J. Korean Math. Soc. 28(1991), No. 2, pp. 245–265

FUNCTIONS IN THE FRESNEL CLASS OF AN ABSTRACT WIENER SPACE

J.M. AHN*, G.W. JOHNSON AND D.L. SKOUG

0. Introduction

Abstract Wiener spaces have been of interest since the work of Gross in the early 1960's (see [18]) and are currently being used as a framework for the Malliavin calculus (see the first half of [20]) and in the study of the Fresnel and Feynman integrals. It is this last topic which concerns us in this paper.

The Feynman integral arose in nonrelativistic quantum mechanics and has been studied, with a variety of different definitions, by mathematicians and, in recent years, an increasing number of theoretical physicists. The Fresnel integral has been defined in Hilbert space [1], classical Wiener space [2], and abstract Wiener space [13] settings and used as an approach to the Feynman integral. Such approaches have their shortcomings; for example, the class of functions that can be dealt with is somewhat limited. However, these approaches also have some distinct advantages including the fact, seen most clearly in [14], that several different definitions of the Feynman integral exist and agree.

A key hypothesis in most of the theorems about the Fresnel and Feynman integrals is that the functions involved are in the 'Fresnel class' of the Hilbert space being used. Theorem 2 below is a general result insuring membership in $\mathcal{F}(B)$, the Fresnel class of an abstract Wiener space B. $\mathcal{F}(B)$ consists of all 'stochastic Fourier transforms' of certain Borel measures of finite variation. (Precise definitions will be given in Section 1 below.)

A surprising number of consequences follow from Theorem 2; indeed, the rest of this paper after Theorem 2 consists of eleven corollaries. Corollaries 3 through 7 are in the setting of classical Wiener space and show, in conjunction with Corollary 1, that various functions of

Received August 23, 1990.
*This research was supported by the Ministry of Education, Korea (1989).

interest in quantum mechanics, are in the Fresnel class of the classical abstract Wiener space. We should mention that all of the corollaries of [6] could have been included in this paper although they are not.

Corollaries 9, 10, and 11 in Section 4 illustrate how Theorem 2 can be applied to abstract Wiener spaces and associated Gaussian stochastic processes other than the classical Wiener process. The processes involved are, respectively, the n parameter Wiener process, the Brownian bridge, and a family of processes described in [20, pp.7–9]. We include, without proof, the necessary background information concerning these processes. Corollary 8, the first corollary of Section 4, is in the setting of the classical abstract Wiener space but is very different from the other corollaries in this paper and from all of the corollaries in the earlier work [4;6].

We close this introduction by remarking that the results of [4] which are in the Hilbert space setting of [1] are less directly comparable to the present paper than are the results of [6]. The Hilbert space and Wiener space settings are closely related however as was shown in [10] and [13] in the classical and abstract Wiener space settings, respectively.

1. Definitions and preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let μ be the cylinder set measure on H defined by

$$\mu(A) = (2\pi)^{-n/2} \int_E \exp\left\{-\frac{|x|^2}{2}\right\} dx,$$

where $A = P^{-1}(E)$, E is a Borel set in the image of an n-dimensional projection P in H and dx is Lebesgue measure in $P(H)$. A norm $\|\cdot\|_0$ on H is called measurable if for every $\epsilon > 0$ there exists a finite dimensional projection P_0 such that $\mu(\{x \in H : \|Px\|_0 > \epsilon\}) < \epsilon$ whenever P is a finite dimensional projection orthogonal to P_0. It is known (see [18]) that H is not complete with respect to $\|\cdot\|_0$. Let B denote the completion of H with respect to $\|\cdot\|_0$. Let i denote the natural injection from H into B. The adjoint operator i^* is one-to-one and maps B^* continuously onto a dense subset of H^*. By identifying H^* with H and B^* with i^*B^*, we have a triple $B^* \subset H^* \equiv H \subset B$ and $\langle x, y\rangle = (x, y)$ for all x in H any y in B^*, where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. By a well-known result of Gross, $\mu \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (B, H, ν) is called an abstract Wiener space, and the Hilbert space H is called the generator of (B, H, ν). Given an arbitrary real separable Banach space B with norm $\|\cdot\|_0$, and a countably additive Gaussian measure ν on $(B, \mathcal{B}(B))$, it is known that there exists a Hilbert space H such that $H \subseteq B$, B is the completion of H under $\|\cdot\|_0$, the norm $\|\cdot\|_0$ is a measurable norm on H with respect to the canonical normal distribution μ and $\nu = \mu \circ i^{-1}$. For more details, see Kuo [18].

