J. Korean Math. Soc. 28(1991), No. 2, pp. 275–283

GENERALIZATIONS OF KY FAN'S MATCHING THEOREMS AND THEIR APPLICATIONS, II

SEHIE PARK

1. Introduction

In our previous work [9], we obtained generalizations of Ky Fan's matching theorems for open [or closed] coverings and their applications. In the present paper, we obtain further generalizations of some main results in [8], [9] by adopting more general coercivity conditions. The necessity of such general conditions is motivated by recent works in [1], [2], [3], [4], [6]. In fact, from our previous matching theorem [9, Theorem 1], we deduce a general matching theorem, general KKM type intersection theorems, the Fan–Browder type fixed point theorems, minimax inequalities, and a geometric property of convex sets.

In Section 2, Theorem 1 is the matching theorem given by the author [9, Theorem 1]. Theorem 2 generalizes Theorem 1 to non–compact case with a new coercivity conditon and includes Fan [5, Theorem 3] and Park [9, Theorem 2].

Section 3 deals with generalizations of the KKM type intersection theorems. Theorem 3 extends Chang [3, Theorem 2.1] and Lassonde [7, Theorems I and III].

In Section 4, we obtain new Fan–Browder type fixed point theorems. In fact, Theorem 5 generalizes Jiang [6, Lemma 2.1], Park [8, Theorem 1], and many others. Corollary 6 is a dual form of Theorem 5 including Park [8, Theorem 4] and others.

Section 5 deals with analytic alternatives generalizing the well–known Ky Fan minimax inequality. In fact, Theorem 7 contains [8, Theorem 9] and various extensions of the Ky Fan inequality due to many authors. From Theorem 7, we obtain Corollary 8, which includes recent results of Bae and Kim [1, Theorem 1], Bae, Kim, and Tan [2, Theorem 1], and Ding and Tan [4, Theorem 1].

Received September 12, 1990.
Supported in part by the Basic Science Research Institute Program, Ministry of Education, 1989.

In Section 6, we give a geometric property of convex sets. Theorem 9 is a far-reaching generalization of Ky Fan's 1961 Lemma and includes results in [2], [8], [10].

Since particular forms of our new results have many applications as shown in the literature, some of them could be improved in view of this paper. For example, most of results in [8] can be improved by adopting the new coercivity condition.

2. Matching theorems for open coverings

A *convex space* X is a nonempty convex set X (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets [7]. A nonempty subset L of X is called a *c–compact set* if for each finite subset $N \subset X$ there is a compact convex subset of X containing $L \cup N$ [7].

For any topological spaces X and Y, let $\mathcal{C}(X,Y)$ denote the class of all continuous functions from X into Y. For other terminology and notations, we follow [7], [9].

We begin with the following [9, Theorem 1] :

THEOREM 1. *Let D be a nonempty subset of a compact convex space X, Y a topological space, and $A : D \to 2^Y$ a multifunction satisfying*

(i) *for each $x \in D$, Ax is compactly open in Y, and*
(ii) $A(D) = Y$.

Then, for each $f \in \mathcal{C}(X,Y)$, there exist a nonempty finite subset $\{x_1, x_2, \ldots, x_n\}$ of D and an $x_0 \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$ such that $fx_0 \in \bigcap_{i=1}^n Ax_i$.

The following consequence of Theorem 1 is the main result of this paper:

THEOREM 2. *Let D be a nonempty subset of a convex space X, Y a topological space, $A : D \to 2^Y$ a multifunction, and $f \in \mathcal{C}(X,Y)$. Suppose that there exists a nonempty compact subset K of Y such that*

(i) *for each $x \in D$, Ax is compactly open;*
(ii) *for each $y \in \overline{f(X)} \cap K$, $A^{-1}y$ is nonempty; and*

(iii) *for each finite subset N of D, there exists a compact convex subset L_N of X containing N such that $x \in L_N \backslash f^{-1}(K)$ implies $fx \in A(L_N \cap D)$.*

Then, there exist a nonempty finite subset $\{x_1, x_2, \dots, x_n\}$ of D and an $x_0 \in \operatorname{co}\{x_1, x_2, \dots, x_n\}$ such that $fx_0 \in \bigcap_{i=1}^n Ax_i$.

