, r, s and t be fixed non-negative integers, and let R be a ring with unity 1. Suppose that R satisfies the polynomial identity (1). Further, if R has property $Q(n)$, then $N(R) \subseteq Z(R)$.*

Proof. Let $a \in N(R)$. Then there exists a positive integer p, such that

$$a^k \in Z(R) \text{ for all } k \geq p, \text{ and } p \text{ minimal}. \tag{3}$$

If $p = 1$, then $a \in Z(R)$. Now, suppose that $p > 1$ and $b = a^{p-1}$. Replace x by b in (1) to obtain $b^t[b^n, y] = y^r[b, y^m]y^s$ for all $x, y \in R$. In view of (3) and the fact that $(p-1)n \geq p$, for $n > 1$, we get

$$y^r[b, y^m]y^s = 0 \text{ for all } y \in R. \tag{4}$$

Now, replace x by $(1+b)$ in (1) to get $(1+b)^t[(1+b)^n, y] = y^r[1+b, y]y^s$ for all $y \in R$. As (1+b) is invertible, using (4) we get form the last identity

$$[(1+b)^n, y] = 0 \text{ for all } y \in R. \tag{5}$$

Combining (3) and (5), we obtain $0 = [(1+b)^n, y] = [1+nb, y] = n[b, y]$ for all $y \in R$. Now, property $Q(n)$ implies that $[b, y] = 0$ for all $y \in R$, that is $a^{p-1} \in Z(R)$. This contradict the minimality of p. So we conclude that $p = 1$ and hence, $a \in Z(R)$. Therefore,

$$N(R) \subseteq Z(R). \tag{2'}$$

REMARK 1. Combining (2) and (2′), one gets

$$C(R) \subseteq N(R) \subseteq Z(R), \tag{6}$$

for any ring R with unity 1, which satisfies the polynomial identity (1) for all fixed non-negative integers n (> 1), m, r, s and t and whenever R has the property $Q(n)$. Hence, in view of (6), it is guaranteed that $[x,[x,y]] = 0$ for all $x, y \in R$ and thus the conclusion of Lemma 2 holds. In the proof of Theorem 1, we shall therefore routinely use Lemma 2 without explicit mention.

Now, we complete the proof of Theorem 1.

Proof of Theorem 1. According to Lemma 6, R is an s-unital ring. Therefore, in view of Proposition 1 of [9], it suffices to prove the theorem for R with unity 1.

If $m = 0$, then (1) gives $x^t[x^n, y] = 0$ for all $x, y \in R$. Hence, $nx^{t+n-1}[x,y] = 0$ for all $x, y \in R$. Replacing x by $(x+1)$ and applying Lemma 1, we obtain $n[x,y] = 0$ for all $x, y \in R$ which by property $Q(n)$, gives $[x,y] = 0$ for all $x, y \in R$. Therefore, R is commutative.

Now, we consider $m \geq 1$. Let $q = (2^{t+n} - 2)$. Then from (1) we have $qx^t[x^n, y] = (2^{t+n} - 2)x^t[x^n, y] = 2^{t+n}x^t[x^n, y] - 2x^t[x^n, y] = (2x)^t[(2x)^n, y] - 2y^r[x, y^m]y^s = (2x)^t[(2x)^n, y] - y^r[(2x), y^m]y^s = 0$. Therefore, $qnx^{t+n-1}[x,y] = 0$ for all $x, y \in R$. Put $k = qn$. Then by Lemma 1, we obtain $k[x,y] = 0$ for all $x, y \in R$. Thus $[x^k, y] = kx^{k-1}[x,y] = 0$ for all $x, y \in R$. So

$$x^k \in Z(R) \text{ for all } x \in R. \tag{7}$$

We distinguish between the two cases:

Case (a): Let $m > 1$. Then from (1) and (6), we have

$$x^t[x^n, y] = m[x,y]y^{r+s+m-1} \text{ for all } x, y \in R.$$

Replacing y by y^m gives $x^t[x^n, y^m] = m[x, y^m]y^{m(r+s+m-1)}$. So

$$mx^t[x^n, y]y^{m-1} = m[x, y^m]y^{m(r+t+m-1)} \text{ for all } x, y \in R.$$

By using (1) again, $my^r[x,y^m]y^{s+m-1} = m[x,y]y^{m(r+s+m-1)}$. This yields $m[x,y^m]y^{r+s+m-1}(1-y^{(m-1)(r+s+m-1)}) = 0$ for all $x,y \in R$. By Lemma 3, we get

$$(8) \quad m[x,y^m]y^{r+s+m-1}(1-y^{k(m-1)(r+s+m-1)}) = 0 \text{ for all } x,y \in R.$$

