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THE CONTACT CONFORMAL CURVATURE TENSOR FIELD AND THE SPECTRUM OF THE LAPLACIAN*

JIN SUK PAK, JANG CHUN JEONG AND WOON-TAEK KIM

1. Introduction

Let (M,g) be an m-dimensional compact orientable Riemannian manifold with metric tensor g. We denote by Δ the Laplacian acting on p-forms on M. Then we have the spectrum for each p:

$$\mathrm{Spec}^p(M,g) := \{0 \le \lambda_{0,p} \le \lambda_{1,p} \le \lambda_{2,p} \le \cdots \uparrow +\infty\}$$

where each eigenvalue $\lambda_{\alpha,p}$ is repeated as many times as its multiplicity indicates. In order to study the relation between $\mathrm{Spec}^p(M,g)$ and the geometry of (M,g) we use Patodi's results ([6]) on coefficients of the Minakshisundaram-Pleijel-Gaffney's asymptotic expansion.

Recently Olszak ([6]) and Yamaguchi and Chuman ([15]) have studied Sasakian analogues for certain results of [1], [7], [12] and [13]. Especially Yamaguchi and Chuman ([15]) proved the following theorem:

THEOREM Y-C. *Let $\mu = (M,g,\phi,\xi,\eta)$ and $\mu' = (M',g',\phi',\xi',\eta')$ be compact Sasakian manifolds with $Spec^2\mu = Spec^2\mu'$. Then* $\dim M = \dim M' = m$. *For $m = 5,7,9$ and 13, or $17 \le m \le 187$, μ is of constant ϕ-sectional curvature k if and only if μ' is of constant ϕ'-sectional curvature $k' = k$.*

In this paper we improve Theorem Y-C as follows:

THEOREM. *Let $\mu = (M,g,\phi,\xi,\eta)$ and $\mu' = (M',g',\phi',\xi',\eta')$ be compact Sasakian manifolds with $Spec^2\mu = Spec^2\mu'$. Then* $\dim M = \dim M' = m$. *For $m = 3,5,7,9,11$ and 13, or $17 \le m \le 187$, μ is of constant ϕ-sectional curvature k if and only if μ' is of constant ϕ'-sectional curvature $k' = k$.*

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2. Preliminaries

By $R = (R_{kji}{}^h)$, $R_1 = (R_{ji})$ and s we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. For a tensor field T on M we denote by $\|T\|$ the norm of T with respect to g. Then the Minakshisundaram-Pleijel-Gaffney's formula for $\mathrm{Spec}^p(M, g)$ is given by

$$\sum_{\alpha=0}^{\infty} \exp(-\lambda_{\alpha,p} t) \underset{t\to 0+}{\sim} (4\pi t)^{-m/2} \sum_{\alpha=0}^{\infty} a_{\alpha,p} t^{\alpha},$$

where the constants $a_{\alpha,p}$ are spectral invariants. In the present paper we are interested in the case of $p = 2$. For $p = 2$, we have (cf. [7], [14], [15])

$$a_{0,2} = \frac{1}{2} m(m-1)\mathrm{Vol}(M, g), \tag{2.1}$$

$$a_{1,2} = \frac{1}{12}(m^2 - 13m + 24) \int_M s dM, \tag{2.2}$$

$$\begin{aligned} a_{2,2} = \frac{1}{720} \int_M &[2(m^2 - 31m + 240)\|R\|^2 \\ &- 2(m^2 - 181m + 1080)\|R_1\|^2 \\ &+ 5(m^2 - 25m + 120)s^2 \\ &+ A_1(m)s + A_0(m)]dM, \end{aligned} \tag{2.3}$$

where dM denotes the natural volume element of (M, g) and $A_i(m)$ $(i = 0, 1)$ is constant depending only on m.