Proof. Since $\overline{f(X)} \cap K$ is compact in Y and is included in $A(D)$ by (ii), there exists a finite subset $N = \{p_1, p_2, \dots, p_m\} \subset D$ such that $\overline{f(X)} \cap K \subset \bigcup_{i=1}^m Ap_i$. Let $X_1 = L_N$ be the compact convex subset of X as in (iii). Let $D_1 = X_1 \cap D$ and $Y_1 = A(D_1)$. We show that $f(X_1) \subset Y_1$. In fact, if $x \in X_1 \cap f^{-1}(K)$, then $fx \in f(X_1) \cap K \subset \overline{f(X)} \cap K \subset \bigcup_{i=1}^m Ap_i \subset A(X_1 \cap D) = Y_1$. On the other hand, if $x \in X_1 \backslash f^{-1}(K) = L_N \backslash f^{-1}(K)$, then, by (iii), $fx \in A(L_N \cap D) = Y_1$. Hence A and f can be regarded as functions $A : D_1 \to 2^{Y_1}$ and $f \in \mathcal{C}(X_1, Y_1)$. By applying Theorem 1 to D_1, X_1, and Y_1, we have the conclusion.

REMARKS. 1. Instead of (ii), we may assume

(ii)$'$ $A(D) = Y$

in Theorem 2 without affecting the conclusion.

2. The coercivity condition (iii) is motivated by Chang [3] and improves that of Jiang [5, Lemma 2.1].

3. If there exists a c-compact subset L of X, then, instead of (iii), we may assume any of the following:

(iii)$_1$ for each $x \in X \backslash f^{-1}(K)$, $fx \in A(\overline{\operatorname{co}}(L \cup \{x\}) \cap D)$.

(iii)$_2$ $Y \backslash A(L \cap D) \subset K$.

In fact, it is easy to check (iii)$_2 \Longrightarrow$ (iii)$_1 \Longrightarrow$ (iii).

Theorem 2 with (iii)$_2$ instead of (iii) reduces to Park [9, Theorem 2], which is a generalization of the matching theorem for open coverings due to Ky Fan [5, Theorem 3].

3. Generalizations of the KKM theorem

Theorem 2 with (ii)$'$ instead of (ii) may be restated in its contrapositive form and in terms of the complement Fx of Ax in Y as follows :

THEOREM 3. *Let D be a nonempty subset of a convex space X, Y a topological space, $F : D \to 2^Y$ a multifunction, and $f \in \mathcal{C}(X, Y)$.*

Suppose that

(i) *for each* $x \in D$, Fx *is compactly closed in* Y;

(ii) *for any finite subset* $\{x_1, x_2, \ldots, x_n\}$ *of* D,

$$f(\operatorname{co}\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} Fx_i;$$

and

(iii) *there exist a nonempty compact subset* K *of* Y *and, for each finite subset* N *of* D, *a compact convex subset* L_N *of* X *containing* N *such that* $x \in L_N \backslash f^{-1}(K)$ *implies* $fx \notin \bigcap\{Fz : z \in L_N \cap D\}$.

Then we have $\bigcap\{Fx : x \in D\} \neq \emptyset$.

REMARK. If there exists a c-compact subset L of X, then, instead of (iii), we may assume the following:

(iii)′ $\bigcap\{Fz : z \in L \cap D\} \subset K$.

Theorem 3 with (iii)′ instead of (iii) is due to Lassonde [7, Theorem I] and, as indicated in [9], contains a number of generalizations of the KKM theorem.