Let $\{e_j\}$ denote a complete orthonormal system on H such that the e_j's are in B^*. For each $h \in H$ and $x \in B$, define a stochastic inner product $(\cdot,\cdot)^{\sim}$ between B and H as follows:

$$(x, h)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

It is well-known that for every $h \in H$, $(x, h)^{\sim}$ exists for ν-almost all x in B and is a Borel measurable function on B having a Gaussian distribution with mean zero and variance $|h|^2$. Furthermore, it is easy to show that $(x, h)^{\sim}$ equals (x, h) almost everywhere on B if $h \in B^*$. Note that if both h and x are in H, then $(x, h)^{\sim} = \langle x, h\rangle$.

Let $M(H)$ denote the space of complex-valued countably additive measures defined on $\mathcal{B}(H)$, the Borel class of H. A complex-valued countably additive measure μ necessarily has finite total variation $\|\mu\|$ [19, p.119]. Under the norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [1, p.17].

Given two complex-valued functions F and G on B, we say that $F = G$ s-almost surely ($F \approx G$) if for each $\alpha > 0$, $\nu(\{x \in B \mid F(\alpha x) \neq G(\alpha x)\}) = 0$, where ν is an abstract Wiener measure. For a function F on B, we will denote by $[F]$ the equivalence class of functions G which are equal to F s-a.e., i.e., $[F] = \{G \mid G = F\ s\text{-a.s.}\} = \{G \mid G \approx F\}$.

We will now introduce the Fresnel class $\mathcal{F}(B)$ of functions on B. The space $\mathcal{F}(B)$ is defined as the space of all stochastic Fourier transforms of elements of $M(H)$, i.e., $\mathcal{F}(B) = \{[F] \mid F(x) = \int_H \exp(i(x, h)^{\sim}) d\mu(h)$, $x \in B$, $\mu \in M(H)\}$. As is customary, we will often identify a function with its s-equivalence class and think of $\mathcal{F}(B)$ as a collection of functions on B rather than as a collection of equivalence classes.

It is well-known (see [13, 14]) that $\mathcal{F}(B)$ is a Banach algebra with the norm $\|F\| = \|\mu\|$ and the mapping $\mu \mapsto F$ is a Banach algebra isomorphism, where μ is related to F by

$$F(x) = \int_H \exp\{i(x, \tilde{h})\} d\mu(h), \ x \in B, \ \mu \in M(H).$$

Although it will not be a direct concern of this paper, we note that for each F in $\mathcal{F}(B)$, the analytic Feynman integral, the sequential Feynman integral and the Fresnel integral of F all exist [14] and are given by $\int_H \exp\{-\frac{i}{2}|h|^2\} d\mu(h)$.

2. Functions in $\mathcal{F}(B)$: a general theorem

We begin by quoting an Unsymmetric Fubini Theorem from [4, pp.310–311] which we use to prove our main theorem.

THEOREM 1. *Let $(Y, \mathcal{Y}, \eta)$ be a measure space, where η is either a nonnegative, σ-finite measure or a complex-valued measure, and let $(Z, \mathcal{Z})$ be a measurable space. For $|\eta|$-a.e. y, let ν_y be a complex-valued measure on $(Z, \mathcal{Z})$. Suppose that for any E in $\mathcal{Z}$, $\nu_y(E)$ is $\mathcal{Y}$-measurable function of y. Then*

A. *for any E in $\mathcal{Y} \times \mathcal{Z}$, $\nu_y(E^{(y)})$ is $\mathcal{Y}$-measurable where $E^{(y)}$ is the y-section of E, and*
B. *for any bounded, complex-valued, $\mathcal{Y} \times \mathcal{Z}$-measurable function θ on $Y \times Z$, $\int_Z \theta(y, z) d\nu_y(z)$ is a $\mathcal{Y}$-measurable functions of y.*