Now, by (6), the polynomial identity (1) becomes

$$(9) \qquad nx^{t+n-1}[x,y] = my^{r+s+m-1}[x,y] = m[x,y]y^{r+s+m-1} \quad \text{for all } x,y \in R.$$

It is well-known that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i \in I$, the index set). Each R_i satisfies (1), (6), (7), (8) and (9) but not necessarily has $Q(n)$ property. We consider the ring R_i ($i \in I$). Let S be the intersection of all non-zero ideals of R_i. Then $S \neq (0)$, and $Sd = 0$ for any central zero-divisor d.

Let $a \in N'(R_i)$. Then by (8) we have

$$m[x,a^m]a^{r+s+m-1}(1-a^{k(m-1)(r+s+m-1)}) = 0 \text{ for all } x \in R_i.$$

Suppose that $m[x,a^m]a^{r+s+m-1} \neq 0$ for $x \in R_i$. So $a^{k(m-1)(r+s+m-1)}$ and $1-a^{k(m-1)(r+s+m-1)}$ are central zero-divisors. So $(0) = S(1-a^{k(m-1)(r+s+m-1)}) = S \neq 0$, which is a contradiction. Therefore,

$$(10) \qquad m[x,a^m]a^{r+s+m-1} = 0 \text{ for all } x \in R_i.$$

From (9) and (10), we have $nx^{t+n-1}[x,a^m] = m[x,a^m]a^{m(r+s+m-1)} = 0$. Therefore by Lemma 1, $n[x,a^m] = 0$ for all $x \in R_i$, and hence $nm[x,a]a^{m-1} = 0$ for all $x \in R_i$. Now, $n^2x^{t+n-1}[x,a] = n(nx^{t+n-1}[x,a]) = nm[x,a]a^{r+s+m-1} = 0$ for all $x \in R_i$. Replacing x by $(x+1)$ and applying Lemma 1, we get $n^2[x,a] = 0$ for all $a \in R_i$. But $[x^{n^2},a] = n^2x^{n^2-1}[x,a]$. Therefore,

$$(11) \qquad [x^{n^2},a] = 0 \text{ for all } x \in R_i \text{ and } a \in N'(R_i).$$

Let $c \in Z(R_i)$. Then by (1), we have $(c^{t+n}-c)x^t[x^n,y] = (cx)^t[(cx)^n, y] - cx^t[x^n,y] = (cx)^t[(cx)^n,y] - y^r[(cx),y^m]y^s = 0$ for all $x,y \in R_i$. Applying Lemma 2, we obtain $n(c^{t+n}-c)x^{t+n-1}[x,y] = 0$ for all $x,y \in R_i$. Now, by Lemma 1, we get $n(c^{t+n}-c)[x,y] = 0$ which implies

$$(12) \qquad (c^{t+n}-c)[x^n,y] = 0 \text{ for all } x,y \in R_i \text{ and } c \in Z(R_i).$$

In particular, by (7) we have,

$$(y^{k(t+n)} - y^k)[x^n, y] = 0 \text{ for all } x, y \in R_i. \tag{13}$$

Now, consider $y \in R_i$. If $[x^n, y] = 0$, then clearly $[x^{n^2}, y^j - y] = 0$ for all positive integers j and $x \in R_i$. If $[x^{n^2}, y] \neq 0$, then $[x^n, y] \neq 0$, for $[x^n, y] = 0$ implies that $[x^{n^2}, y] = 0$, which is a contradiction. Since $[x^n, y] \neq 0$, then (13) implies that $(y^{k(t+n)} - y^k)$ is a zero-divisor. Therefore, $(y^{k(t+n-1)+1} - y)$ is also a zero-divisor. Hence, by (11)

$$[x^{n^2}, y^{k(t+n-1)+1} - y] = 0 \text{ for all } x, y \in R_i. \tag{14}$$

As each R_i satisfies (14), the original ring R also satisfies (14). But R has property $Q(n)$, therefore, combining (14) with Lemma 2, we finally obtain

$$[x, y^{k(t+n-1)+1} - y] = 0 \text{ for all } x, y \in R.$$

Thus, R is commutative by Theorem H.

