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# EXISTENCE OF MAXIMIZABLE QUASICONCAVE FUNCTIONS ON CONVEX SPACES

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## 0. Introduction

Recently, J.C. Bellenger [1] gave conditions for the existence of an upper semicontinuous (u.s.c.) quasiconcave real function which attains a global maximum on a given compact subset of a paracompact convex space. This is also a coincidence theorem for multifunctions involving a fixed point for the composition of two multifunctions. Bellenger's result includes well-known theorems of Ky Fan [6] and S. Simons [17], which were used to deduce generalized Brouwer–Kakutani type fixed point theorems, matching theorems, and others.

The main purpose in the present paper is to show that Bellenger's theorem holds without assuming the paracompactness of the convex space, as he raised the question at the end of [1]. Some direct applications of our result to the Fan type nonseparation theorems and coincidence theorems are also given.

In Section 1, we give some basic definitions and two lemmas. Lemma 1 gives the crucial point in order to remove the paracompactness in Bellenger's theorem. Lemma 2 is the celebrated Fan–Browder fixed point theorem [2].

Section 2 deals with the proof of Bellenger's theorem without assuming the paracompactness. The proof is based on the partition of unity argument and Lemmas 1 and 2. We also obtain Fan's theorem [6, Theorem 8] without assuming the paracompactness.

In Section 3, we give a variation of the Fan type nonseparation theorems and a coincidence theorem, which are bases of the theory of the Brouwer–Kakutani type fixed point theorems in Fan [6], J. Jiang [7], H.-M. Ko and K.-K. Tan [8], M.-H. Shih and K.-K. Tan [14], and K.-K. Tan [19]. Actually, in these papers, the paracompactness

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assumption can be removed. Note that the first author's preprints [11, 12] were improved in this direction.

## 1. Preliminaries

A *convex space* $X$ is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset $L$ of a convex space $X$ is called a *c–compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$ [9].

A real-valued function $f$ on $X$ is *quasiconcave* if the set $\{x \in X \mid f(x) > a\}$ is convex for every real $a$. Also for any finite subset $\{x_1, \ldots, x_n\}$ of $X$ and a subset $L$ of $X$, $[x_1, \ldots, x_n, L]$ is the closed convex hull of $\{x_1, \ldots, x_n\} \cup L$ in $X$. Note that if $L$ is $c$–compact, then $[x_1, \ldots, x_n, L]$ is a compact convex subset of $X$.

Throughout this paper, we assume that any topological space is Hausdorff.

The following lemma has a crucial role in this paper.

LEMMA 1. *Let $\tilde{K}$ be a normal space, $K$ a compact subset of $\tilde{K}$, and $U$ an open subset of $\tilde{K}$. Suppose that $\alpha : K \to [0,1]$ is a continuous function such that* $\operatorname{supp} \alpha \subset U \cap K$. *Then there exists a continuous extension $\tilde{\alpha} : \tilde{K} \to [0,1]$ of $\alpha$ such that* $\operatorname{supp} \tilde{\alpha} \subset U$.

*Proof.* From the Tietze extension theorem, we have a continuous extension $\alpha_1 : \tilde{K} \to [0,1]$ of $\alpha$. Since $\operatorname{supp} \alpha \subset U$, we can find an open set $V$ such that $\operatorname{supp} \alpha \subset V \subset \overline{V} \subset U$. Since $\tilde{K}$ is normal, by Uryshon's Lemma there is a continuous function $\beta : \tilde{K} \to [0,1]$ satisfying $\beta(x) = 1$ for $x \in \operatorname{supp} \alpha$ and $\beta(x) = 0$ for $x \in \tilde{K} \backslash V$. Then $\tilde{\alpha} = \alpha_1 \cdot \beta$ is the desired extension of $\alpha$.

The following classical result is known as the Fan–Browder fixed point theorem [2] :

LEMMA 2. *Let $X$ be a compact convex space and $T : X \to 2^X$ a multifunction satisfying*

(i) *for each $x \in X$, $Tx$ is nonempty and convex; and*
(ii) *for each $y \in X$, $T^{-1}y = \{x \in X : y \in Tx\}$ is open.*

*Then $T$ has a fixed point $x_0 \in X$, that is, $x_0 \in Tx_0$.*

## 2. Existence of maximizable quasiconcave functions

We give the affirmative answer to Bellenger's question as follows :

THEOREM 1. *Let $X$ be a convex space. Suppose that*

(1.1) *for each $x \in X$, $Tx$ is a nonempty convex subset of u.s.c. quasiconcave real functions on $X$ (that is, every convex combination of two functions in $Tx$ is in $Tx$);*