If we add the assumption that $\|\nu_y\| \leq h(y)$, where h is in $L^1(Y, \mathcal{Y}, |\eta|)$, and define ρ on $\mathcal{Y} \times \mathcal{Z}$ by the formula

$$\rho(E) := \int_Y \nu_y(E^{(y)}) d\eta(y), \tag{2.1}$$

then

C. *ρ is a complex-valued, countably additive measure on $\mathcal{Y} \times \mathcal{Z}$ with $\|\rho\| \leq \|h\|_1$, and*
D. *if $\theta(y, z)$ is bounded and $\mathcal{Y} \times \mathcal{Z}$-measurable, then $\int_Z \theta(y, z) d\nu_y(z)$ is in $L^1(Y, \mathcal{Y}, |\eta|)$, and we have*

$$\int_Y \left[\int_Z \theta(y,z) d\nu_y(z) \right] d\eta(y) = \int_{Y \times Z} \theta(y,z) d\rho(y,z). \tag{2.2}$$

We are now ready to establish the general theorem discussed in the introduction. This theorem shows (as we will see in the next section) that $\mathcal{F}(B)$ contains many functions of interest in connection with Feynman integration theory and quantum mechanics.

THEOREM 2. (1) *Let (B, H, ν) be an abstract Wiener space.*

(2) *Let $(Y, \mathcal{Y}, \eta)$ be a measure space, where η is either a nonnegative, σ-finite measure or a complex-valued measure.*

(3) *Let $\phi_j : Y \to H$ be $\mathcal{Y} - \mathcal{B}(H)$-measurable for $j = 1, 2, \cdots, d$.*

(4) *Let $\theta : Y \times \mathbf{R}^d \to \mathbf{C}$ be given by $\theta(y; \cdot) = \hat{\nu}_y(\cdot)$, where ν_y is in $M(\mathbf{R}^d)$ for every y in Y, and where the family $\{\nu_y : y \in Y\}$ satisfies (i) $\nu_y(E)$ is a $\mathcal{Y}$-measurable function of y for every E in $\mathcal{B}(\mathbf{R}^d)$, and (ii) $\|\nu_y\|$ is in $L^1(Y, \mathcal{Y}, |\eta|)$.*

Under these hypotheses, $f : B \to \mathbf{C}$ given by

$$f(x) := \int_Y \theta(y; ((x, \phi_1(y))\tilde{}, \cdots, (x, \phi_d(y))\tilde{})) d\eta(y) \tag{2.3}$$

belongs to $\mathcal{F}(B)$ and satisfies the inequality

$$\|f\| \le \int_Y \|\nu_y\| d|\eta|(y). \tag{2.4}$$

Proof. Using techniques similar to those used in [4] we can show that $\|\nu_y\|$ is measurable as a function of y, that θ is $\mathcal{Y} \times \mathcal{B}(\mathbf{R}^d)$-measurable, and that the integrand in equation (2.3) is a measurable function of y for every x in B.

To show that f given by (2.3) belongs to $\mathcal{F}(B)$ we need to show that there exists a measure μ in $M(H)$ such that for each $\alpha > 0$,

$$\tilde{\mu}(\alpha x) := \int_H \exp\{i(\alpha x, h)\tilde{}\} d\mu(h)$$

equals $f(\alpha x)$ for a.e. x in B. First, we use Theorem 1 to define a measure ρ on $\mathcal{Y} \times \mathcal{B}(\mathbf{R}^d)$ by letting

$$\rho(E) := \int_Y \nu_y(E^{(y)}) d\eta(y)$$

for each set E in $\mathcal{Y} \times \mathcal{B}(\mathbf{R}^d)$. Note that, by C of theorem 1, ρ satisfies

$$\|\rho\| \leq \int_Y \|\nu_y\| d|\eta|(y). \tag{2.5}$$

Next we define $\Phi : Y \times \mathbf{R}^d \to H$ by

$$\Phi(y; (v_1, \cdots, v_d)) := v_1\phi_1(y) + \cdots + v_d\phi_d(y).$$

Then, using hypothesis (3), we see that Φ is $\mathcal{Y} \times \mathcal{B}(\mathbf{R}^d) - \mathcal{B}(H)$-measurable. Finally we define μ as the image measure $\mu := \rho \circ \Phi^{-1}$. Then clearly μ is in $M(H)$ and satisfies $\|\mu\| \leq \|\rho\|$. Hence (2.4) will follow immediately from (2.5) once we show that $f \approx \tilde{\mu}$.