Case (b)**:** Let $m = 1$. Then we get $x^t[x^n, y] = y^r[x, y]y^s$ for all $x, y \in R$. Thus, $nx^{t+n-1}[x, y] = [x, y]y^{r+s}$ for all $x, y \in R$. Replace x by x^n in the last identity to get $nx^{n(t+n-1)}[x^n, y] = [x^n, y]y^{r+s} = nx^{n-1}[x, y]y^{r+s} = nx^{t+n-1}[x^n, y]$ for all $x, y \in R$. Therefore, $n(1 - x^{(n-1)(t+n-1)})x^{t+n-1}[x^n, y] = 0$, which in view of Lemma 3, yields

$$n(1 - x^{k(n-1)(t+n-1)})x^{t+n-1}[x^n, y] = 0 \text{ for all } x, y \in R. \tag{15}$$

As in case (a) if $a \in N'(R_i)$, then by (15) we obtain

$$n(1 - a^{k(n-1)(t+n-1)})a^{t+n-1}[a^n, y] = 0 \text{ for all } y \in R_i.$$

By a similar argument as in case (a), we can prove that

$$na^{t+n-1}[a^n, y] = 0 \text{ for all } y \in R_i. \tag{16}$$

Now, we have $[a^n, y]y^{r+s} = na^{n(t+n-1)}[a^n, y] = 0$, and by Lemma 1,we get $[a^n, y] = 0$ for all $y \in R_i$. Therefore, $[a, y]y^{r+s} = a^t[a^n, y] = 0$. So

$$[a, y] = 0 \text{ for all } y \in R_i, \text{ and } a \in N'(R_i). \tag{17}$$

If $c \in Z(R_i)$, then as in case (a), we obtain $(c^{t+n} - c)[x, y] = 0$ for all $x, y \in R_i$. In particular, by (7), we have $(x^{k(t+n)} - x^k)[x, y] = 0$ for all $x, y \in R_i$. If $[x, y] = 0$ for all $x, y \in R_i$, then R satisfies $[x, y] = 0$ for all $x, y \in R$. Therefore, R is commutative. Now, if for each $x, y \in R_i$, $[x, y] \neq 0$, then $(x^{k(t+n-1)+1} - x) \in N'(R_i)$, and hence $(x^{k(t+n-1)+1} - x) \in N'(R)$. But the identity (17) is satisfied by the original ring R. Therefore, $[x^{k(t+n-1)+1} - x, y] = 0$ for each $x, y \in R$. Hence R is commutative by Theorem H. This completes the proof.

In Theorem 1, $Q(n)$ property is essential. To see this, consider, the following example:

EXAMPLE 1. Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over Z_2, the ring of integers *mod* 2. If R is the ring generated by the matrices A_1, B_1 and C_1, then using Dorroh construction with Z_2 (see [4, Remark]), we obtain a ring R with unity 1. Then R is non-commutative and satisfies $[x^2, y] = [x, y^2]$ for all $x, y \in R$.

The presence of the identity in Theorem 1 is not superfluous, as is shown by Example 1, and the following example.

EXAMPLE 2. Let

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over Z_2. If R is the ring generated by the matrices A_2, B_2 and C_2, then for each integer $n \geq 1$, the ring R satisfies the identity $[x^n, y] = [x, y^n]$ for all $x, y \in R$, but R is not commutative.

The following results are consequences of Theorem 1.

COROLLARY 1 ([4, THEOREM 5]). *Let R be a ring with unity 1, and $n > 1$ be a fixed integer. If R^+ is n-torsion free and R satisfies the identity*

$$x^n y - y x^n = x y^n - y^n x \text{ for all } x, y \in R,$$

then R is commutative.

COROLLARY 2 ([15, THEOREM 2]). *Let $n \geq m \geq 1$ be fixed integers such that $mn > 1$, and let R be an s-unital ring. Suppose that every commutator in R is $m!$-torsion free. Further, if R satisfies the polynomial identity*

$$[x^n, y] = [x, y^m] \text{ for all } x, y \in R, \tag{18}$$

then R is commutative.