From Theorem 3, we obtain the following :

COROLLARY 4. *Let* D *be a nonempty subset of a convex space* X, *and* $F : D \to 2^X$ *a multifunction. Suppose that*

(i) *for each* $x \in D$, Fx *is compactly closed in* X;

(ii) *for any finite subset* $\{x_1, x_2, \ldots, x_n\}$ *of* D,

$$\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} Fx_i;$$

and

(iii) *there exist a nonempty compact subset* K *of* X *and, for each finite subset* N *of* D, *a compact convex subset* L_N *of* X *containing* N *such that*

$$L_N \cap \bigcap\{Fx : x \in L_N \cap D\} \subset K.$$

Then $\bigcap\{Fx : x \in D\} \neq \emptyset$.

Proof. In Theorem 3, put $X = Y$ and $f = 1_X$.

REMARK. Corollary 4 is due to Chang [3, Theorem 2.1] and improves Lassonde [6, Theorem III] who assumed the Hausdorffness of the underlying topological vector space and adopted a stronger condition than (iii).

4. The Fan–Browder type fixed point theorems

As an application of Theorem 2, we give the following generalization of the Fan–Browder fixed point theorem:

THEOREM 5. *Let D be a nonempty subset of a convex space X, Y a topological space, $A : X \to 2^Y$, $B : D \to 2^Y$ multifunctions, and $f \in \mathcal{C}(X, Y)$. Suppose that there exists a nonempty compact subset K of Y such that*

(a) *$Bx \subset Ax$ for each $x \in D$;*
(b) *$A^{-1}(fx)$ is convex for each $x \in X$;*
(c) *$B^{-1}y \neq \emptyset$ for each $y \in \overline{f(X)} \cap K$;*
(d) *Bx is compactly open for each $x \in D$; and*
(e) *for each finite subset N of D, there exists a compact convex subset L_N of X containing N such that $x \in L_N \backslash f^{-1}(K)$ implies $fx \in B(L_N \cap D)$.*

Then there exists an $x_0 \in X$ such that $fx_0 \in Ax_0$.

Proof. Since the conditions (d), (c), and (e) imply (i), (ii), and (iii) of Theorem 2, respectively, there exist $\{x_1, x_2, \ldots, x_n\} \subset D$ and $x_0 \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$ such that $fx_0 \in \bigcap_{i=1}^n Bx_i \subset \bigcap_{i=1}^n Ax_i$ by (a). We have $x_i \in A^{-1}(fx_0)$ for all $i = 1, \ldots, n$, and hence by (b), $\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset A^{-1}(fx_0)$. In particular, $x_0 \in A^{-1}(fx_0)$, that is, $fx_0 \in Ax_0$. This completes our proof.

REMARKS. 1. For $D = X$ and $A = B$, Theorem 5 includes Jiang [6, Lemma 2.1]. Some particular forms and applications to fixed point theorems are also given in [6].

2. Particular versions of Theorem 5 are due to Park [8, Theorem 1], [9, Theorem 6], where A should be $A : X \to 2^Y$ and $x_0 \in X$. As we noted in [8], Theorem 5 generalizes a number of results due to Takahashi, Lassonde, Ben–El–Mechaiekh–Deguire–Granas, Simons, Ko–Tan, and Browder.

3. In Theorem 5, we may replace (a) and (b) by $\operatorname{co} B^{-1}y \subset A^{-1}y$ for each $y \in Y$.

From Theorem 5, we have the following "dual" form:

COROLLARY 6. *Let Y be a topological space, X a convex space, $S, T : Y \to 2^X$ multifunctions, and $f \in \mathcal{C}(X, Y)$. Suppose that there exists a nonempty compact subset K of Y such that*

(a) *$Ty \subset Sy$ for each $y \in Y$;*
(b) *$S(fx)$ is convex for each $x \in X$;*
(c) *$Ty \neq \emptyset$ for each $y \in \overline{f(X)} \cap K$;*
(d) *$T^{-1}x$ is compactly open for each $x \in X$; and*
(e) *for each finite subset N of X, there exists a compact convex subset L_N of X containing N such that $x \in L_N \backslash f^{-1}(K)$ implies $T(fx) \subset L_N$.*

Then there exists an $x_0 \in X$ such that $x_0 \in (Sf)x_0$.