3. Contact conformal curvature tensor field

Let $\mu = (M, g, \phi, \xi, \eta)$ be an m-dimensional Sasakian manifold (cf. [9]). This means that M is an $m(= 2n + 1)$-dimensional differentiable manifold with a normal contact metric structure (ϕ, g, ξ, η). Thus, $\phi = (\phi_j{}^i)$, $\xi = (\xi^i)$, $\eta = (\eta_j)$ are tensor fields of type (1,1), (1,0), (0,1), respectively, and $g = (g_{ji})$ is a Riemannian metric, on M such that

$$\phi_s{}^i \phi_j{}^s = -\delta_j^i + \eta_j \xi^i,\ \eta_s \xi^s = 1,\ \phi_s{}^i \xi^s = 0,\ \eta_s \phi_i{}^s = 0,$$
$$g_{ts}\phi_j{}^t \phi_i{}^s = g_{ji} - \eta_j \eta_i,\ \eta_i = \xi^s g_{si},$$
$$\nabla_k \phi_j{}^i = \eta_j \delta_k^i - \xi^i g_{kj},\ \nabla_j \eta_i = \phi_{ji},$$

where $\phi_{ji} = \phi_j{}^s g_{si} (= -\phi_{ij})$ and ∇_k denotes the operator of covariant differentiation with respect to the Levi-Civita connection. Then we have (cf. [9])

$$
\begin{aligned}
&R_{kjit}\xi^t = \eta_k g_{ji} - \eta_j g_{ki}, \\
(3.1)\qquad &R_{kjts}\phi_i{}^t\phi_h{}^s = R_{kjih} - g_{kh}g_{ji} + g_{ki}g_{jh} + \phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh}, \\
&\frac{1}{2}R_{tsji}\phi^{ts} = R_{jt}\phi_i{}^t + (2n-1)\phi_{ji} = R_{tjsi}\phi^{ts},
\end{aligned}
$$

$$
\begin{aligned}
(3.2)\qquad &R_{it}\xi^t = 2n\eta_i, \ R_{ts}\phi_j{}^t\phi_i{}^s = R_{ji} - 2n\eta_j\eta_i, \\
&S_{ji} + S_{ij} = 0,
\end{aligned}
$$

where $\phi^{ji} = \phi_t{}^i g^{jt}$, $R_{kjih} = R_{kji}{}^t g_{th}$ and $S_{ji} = \phi_j{}^t R_{ti}$.

Define on M a tensor field $Q = (Q_{ji})$ by

$$Q_{ji} = R_{ji} - \alpha g_{ji} - \beta \eta_j \eta_i,$$

where $\alpha = \frac{s}{2n} - 1$ and $\beta = 2n + 1 - \frac{s}{2n}$. By direct calculations we have

$$(3.3)\qquad \|Q\|^2 = \|R_1\|^2 - \frac{1}{2n}s^2 + 2s - 2n(2n+1)$$

with the help of (3.2). A Sasakian manifold is said to be *η-Einsteinian* if $Q = 0$. For any η-Einstein Sasakian manifold of dimension ≥ 5, s is constant. Any 3-dimensional Sasakian manifold is η-Einsteinian, but in this case s may not be constant.

Define on M a tensor field $H = (H_{kjih})$ by

$$
\begin{aligned}
(3.4)\qquad H_{kjih} = R_{kjih} &- \frac{k+3}{4}(g_{kh}g_{ji} - g_{ki}g_{jh}) \\
&- \frac{k-1}{4}(\phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih} \\
&- g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - \eta_k\eta_h g_{ji} + \eta_k\eta_i g_{jh}),
\end{aligned}
$$

where $k = \frac{s - n(3n+1)}{n(n+1)}$. By using (3.2) and (3.3) we can easily verify that

$$(3.5)\quad \|H\|^2 = \|R\|^2 - \frac{2}{n(n+1)}s^2 + \frac{4(3n+1)}{n+1}s - \frac{4n(2n+1)(3n+1)}{n+1}.$$

A Sasakian manifold μ is said to be *of constant ϕ-sectional curvature* if the sectional curvature $K(X, \phi X)$ is independent of the choice of a vector $X \neq 0$ tangent to M and orthogonal to ξ. A Sasakian manifold of dimension ≥ 5 is of constant ϕ-sectional curvature if and only if $H = 0$ (cf. [5]). In this case $K(X, \phi X) = k$ which is constant for any tangent vector $X(\neq 0)$ orthogonal to ξ. On any 3-dimensional Sasakian manifold the tensor field H vanishes, but in this case the scalar curvature s may not be constant. Therefore, in M of dimension 3 it is of constant ϕ-sectional curvature if and only if $s =$ constant. For the model spaces of Sasakian manifolds with constant ϕ-sectional curvature, see e.g. [10], [11].