COROLLARY 3 ([16, THEOREM 1]). *Let $n > 1$ and m be positive integers, and let s and t be any non-negative integers. Let R be an associative ring with unity 1. Suppose*

$$x^t [x^n, y] = [x, y^m] y^s \text{ for all } x, y \in R. \tag{19}$$

Further, if R is n-torsion free, then R is commutative.

Next theorem shows that the conclusion of Theorem 1 is still valid if the property $Q(n)$ is replaced by requiring m and n to be relatively prime positive integers.

THEOREM 3. *Let $m > 1$, and $n > 1$ be relatively prime integers, and let r, s, and t be non-negative integers. If R is a left s-unital ring satisfying the polynomial identity (1), then R is commutative.*

Proof. According to Lemma 6, R is an s-unital ring. Therefore, in view of Proposition 1 of [9], it is sufficient to prove the theorem for R with unity 1.

Now, without loss of generality, we can assume that R is subdirectly irreducible. Let $a \in N(R)$. Consider p and b as in Lemma 7. Then following the arguments of the proof of Lemma 7, we get $n[b, y] = 0$ and $m[b, y] = 0$ for each $x \in R$. Now, Lemma 4 gives $[b, y] = 0$ for all $y \in R$. Hence $a^{p-1} \in Z(R)$, which contradicts the minimality of p.

Therefore, $p = 1$, and so $a \in Z(R)$. Thus $N(R) \subseteq Z(R)$. This together with Lemma 6, shows that

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{20}$$

The proof of (7) also works in the present situation. So there exists an integer k (as in the proof of Theorem 1) such that

$$x^k \in Z(R) \text{ for each } x \in R. \tag{21}$$

Using argument similar to one as in the proof of Theorem 1 (see (11)), we obtain $[x^{n^2}, u] = 0$ and $[x^{m^2}, u] = 0$ for all $x \in R$ and $u \in N'(R)$. Then by Lemma 4 we get

$$[x, u] = 0 \text{ for all } x \in R, \text{ and } u \in N'(R). \tag{22}$$

As is observed in the proof followed by (11), we can prove that $n(c^{t+n} - c)[x, y] = 0$ and $m(c^{t+n} - c)[x, y] = 0$ for all $x, y \in R$, and $c \in Z(R)$. Again Lemma 4 gives

$$(c^{t+n} - c)[x, y] = 0 \text{ for all } x, y \in R, \text{ and } c \in Z(R). \tag{23}$$

As $y^k \in Z(R)$, by (21),we get $(y^{k(t+n)} - y^k)[x, y] = 0$ for all $x, y \in R$.

Arguing as in the proof of Theorem 1, we finally get $(y^{k(t+n-1)+1} - y) \in N'(R)$. Hence (22) yields

$$(y^{k(t+n-1)+1} - y) \in Z(R) \text{ for all } y \in R.$$

Now Theorem H implies the commutativity of R.

COROLLARY 4 ([16, THEOREM 2]). *Let m and n be relatively prime positive integers, and let s and t be any non-negative integers. Suppose that R is an associative ring with unity 1, satisfying the polynomial identity (19). Then R is commutative.*

Next result deals with the commutativity of R satisfying (1) for the case $n = 1$.

THEOREM 4. *Let R be a left s-unital ring, and let m, r, s and t be fixed non-negative integers such that $(t,m,r,s) \neq (0,1,0,0)$. If R satisfies the polynomial identity*

$$(24) \qquad x^t[x,y] = y^r[x,y^m]y^s \text{ for all } x,y \in R,$$

then R is commutative.

Proof. According to Lemma 6, R is an s-unital ring. Hence, in view of [9, proposition 1], we can prove the result for R with unity 1.

Case (I): If $m = 0$, then the identity (24) becomes

$$x^t[x,y] = 0 \text{ for all } x,y \in R,$$

giving there by, $(x+1)^t[x,y] = 0$ for all $x,y \in R$. By Lemma 1, $[x,y] = 0$ for each $x,y \in R$. Therefore, R is commutative.

Case (II): Suppose $m > 1$. Let $x = e_{11}$, and $y = e_{12}$. Then x and y fails to satisfy the identity (24). So by Theorem KB, $C(R) \subseteq N(R)$.