(1.2) *for each u.s.c. quasiconcave real function $f$ on $X$, $T^{-1}f$ is compactly open in $X$; and*

(1.3) *there exist a c-compact subset $L$ of $X$ and a nonempty compact subset $K$ of $X$ such that for every $x \in X \backslash K$ and $f \in Tx$, we have $fx < \max f[x, L]$.*

*Then there exist an $\hat{x} \in K$ and an $f \in T\hat{x}$ such that $f\hat{x} = \max f(X)$.*

*Proof.* Let $\hat{X}$ denote the set of all u.s.c. quasiconcave real functions on $X$.

Step 1. $T|K$ has a selection $\phi : K \to \hat{X}$.

Since $Tx$ is nonempty for each $x \in X$, there exists an $f_x \in \hat{X}$ such that $x \in T^{-1}f_x$. Then $\{T^{-1}f_x : x \in X\}$ is a compactly open cover of $X$. Since $K$ is compact, we have a finite number of functions $f_1, f_2, \ldots, f_n$ in $\hat{X}$ such that $K \subset \bigcup_{i=1}^n T^{-1}f_i$. Let $\{\alpha_i\}_{i=1}^n$ be a continuous partition of unity subordinate to $\{T^{-1}f_i \cap K\}_{i=1}^n$ such that

(1.4) supp $\alpha_i \subset T^{-1}f_i \cap K$ and $\sum_{i=1}^n \alpha_i(x) = 1$ for $x \in K$.

For each $x \in K$, define $\phi_x = \phi(x) = \sum_{i=1}^n \alpha_i(x) f_i$. Then by (1.1) and (1.4), we see that $\phi_x \in Tx$ and $\phi_x$ is u.s.c. and quasiconcave for each $x \in K$. Then we know that

(1.5) for each $y \in X$, the function $x \mapsto \phi_x(y)$ is continuous on $K$; and

(1.6) the function $x \mapsto \phi_x(x)$ is u.s.c. on $K$.

Step 2. For any finite number of points $y_1, y_2, \ldots, y_m$ of $X$, let $\tilde{L} = [y_1, y_2, \ldots, y_m, L]$. Then $T|\tilde{L}$ has a selection $\tilde{\phi} : \tilde{L} \to \hat{X}$ such that $\tilde{\phi} = \phi$ on $\tilde{L} \cap K$.

Since $\tilde{L}$ is a compact subset of $X$, by (1.1) and (1.2), we have a finite number of functions $f_{n+1}, f_{n+2}, \ldots, f_{n+k}$ in $\hat{X}$ such that $\tilde{L} \subset \bigcup_{j=1}^k T^{-1}f_{n+j}$. Now by Lemma 1, we have continuous extensions $\tilde{\alpha}_i : \tilde{L} \cup K \to [0,1]$ of $\alpha_i$ such that supp $\tilde{\alpha}_i \subset T^{-1}f_i$ for $1 \leq i \leq n$.

Let $L_1 = \{x \in \tilde{L} : \sum_{i=1}^{n} \tilde{\alpha}_i(x) = 0\}$ and let $U_{n+j} = T^{-1}f_{n+j} \cap (\tilde{L}\backslash K)$ for $1 \le j \le k$. Then $L_1 \subset \bigcup_{j=1}^{k} U_{n+j}$ and hence we have a continuous partition of unity $\{\alpha_{n+j} : 1 \le j \le k\}$ defined on $L_1$ satisfying

(1.7) $\operatorname{supp} \alpha_{n+j} \subset U_{n+j} \cap L_1$ for each $1 \le j \le k$ and $\sum_{j=1}^{k} \alpha_{n+j}(x) = 1$ for every $x \in L_1$.

Now also by Lemma 1, we have continuous extensions $\tilde{\alpha}_{n+j} : \tilde{L} \cup K \to [0,1]$ of $\alpha_{n+j}$ such that $\operatorname{supp} \tilde{\alpha}_{n+j} \subset U_{n+j}$ for all $1 \le j \le k$. Then by (1.7) we know that $\sum_{i=1}^{n+k} \tilde{\alpha}_i(x) \neq 0$ for all $x \in \tilde{L} \cup K$. Now for each $1 \le i \le n+k$, define $\beta_i : \tilde{L} \cup K \to [0,1]$ by $\beta_i(x) = \tilde{\alpha}_i(x) / \sum_{j=1}^{n+k} \tilde{\alpha}_j(x)$. Then we have continuous functions $\beta_i : \tilde{L} \cup K \to [0,1]$, $1 \le i \le n+k$, such that

(1.8) $\operatorname{supp} \beta_i \subset T^{-1} f_i$ for each $1 \le i \le n+k$;

(1.9) for $x \in K$, $\beta_i(x) = \alpha_i(x)$ for $1 \le i \le n$ and $\beta_{n+j}(x) = 0$ for $1 \le j \le k$; and

(1.10) $\sum_{i=1}^{n+k} \beta_i(x) = 1$ for all $x \in \tilde{L} \cup K$.