We also consider the so-called contact conformal curvature tensor field $C_0 = (C_{0,kjih})$ defined by ([2])

(3.6)

$$\begin{aligned} C_{0,kjih} = R_{kjih} &+ \frac{1}{2n}(g_{kh}R_{ji} - g_{jh}R_{ki} \\ &+ R_{kh}g_{ji} - R_{jh}g_{ki} - R_{kh}\eta_j\eta_i + R_{jh}\eta_k\eta_i \\ &- \eta_k\eta_h R_{ji} + \eta_j\eta_h R_{ki} - \phi_{kh}S_{ji} + \phi_{jh}S_{ki} \\ &- S_{kh}\phi_{ji} + S_{jh}\phi_{ki} + 2\phi_{kj}S_{ih} + 2S_{kj}\phi_{ih}) \\ &+ \frac{1}{2n(n+1)}[2n^2 - n - 2 + \frac{(n+2)s}{2n}](\phi_{kh}\phi_{ji} \\ &- \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) + \frac{1}{2n(n+1)}\times \\ &[n + 2 - \frac{(3n+2)s}{2n}](g_{kh}g_{ji} - g_{jh}g_{ki}) \\ &+ \frac{1}{2n(n+1)}[-(4n^2 + 5n + 2) + \frac{(3n+2)s}{2n}] \\ &\times (g_{kh}\eta_j\eta_i - g_{jh}\eta_k\eta_i + \eta_k\eta_h g_{ji} - \eta_j\eta_h g_{ki}), \end{aligned}$$

where $S_{ji} = \phi_j{}^t R_{ti}$. The tensor field C_0 satisfies, among others, the following identities:

(3.7)

$$\begin{aligned} C_{0,kjih} &= C_{0,ihkj} = -C_{0,jkih} = -C_{0,kjhi}, \\ C_{0,kjih} &+ C_{0,jikh} + C_{0,ikjh} = 0, \\ g^{kh}C_{0,kjih} &= \frac{2(n-2)}{n}R_{ji} + \frac{1}{n}[2(n-2) - \frac{n-2}{n}s]g_{ji} \end{aligned}$$

$$+\frac{1}{n}[-2(2n+1)(n-2)+\frac{n-2}{n}s]\eta_j\eta_i,$$

$$\xi^h C_{0,kjih}=0, \phi^{kj}C_{0,kjih}=0,\ \phi^{kh}C_{0,kjih}=0.$$

Using those identities and (3.1), (3.2), (3.5), (3.6), we can check that

(3.8)

$$\begin{aligned}\|C_0\|^2 &= \|R\|^2-\frac{8}{n^2}\|R_1\|^2+\frac{2(-n^2+2n+2)}{n^3(n+1)}s^2\\ &+\frac{4(3n^3+n^2-4n-4)}{n^2(n+1)}s+\frac{1}{n(n+1)}(-24n^4\\ &-20n^3+28n^2+48n+16),\end{aligned}$$

$$\|C_0\|^2=\|H\|^2-\frac{8}{n^2}\|Q\|^2. \tag{3.9}$$

By the way it is already shown in [2] that $C_0=0$ if and only if $H=0$, provided $2n+1>5$. In this case it holds that $Q=0$.