Let $a \in N(R)$. Then there exists a positive integer p such that

$$(25) \qquad a^k \in Z(R) \text{ for all } k \geq p, \text{ and } p \text{ minimal}.$$

If $p = 1$, then $a \in Z(R)$. Now, let $p > 1$, and let $b = a^{p-1}$. Replace y by b in (24) to get $x^t[x,b] = b^r[x,b^m]b^s$ for all $x \in R$. In view of (25) and $(p-1)m \geq p$, for $m > 1$, we obtain

$$x^t[x,b] = 0 \text{ for all } x \in R.$$

By Lemma 1, we get $[x,b] = 0$ for all $x \in R$. Therefore, $a^{p-1} \in Z(R)$ which is a contradiction. Thus $p = 1$, and hence $N(R) \subseteq Z(R)$. So

$$C(R) \subseteq N(R) \subseteq Z(R).$$

The method of proof of Theorem 1 enables us to establish the commutativity of R.

Case (III): Let $m = 1$. Then identity (24) becomes

$$(26) \qquad x^t[x,y] = y^r[x,y]y^s \text{ for all } x,y \in R.$$

We consider the following cases.

(i) : If $r=0$, then (26) becomes

$$x^t[x,y]=[x,y]y^s \text{ for all } x,y\in R. \tag{27}$$

Hence, if $s=0$, then $t>0$. Thus, $x^t[x,y]=[x,y]$ for all $x,y\in R$. Therefore, R is commutative [11, Theorem]. Similarly, if $t=0$ in (27), then R is commutative by [11, Theorem]. Let $t>0$ and $s>0$. Then $x=e_{11}$, and $y=e_{12}$ fail to satisfy the identity (27). By Theorem KB, $C(R)\subseteq N(R)$. Now, for any positive integer q, we can easily see that

$$x^{qt}[x,y]=[x,y]y^{qt} \text{ for all } x,y\in R. \tag{28}$$

If $a\in N(R)$, then for sufficiently large q, we get $x^{qt}[x,a]=0$ for all $x,y\in R$. Then by Lemma 1, $a\in Z(R)$. Therefore, $N(R)\subseteq Z(R)$. Thus

$$C(R)\subseteq N(R)\subseteq Z(R).$$

Let $l=(2^{s+1}-2)>0$, for $s>0$. Then $l[x,y]y^s=[x,(2y)](2y)^s-2[x,y]y^s=[x,(2y)](2y)^s-x^t[x,(2y)]=0$, and hence, $l[x,y]=0$ for all $x,y\in R$. Thus, $[x^l,y]=lx^{l-1}[x,y]=0$, and

$$x^l\in Z(R) \text{ for all } x\in R. \tag{29}$$

Therefore, by (28) and (29), we get $[x^{lt+1},y]=[x,y^{ls+1}]$ for all $x,y\in R$. In view of Proposition 3 (ii) of [9], there exists positive integer j such that $[x,y^{(ls+1)^j}]=0$ for each $x,y\in R$. But $(ls+1)^j=lk+1$. Then (28) yields $[x,y]y^{lk}=0$, and so by Lemma 1, we obtain $[x,y]=0$ for all $x,y\in R$. Therefore, R is commutative.

(ii) : If $s=0$, then (26) becomes

$$x^t[x,y]=y^r[x,y] \text{ for all } x,y\in R, \tag{30}$$

and so either $t>0$ or $r>0$. Thus without loss of generality, we can suppose that $r>0$. Clearly, $x=e_{11}$ and $y=e_{12}$ fail to satisfy (30). Then by Theorem KB, $C(R)\subseteq N(R)$. Following the same argument as in (i) we can prove the commutativity of R.

(iii) : If $t=0$, then (26) gives

$$[x,y]=y^r[x,y]y^s \text{ for all } x,y\in R. \tag{31}$$

Then either $r > 0$ or $s > 0$. Then clearly $x = e_{11}$ and $y = e_{12}$ fail to satisfy (31). Therefore, $C(R) \subseteq N(R)$. Let p and b as defined in case (II). Then (31) holds and $[x, b] = b^r[x, b]b^s = 0$ for all $x \in R$, which is a contradiction. Therefore $a \in Z(R)$, and hence, $N(R) \subseteq Z(R)$. Thus

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{32}$$

By (32), and Lemma 2, we obtain

$$[x, y] = y^{r+s}[x, y] \text{ for all } x, y \in R.$$

Therefore, R is commutative [11, Theorem].