Proof. Put $D = X$, $A^{-1} = S$, and $B^{-1} = T$ in Theorem 5 replacing (e) by $(\mathrm{iii})_1$ in the Remarks of Theorem 2.

REMARK. Corollary 6 extends Chang [3, Theorems 2.4 and 2.7]. Instead of (e), if there exists a c-compact subset L of X, then we may assume

(e)′ $Y \backslash T^{-1}(L) \subset K$

in Corollary 6 without affecting the conclusion.

Corollary 6 with (e)′ instead of (e) is due to Park [8, Theorem 4] and, as we noted there, includes a number of results due to Ben–El–Mechaiekh–Deguire–Granas, Ko–Tan, Browder, and Tarafdar–Husain.

5. Analytic alternatives

In our previous work [8], a particular form of Theorem 5 with $D = X$ is shown to have various equivalent formulations and applications, that is, alternative forms, geometric forms, fixed point theorems, coincidence theorems, minimax and variational inequalities, etc. This can be done for Theorem 5 with $D = X$.

The following is an analytic alternative and a far-reaching generalization of the Ky Fan minimax inequality.

THEOREM 7. *Let X be a convex space, Y a topological space, $\alpha \geq \beta$, $f, g : X \times Y \to [-\infty, +\infty]$ functions, and $s \in \mathcal{C}(X, Y)$. Suppose that*

(i) *for each $x \in X$, $\{y \in Y : g(x, y) > \alpha\}$ is compactly open;*
(ii) *for each $y \in Y$, $\{x \in X : f(x, y) > \beta\} \supset \operatorname{co}\{x \in X : g(x, y) > \alpha\}$; and*
(iii) *there exist a nonempty compact subset K of Y and, for each finite subset N of X, a compact convex subset L_N of X containing N such that, for each $x \in L_N \backslash s^{-1}(K)$, there exists an $x_1 \in L_N$ such that $g(x_1, sx) > \alpha$.*

Then either (1) there exists a $y_0 \in K$ such that $g(x, y_0) \leq \alpha$ for all $x \in X$, or (2) there exists an $x_0 \in X$ such that $f(x_0, sx_0) > \beta$.

Proof. For each $x \in X$, let

$$\begin{aligned} Ax &= \{y \in Y : f(x, y) > \beta\}, \\ Bx &= \{y \in Y : g(x, y) > \alpha\}. \end{aligned}$$

Then Bx is compactly open by (i), $A^{-1}y \supset \operatorname{co} B^{-1}y$ for each $y \in Y$ by (ii), and for each finite subset N of X and each $x \in L_N \backslash s^{-1}(K)$, we have $sx \in B(L_N)$ by (iii). Suppose that (2) does not hold, that is, $sx \notin Ax$ for any $x \in X$. Then by Theorem 5 (with Remark 3 there), there is a $y_0 \in \overline{s(X)} \cap K$ such that $B^{-1}y_0 = \emptyset$, that is, $g(x, y_0) \leq \alpha$ for all $x \in X$. This completes our proof.

REMARK. Theorem 7 generalizes [8, Theorem 9]. As we noted there, [8, Theorem 9] includes various extensions of Ky Fan's 1972 minimax inequality due to Takahashi, Ben-El-Mechaiekh *et al.*, Lassonde, Aubin, Allen, Lin, Tan, Yen, Shih and Tan, and others.

From Theorem 7, we obtain the following practical form:

COROLLARY 8. *Let X be a convex space and $f, g : X \times X \to [-\infty, +\infty]$ functions. Suppose that*

(0) *$f(x, x) \leq 0$ for all $x \in X$;*
(i) *for each $x \in X$, $\{y \in X : g(x, y) > 0\}$ is compactly open;*
(ii) *for each $y \in X$, $\{x \in X : f(x, y) > 0\} \supset \operatorname{co}\{x \in X : g(x, y) > 0\}$; and*

(iii) *there exist a nonempty compact subset K of X and, for each finite subset N of X, a compact convex subset L_N of X containing N such that, for each $y \in L_N \backslash K$, there exists an $x \in L_N$ with $g(x,y) > 0$.*

Then there exists a $y_0 \in K$ such that $g(x,y_0) \leq 0$ for all $x \in X$.