The contact Bochner curvature tensor field $B=(B_{kjih})$ is defined on M by (cf. [3])

(3.10)

$$\begin{aligned}B_{kjih} &= R_{kjih}-\frac{1}{2n+4}(g_{kh}R_{ji}-g_{ki}R_{jh}+g_{ji}R_{kh}\\ &-g_{jh}R_{ki}+\phi_{kh}S_{ji}-\phi_{ki}S_{jh}+\phi_{ji}S_{kh}\\ &-\phi_{jh}S_{ki}-2\phi_{kj}S_{ih}-2\phi_{ih}S_{kj}-R_{kh}\eta_j\eta_i\\ &+R_{ki}\eta_j\eta_h-R_{ji}\eta_k\eta_h+R_{jh}\eta_k\eta_i)\\ &+\frac{k-4}{2n+4}(g_{kh}g_{ji}-g_{ki}g_{jh})+\frac{k+2n}{2n+4}(\phi_{kh}\phi_{ji}\\ &-\phi_{ki}\phi_{jh}-2\phi_{kj}\phi_{ih})-\frac{k}{2n+4}(g_{kh}\eta_j\eta_i\\ &-g_{ki}\eta_j\eta_h+g_{ji}\eta_k\eta_h-g_{jh}\eta_k\eta_i),\end{aligned}$$

where $k=\frac{s+2n}{2n+2}$. The tensor field B satisfies, among others, the following identities (cf. [3]):

$$B_{kjih}=-B_{jkih}=B_{ihkj},\ g^{ts}B_{tjis}=0,\ B_{kjih}\xi^h=0,$$

$$\phi^{ts}B_{tjis}=0,\ \phi^{ts}B_{tsih}=0.$$

Using these identities and (3.1), (3.2), (3.10) we can easily check (cf. [6], [15]) that

$$\begin{aligned}(3.11)\qquad \|B\|^2 = \|R\|^2 - \frac{8}{n+2}\|R_1\|^2 \\ + \frac{2}{(n+1)(n+2)}s^2 + \frac{4(3n^2+3n-2)}{(n+1)(n+2)}s \\ - 24n^2 + 36n - 56 + \frac{8(13n+14)}{(n+1)(n+2)}.\end{aligned}$$

Moreover, it may be easily seen that $H = 0$ if and only if $B = 0$ and $Q = 0$.

It is also already shown in [2] that $C_0 = B$ if and only if (M, g) is η-Einsteinian.

4. Spec$^2\mu$ and the geometry of μ

Assume that μ is a compact Sasakian manifold of dimension $m(= 2n+1)$ and consider Spec$^2\mu$. With the help of (3.3) and (3.8) the coefficient $a_{2,2}$, given by (2.3), may be written as follow:

$$\begin{aligned}(4.1)\qquad a_{2,2} = \frac{1}{720}\int_M [2(n-7)(4n-30)\|C_0\|^2 \\ + \frac{2}{n^2}(-4n^4 + 358n^3 - 868n^2 - 464n \\ + 1680)\|Q\|^2 + C(n)s^2 + D(n)s + A(n)]dM,\end{aligned}$$

where $C(n)$, $D(n)$, $A(n)$ are constants depending only on n, and $C(n) > 0$.

We shall often use the following lemma, which is a consequence of the Schwarz inequality (cf. [12], p.394).

LEMMA. *Let (M, g) and (M', g') be compact orientable Riemannian manifolds with Vol(M) = Vol(M') and $\int_M s dM = \int_{M'} s' dM'$. If $s' =$ const., then $\int_M s^2 dM \geq \int_{M'} s'^2 dM'$ with equality if and only if $s =$ const. $= s'$.*

From now on we assume that $\mu = (M, g, \phi, \xi, \eta)$ and $\mu' = (M', g', \xi', \eta')$ are m-dimensional compact Sasakian manifolds with Spec$^2\mu$ =

$\mathrm{Spec}^2\mu'$. Then $a_{0,2} = a'_{0,2}$ and $a_{1,2} = a'_{1,2}$ imply $\mathrm{Vol}(\mu) = \mathrm{Vol}(\mu')$ and $\int_M s dM = \int_{M'} s' dM'$ respectively. Therefore, $a_{2,2} = a'_{2,2}$ gives, in view of (4.1),

$$\begin{aligned}(4.2)\qquad &\int_M [2(n-7)(4n-30)\|C_0\|^2 + \frac{2}{n^2}(-4n^4 + 358n^3 \\ &- 868n^2 - 464n + 1680)\|Q\|^2 + C(n)s^2]dM \\ &= \int_{M'} [2(n-7)(4n-30)\|C'_0\|^2 \\ &+ \frac{2}{n^2}(-4n^4 + 358n^3 - 868n^2 - 464n \\ &+ 1680)\|Q'\|^2 + C(n)s'^2]dM'.\end{aligned}$$