(iv): Suppose $r > 0$, $s > 0$ and $t > 0$. Then $x = e_{11}$ and $y = e_{12}$ fail to satisfy (26). Therefore, $C(R) \subseteq N(R)$. If p and b be as defined in case (II), then $x^t[x, b] = b^r[x, b]b^s = 0 = (x+1)^t[x, b]$ for all $x \in R$. So by Lemma 1, $[x, b] = 0$ for all $x \in R$, which contradicts the minimality of p. Therefore, $N(R) \subseteq Z(R)$, and thus

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{33}$$

By (33), the identity (26) becomes

$$x^t[x, y] = [x, y]y^{r+s} \text{ for all } x, y \in R. \tag{34}$$

Following the proof of (i), we can establish the commutativity of a ring R with unity 1, which satisfies the identity (26). This completes the proof.

COROLLARY 5 ([12, THEOREM]). *Let t and m be two fixed non-negative integers. Suppose that R satisfies the polynomial identity*

$$x^t[x, y] = [x, y^m] \textit{ for all } x, y \in R. \tag{35}$$

- (i) *If R is a left s-unital, then R is commutative except when $(m, t) = (1, 0)$.*
- (ii) *If R is a right s-unital, then R is commutative except when $(m, t) = (1, 0)$; and $m = 0$ and $t > 0$.*

REMARK 2. In the above Corollary, for $m > 1$, R is commutative by Theorem 1. However, for $m = 0$ (resp. $m = 1$ and $t > 0$), it is easy to prove the commutativity of R.

COROLLARY 6. *Let $n > 0$ and m (resp. $m > 0$, and n) be fixed non-negative integers. Suppose that a left (resp. right) s-unital ring R satisfies the polynomial identity*

$$[xy, x^n + y^m] = 0 \text{ for all } x, y \in R. \tag{36}$$

If R has property $Q(n)$, then R is commutative.

Proof. Actually, R satisfies the identity

$$x[x^n, y] = [x, y^m]y \text{ for all } x, y \in R.$$

Therefore, R is commutative by Theorem 1 and Theorem 4.

As a consequence of Theorem 3, we have the following:

COROLLARY 7. *Let $m > 1$ and $n > 1$ be relatively prime fixed integers, and let R be a left s-unital ring satisfying the polynomial identity (36). Then R is commutative.*

In [6, Theorem B], A. Harmanci proved that "If $n > 1$ is a fixed integer and R is a ring with unity 1 which satisfies the identities $[x^n, y] = [x, y^n]$ and $[x^{n+1}, y] = [x, y^{n+1}]$ for each $x, y \in R$, then R must be commutative." In [5, Theorem 6], H. E. Bell generalized this result. The following theorem further extends the result of Bell.

THEOREM 5. *Let $m > 1$ and $n > 1$ be fixed relatively prime integers, and let r, s and t be fixed non-negative integers. If R is a left s-unital ring satisfying both the identities*

$$x^t[x^n, y] = y^r[x, y^n]y^s \text{ and } x^t[x^m, y] = y^r[x, y^m]y^s \quad \text{for all } x, y \in R, \tag{37}$$

then R is commutative.

Proof. According to Proposition 1 of [9], we prove the theorem for R with 1. Let b as in the proof of Lemma 8. Following the proof of Theorem 1 and Theorem 2 of [16], we can prove that $n[b,y] = 0$ and $m[b,y] = 0$ for all $y \in R$. By Lemma 4, we get $[b,y] = 0$ for all $y \in R$. The argument in the proof of Lemma 8, gives $N(R) \subseteq Z(R)$. Also, $x = e_{22}$ and $y = e_{21} + e_{22}$ fail to satisfy the polynomial identities in (37). Hence, by Theorem KB, $C(R) \subseteq N(R)$, and thus $C(R) \subseteq N(R) \subseteq Z(R)$. The argument of subdirectly irreducible rings can then be carried out both for n and m, yielding integers $j > 1$ and $k > 1$ such that R satisfies the identities $[x^j - x, y^{n^2}] = 0$ and $[x^k - x, y^{m^2}] = 0$ for all $x,y \in R$. Let $p(x) = (x^j - x)^k - (x^j - x)$. Then $0 = [p(x), y^{n^2}] = n^2[p(x),y]y^{n^2-1}$ for all $x,y \in R$, and $0 = [p(x), y^{m^2}] = m^2[p(x),y]y^{m^2-1}$ for all $x,y \in R$. Then Lemma 4 and Lemma 5 yield $[p(x),y]y^r = 0$ for all $x,y \in R$, and $r = max\{m^2 - 1,\ n^2 - 1\}$. Therefore, by Lemma 1, we get $p(x) \in Z(R)$. Since $p(x)$ has the form $x^2q(x) - x$ with $q(x)$ having integral coefficients, Theorem H' shows that R is commutative.