For each $x \in \tilde{L}$, define $\tilde{\phi}_x = \sum_{i=1}^{n+k} \beta_i(x) f_i$. Then by (1.8) and (1.10), $\tilde{\phi}_x \in Tx$ for each $x \in \tilde{L}$ and $\tilde{\phi}_x$ is an u.s.c. quasiconcave function on $X$. Therefore (1.5) and (1.6) still hold for $\tilde{\phi}$ and $\tilde{L}$ instead of $\phi$ and $K$, resp. Note that by (1.9), $\tilde{\phi}_x = \phi_x$ for each $x \in \tilde{L} \cap K$.

Step 3. Let $Ay = \{x \in K : \phi_x(y) \le \phi_x(x)\}$ for each $y \in X$. Then $\{Ay : y \in X\}$ has the finite intersection property.

Note that by (1.5) and (1.6), each $Ay$ is closed in $K$. For any finite number of points $y_1, y_2, \ldots, y_m$, consider $\tilde{L} = [y_1, y_2, \ldots, y_m, L]$ and $\tilde{\phi} : \tilde{L} \to \hat{X}$ as in Step 2. Define

$$Wx = \{y \in \tilde{L} : \tilde{\phi}_x(y) > \tilde{\phi}_x(x)\}$$

for $x \in \tilde{L}$, and hence we have

$$W^{-1}y = \{x \in \tilde{L} : \tilde{\phi}_x(y) > \tilde{\phi}_x(x)\}$$

for $y \in \tilde{L}$. Since $\tilde{\phi}_x$ is quasiconcave, $Wx$ is convex. Note that $x \notin Wx$ for each $x \in \tilde{L}$. Moreover, by (1.5) and (1.6) for $\tilde{\phi}_x$ and $\tilde{L}$, $W^{-1}y$ is open in $\tilde{L}$. Therefore, by Lemma 2, we should have an $\overline{x} \in \tilde{L}$ such that $W\overline{x} = \emptyset$, or equivalently,

$$\tilde{\phi}_{\overline{x}}(y) \le \tilde{\phi}_{\overline{x}}(\overline{x}) \quad \text{for all} \quad y \in \tilde{L}.$$

This implies $\tilde{\phi}_{\overline{x}}(\overline{x}) = \max \tilde{\phi}_{\overline{x}}[\overline{x}, L]$. Since $\tilde{\phi}_{\overline{x}} \in T\overline{x}$, by (1.3), we should have $\overline{x} \in K$. Since $\tilde{\phi}_{\overline{x}} = \phi_{\overline{x}}$, we have

$$\phi_{\overline{x}}(y) \leq \phi_{\overline{x}}(\overline{x}) \quad \text{for all} \quad y \in \tilde{L},$$

and hence

$$\overline{x} \in \bigcap_{y \in \tilde{L}} Ay \subset \bigcap_{i=1}^{m} Ay_i.$$

Step 4. Since $K$ is compact, we have a point $\hat{x} \in \bigcap\{Ay : y \in X\}$. Let $f = \phi_{\hat{x}} \in T\hat{x}$. Then $f\hat{x} = \max f(X)$. This completes our proof.

REMARK. If $X$ is paracompact, then Theorem 1 reduces to Bellenger [1, Theorem 1]. Note that we can relax the requirement that $Tx$ be convex by considering $Sx = \operatorname{conv} Tx$ as in [1]. If $X$ is compact, that is, $X = K$, then Theorem 1 reduces to Simons [17, Theorem 0.1], whose particular forms appeared in Simons [15, 16]. Those results are used to obtain fixed point theorems, approximation theorems, and other consequences in [3, 15, 16, 17].

Moreover, from Theorem 1, we obtain the following version of Fan [6, Theorem 8] without assuming the paracompactness of $X$.