On the other hand, as already mentioned in section 3, $H = 0$ implies $C_0 = 0$ and $Q = 0$. Therefore, (4.2) and lemma yield that, for $m = 2n+1 = 3,\ 11$, μ is of constant ϕ-sectional curvature k if and only if μ' is of constant ϕ'-sectional curvature $k' = k$. Combining with Theorem Y-C, we have the main theorem stated in section 1.

REMARK. Let (S^{2n+1}, g_0) be a Euclidean $(2n+1)$-dimensional sphere with constant curvature 1. We denote by $\varphi = (S^{2n+1}, g_0, \phi_0, \xi_0, \eta_0)$ the standard Sasakian structure on S^{2n+1}. Using our main theorem and Takahashi's Theorem 2 in [10], we can deduce the following characterization of sphere among Sasakian manifolds.

COROLLARY. *Let μ be a compact Sasakian manifold of dimension $m = 3, 5, 7, 9, 11$ and 13, or $17 \le m \le 187$. If $\mathrm{Spec}^2\mu = \mathrm{Spec}^2\varphi$, then μ is isometric to φ, that is, there is an isometry $F : (M, g) \to (S^{2n+1}, g_0)$ such that $F_*\xi = \xi_0$, $F^*\eta_0 = \eta$ and $F_* \cdot \phi = \phi_0 \cdot F_*$.*

References

1. M. Berger, P. Gauduchon et E. Mazet, *Le spectre d'une variètè Riemanniene*, Lecture Notes in Mathematics **194**, Springer-Verlag, 1971.
2. J. C. Jeong, J. D. Lee, G. H. Oh and J. S. Park, *On the contact conformal curvature tensor*, preprint.
3. M. Matsumoto and G. Chuman, *On the C-Bochner curvature tensor*, **5** (1969), 21–30.
4. H. F. Mckean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geometry, **1** (1967), 43–69.

5. K. Ogiue, *On almost contact manifolds admitting exiom of planes or axiom of free mobility*, Kodai Math. Sem. Rep., **16** (1964), 223–232.
6. Z. Olszak, *The spectrum of the Laplacian and the curvature of Sasakian manifolds*, Lecture Notes in Mathematics 838, Springer-Verlag, 1981, 221–228.
7. V. K. Patodi, *Curvature and the fundamental solution of the heat operator*, J. Indian Math. Soc., —bf 34 (1970), 269–285.
8. ———, *Curvature and the eigenforms of the Laplacian operator*, J. Diff. Geometry, **5** (1971), 233–249.
9. S. Sasaki, *Almost contact manifolds*, Lecture Notes, I, II, Tohoku Univ., 1965.
10. T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Tohoku Math. J., **21** (1969), 271–290.
11. S. Tanno, *Sasakian manifolds with constant ϕ-holomorphic sectional curvature*, Tohoku Math. J., **21** (1969), 501–507.
12. ———, *Eingenvalues of the Laplacian of Riemannian manifolds*, Tohoku Math. J., **21** (1973), 391–403.
13. ———, *The spectrum of the Laplacian for 1-forms*, Proc. Amer. Math. Soc., **45** (1974), 125–129.
14. Gr. Tsagas, *On the spectrum of the Laplace operator for the exterior 2-forms*, Tensor N. S., **33** (1979), 94–96.
15. S. Yamaguchi and G. Chuman, *Eingenvalues of the Laplacian of Sasakian manifolds*, TRU Math. **15** (1979), 31–41.

Department of Mathematical Education
Kyungpook National University
Taegu 702–701, Korea

Yeong Jin Junior College
Taegu 702-020, Korea

Department of Mathematics
Taegu University
Taegu 705–714, Korea