REMARK 3. In case $m = 0$ and $n \geq 1$, Theorem 1 need not be true for right s-unital ring. Also, when $m = 0$ and $t = 1$, Corollary 4 is not valid for s-unital ring. In fact we have the following example.

EXAMPLE 3. Let K be a field. Then, the non-commutative ring $R = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$, has a right identity element and satisfies the polynomial identity $x[x,y] = 0$ for all $x,y \in R$. Hence, in the case $m = 0$ and $n > 0$, Theorem 1 need not be true for right s-unital rings. As a matter of fact, Example 3 disproves Theorems 1,3,4, and 5 for right s-unital case whenever both r and t are positive.

COROLLARY 8 ([4, THEOREM 6]). *Let $m > 1$ and $n > 1$ be relatively prime positive integers. If R is any ring with unity 1 satisfying the identities $[x^m, y] = [x, y^m]$ and $[x^n, y] = [x, y^n]$ for all $x,y \in R$, then R is commutative.*

References

1. H. Abu-Khuzam, H. Tominaga and A. Yaqub, *Commutativity theorems for s-unital rings satisfying polynomial identities*, Math. J. Okayama Univ., **22** (1980), 111–114.
2. H. Abu-Khuzam and A. Yaqub, *Rings and groups with commuting powers*, Internat. J. Math. and Math. Sci., **4**, (1981), 101–107.

3. H. E. Bell, *On some commutativity theorems of Herstein*, Arch. Math., **24** (1973), 34–38.
4. ———, *On the power map and ring commutativity*, Canad. Math. Bull., **21** (1978), 399–404.
5. ———, *On rings with commuting powers*, Math. Japon., **24** (1979), 473–478.
6. A. Harmanci, *Two elementary commutativity theorems for rings*, Acta. Math. Acad. Sci. Hungar., **29** (1977), 23–29.
7. I. N. Herstein, *A generalization of a theorem of Jacobson*, Amer. J. Math., **73** (1951), 756–762.
8. ———, *The structure of a certain class of rings*, Amer. J. Math., **75** (1953), 864–871.
9. Y. Hirano, Y. Kobayashi and H. Tominaga, *Some polynomial identities and commutativity of s-unital rings*, Math. J. Okayama Univ., **24** (1982), 7–13.
10. T. P. Kezlan, *A note on commutativity of semiprime PI-rings*, Math. Japon., **27** (1982), 267–268.
11. ———, *On identities which are equivalent with commutativity*, Math. Japon., **29** (1984), 135–139.
12. H. Komatsu, *A commutativity theorem for rings*, Math. J. Okayama Univ., **26** (1984), 109–111.
13. W. K. Nicholson and A. Yaqub, *A commutativity theorem for rings and groups*, Canad. Math. Bull., **22** (1979), 419–423.
14. ———, *A commutativity theorem*, Algebra Universalis, **10** (1980), 260–263.
15. E. Psomopoulos, H. Tominaga and A. Yaqub, *Some commutativity theorems for n-torsion free rings*, Math. J. Okayama Univ., **23** (1981), 37–39.
16. E. Psomopoulos, *Commutativity theorems for rings and groups with constraints on commutators*, Internat. J. Math. and Math. Sci., **7** (1984), 513–517.
17. M. A. Quadri and M. A. Khan, *A commutativity theorem for left s-unital rings*, Bull. Inst. Math. Acad. Sinica, **15** (1987), 323–327.

Department of Mathematics
Faculty of Science
King Abdul Aziz University
P.O.Box 31464, Jeddah 21497
Saudi Arabia

Department of Mathematics
Sultan Qaboos University
P.O.Box 32500, Muscat Oman