THEOREM 2. *Let $X$ be a convex space and $\Phi$ a nonempty convex set of lower semicontinuous convex real functions on $X$. Let $S$ be a subset of $X \times \Phi$ such that*

(2.1) *for each $\phi \in \Phi$, $\{x \in X : (x, \phi) \in S\}$ is compactly open in $X$; and*

(2.2) *for each $x \in X$, $\{\phi \in \Phi : (x, \phi) \in S\}$ is nonempty and convex. Then either*

(2.3) *there exists $(y, \phi) \in S$ such that $\phi(y) = \min \phi(X)$; or*

(2.4) *for each c–compact set $L$ in $X$ and each nonempty compact set $K$ in $X$, there exists a $(y, \phi) \in S$ such that*

$$y \in X \backslash K \quad \textit{and} \quad \phi(y) = \min \phi[y, L].$$

*Proof.* For each $x \in X$, we define $Tx = \{-\phi : (x, \phi) \in S\}$. One can easily verify that (1.1) and (1.2) are satisfied by $T$. Now Theorem 2 follows from Theorem 1.

## 3. Coincidence and fixed point theorems

Let $E$ be a real topological vector space (t.v.s.) and $E^*$ its dual space. For any $f \in E^*$ and $K \subset X \subset E$, let $Mf = \{x \in X : fx = \max f(X)\}$ and $M_f = \{x \in K : fx = \max f(X)\} = Mf \cap K$.

The following is a variation of the Fan type nonseparation theorems in [4, 5, 10, 13, 18] and a generalization of Simons [17, Theorem 2.2].

THEOREM 3. *Let $X$ be a nonempty convex subset of a t.v.s. $E$ and $P, Q : X \to 2^E \backslash \{\emptyset\}$ multifunctions. Let $K$ be a nonempty compact subset of $X$ and $L$ a c–compact subset of $X$. Suppose that, for each $f \in E^*$,*

(3.1) $X_f = \{x \in X : \sup f(Px) \geq \inf f(Qx)\}$ *is compactly closed;*

(3.2) $M_f \subset X_f$; *and*

(3.3) *for each $x \in X \backslash K$, $fx = \max f[x, L]$ implies $x \in X_f$.*

*Then there exists an $\hat{x} \in \bigcap\{X_f : f \in E^*\}$.*

*Proof.* For each $x \in X$, define

$$Tx = \{f \in E^* : \sup f(Px) < \inf f(Qx)\}.$$

Then $Tx$ is a convex subset of continuous linear functionals on $X$. Note that for each $f \in E^*$, $T^{-1}f = X_f^C$ is compactly open in $X$ and (3.3) implies (1.3). Suppose that $Tx \neq \emptyset$ for each $x \in X$. Then, by Theorem 1, there exist an $\hat{x} \in K$ and $f \in T\hat{x}$ such that $\hat{x} \in M_f$. However, (3.2) implies $\hat{x} \in X_f$, a contradiction. Therefore, we conclude that $T\hat{x} = \emptyset$ for some $\hat{x} \in X$, and hence $\hat{x} \in \bigcap\{X_f : f \in E^*\}$. This completes our proof.

Theorem 3 is the key result in the first author's forthcoming work [12]. For exmaple, a direct application of Theorem 3 is as follows:

THEOREM 4. *Under the same hypotheses of Theorem 3, suppose that either*

(A) *$E^*$ separates points of $E$ and $P, Q : X \to kc(E)$; or*

(B) *$E$ is locally convex, $P, Q : X \to cc(E)$, and one of $Px$ and $Qx$ is compact in $E$ for each $x \in X$,*

*where $kc(E)$ [$cc(E)$] denotes the set of all nonempty compact convex [closed convex, resp.] subsets of $E$. Then there is an $\hat{x} \in X$ such that $P\hat{x} \cap Q\hat{x} \neq \emptyset$.*

*Proof.* Suppose that $Px \cap Qx = \emptyset$ for all $x \in X$. Then each of (A) and (B) implies that for each $x \in X$ there exists an $f \in E^*$ with $\inf f(Qx) > \sup f(Px)$, by the standard separation theorems in a t.v.s. This contradicts Theorem 3.

REMARK. Note that if $P$ and $Q$ are upper hemicontinuous (see [11]), then (3.1) holds. Moreover, if $P$ or $Q$ is the identity on $X$, then we have a fixed point theorem. Actually Theorem 4 includes results of Jiang [7, II, Theorem 2.2], Ko and Tan [8, Lemma 1.2], Ky Fan [6, Theorem 9], Shih and Tan [14, Theorems 4 and 5], and Tan [19, Theorem 3.2]. For details, see [11, 12].

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