Proof. Put $X = Y$, $s = 1_X$, and $\alpha = \beta = 0$ in Theorem 7.

REMARKS. 1. Particular forms of Corollary 8 are due to Bae and Kim [1, Theorem 1], Bae, Kim, and Tan [2, Theorem 1], and Ding and Tan [4, Theorem 1]. These authors applied Corollary 8 to (quasi-) variational inequalities, fixed point theorems, and other results. Such results can be improved by adopting the more general coercivity condition (iii) as in Corollary 8 instead of their condition.

2. As the authors of [1], [2], [4] noted, Corollary 8 includes minimax inequalities due to Shih and Tan, Fan, Tan, Allen, and others.

6. A geometric form

Theorems 5 and 7 can be expressed in a geometric form. In fact, the following is a far-reaching generalization of Ky Fan's 1961 Lemma.

THEOREM 9. *Let X be a convex space, Y a topological space, $A, B \subset X \times Y$, and $s \in \mathcal{C}(X,Y)$. Suppose that*

(i) *for each $x \in X$, $\{y \in Y : (x,y) \in B\}$ is compactly open;*
(ii) *for each $y \in Y$, $\{x \in X : (x,y) \in A\} \supset \mathrm{co}\{x \in X : (x,y) \in B\}$; and*
(iii) *there exist a nonempty compact subset K of Y and, for each finite subset N of X, a compact convex subset L_N of X containing N such that for each $x \in L_N \backslash s^{-1}(K)$, $L_N \bigcap \{z \in X : (z, sx) \in B\} \neq \emptyset$.*

Then either (1) there exists a $y_0 \in K$ such that $\{x \in X : (x,y_0) \in B\} = \emptyset$, or (2) there exists an $x_0 \in X$ such that $(x_0,\ sx_0) \in A$.

Proof. Use Theorem 5 with $D = X$ as in the proof of Theorem 7.

REMARKS. 1. A number of particular forms of Theorem 9 can be found in [8].

2. For $X = Y$ and $s = 1_X$, particular forms of Theorem 9 were due to Shih and Tan [10], and Bae, Kim, and Tan [2]. These authors applied

Theorem 9 to obtain minimax inequalities of the von Neumann type, results on systems of convex inequalities, and fixed point theorems. Some of these results could be improved in view of Theorem 9.

References

1. J.S. Bae and W.K. Kim, *Variational inequalities of pseudo-monotone maps and fixed point theorems of pseudo-contractive maps,* to appear.
2. J.S. Bae, W.K. Kim, and K.-K. Tan, *Another generalization of Ky Fan's minimax inequality and its applications*, to appear.
3. S.Y. Chang, *A generalization of KKM principle and its applications*, Soochow J. Math. **15** (1989), 7-17.
4. X.P. Ding and K.-K. Tan, *Generalized variational inequalities and generalized quasi-variational inequalities*, J. Math. Anal. Appl. **148** (1990), 497-508.
5. Ky Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519-537.
6. J. Jiang, *Fixed point theorems for convex sets*, Acta Math. Sinica **4** (1988), 356-363.
7. M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151-201.
8. Sehie Park, *Generalized Fan-Browder fixed point theorems and their applications*, Collec. of Papers Dedicated to J.G. Park, 1989, 51-77.
9. ———, *Generalizations of Ky Fan's matching theorems and their applications*, J. Math. Anal. Appl. **141** (1989), 164-176.
10. M.-H. Shih and K.-K. Tan, *A geometric property of convex sets with applications to minimax type inequalities and fixed point theorems*, J. Austral. Math. Soc. **45** (1988), 169-183.

Department of Mathematics
Seoul National University
and
The Mathematical Sciences Research Institute of Korea
Seoul 151-742, Korea