It remains only to show that for every $\alpha > 0$, $\tilde{\mu}(\alpha x) = f(\alpha x)$ for a.e. x in B. But using the Change of Variables Theorem and equation (2.2) above, we see that for a.e. $x \in B$,

(2.6)

$$\begin{aligned}
\tilde{\mu}(\alpha x) &= \int_H \exp\{i(\alpha x, h)\tilde{}\} d\mu(h) \\
&= \int_H \exp\{i\alpha(x, h)\tilde{}\} d(\rho \circ \Phi^{-1})(h) \\
&= \int_{Y \times \mathbf{R}^d} \exp\{i\alpha(x, \Phi(y; (v_1, \cdots, v_d)))\tilde{}\} d\rho(y, (v_1, \cdots, v_d)) \\
&= \int_{Y \times \mathbf{R}^d} \exp\{i\alpha(x, v_1\phi_1(y) + \cdots + v_d\phi_d(y))\tilde{}\} d\rho(y, (v_1, \cdots, v_d)) \\
&= \int_{Y \times \mathbf{R}^d} \exp\{i\alpha\langle(v_1, \cdots, v_d), ((x, \phi_1(y))\tilde{}, \cdots, (x, \phi_d(y))\tilde{})\rangle\} \\
&\qquad\qquad d\rho(y, (v_1, \cdots, v_d)) \\
&= \int_Y \Big\{ \int_{\mathbf{R}^d} \exp\{i\alpha\langle(v_1, \cdots, v_d), ((x, \phi_1(y))\tilde{}, \cdots, (x, \phi_d(y))\tilde{})\rangle\} \\
&\qquad\qquad d\nu_y(v_1, \cdots, v_d)\Big\} d\eta(y) \\
&= \int_Y \hat{\nu}_y((\alpha x, \phi_1(y))\tilde{}, \cdots, (\alpha x, \phi_d(y))\tilde{}) d\eta(y) \\
&= \int_Y \theta(y; ((\alpha x, \phi_1(y))\tilde{}, \cdots, (\alpha x, \phi_d(y))\tilde{})) d\eta(y) \\
&= f(\alpha x).
\end{aligned}$$

Thus f is in $\mathcal{F}(B)$ which completes the proof of Theorem 2.

3. Corollaries of Theorem 2

The corollaries and remarks in this section and the next show that Theorem 2 has many consequences. Our first result comes from the fact that $\mathcal{F}(B)$ is a Banach algebra. This corollary is relevant to quantum mechanics where exponential functions play a prominent role. In fact, in several of the later corollaries, it is the exponential of the function considered which is physically relevant. In light of Corollary 1, it will not be necessary to mention that every time.

COROLLARY 1. *Let f be as in Theorem 2 and let g be an entire function on $\mathbf{C}$. Then $(g \circ f)(x)$ is in $\mathcal{F}(B)$. In particular,* $\exp\{f(x)\}$ *is in $\mathcal{F}(B)$.*

COROLLARY 2. *Let (B, H, ν) be an abstract Wiener space and let $\{h_1, \cdots, h_d\}$ be a finite subset of H. Given $\psi = \hat{\mu}$ where μ is in $M(\mathbf{R}^d)$, define $f_1 : B \to \mathbf{C}$ by*

$$f_1(x) := \psi((x, h_1)\tilde{}, \cdots, (x, h_d)\tilde{}). \tag{3.1}$$

Then f_1 belongs to $\mathcal{F}(B)$.

Proof. Let $(Y, \mathcal{Y}, \eta)$ be a probability space and for $i = 1, \cdots, d$, let $\phi_i(y) \equiv h_i$. Take $\theta(y; \cdot) = \psi(\cdot) = \hat{\mu}(\cdot)$. then for all $\alpha > 0$ and for a.e. $x \in B$,

$$\begin{aligned}
&\int_Y \theta(y; ((\alpha x, \phi_1(y))\tilde{}, \cdots, (\alpha x, \phi_d(y))\tilde{}))d\eta(y) \\
&= \int_Y \psi((\alpha x, \phi_1(y))\tilde{}, \cdots, (\alpha x, \phi_d(y))\tilde{})d\eta(y) \\
&= \int_Y \psi(\alpha((x, \phi_1(y))\tilde{}, \cdots, (x, \phi_d(y))\tilde{}))d\eta(y) \\
&= \psi(\alpha((x, h_1)\tilde{}, \cdots, (x, h_d)\tilde{})) \\
&= f_1(\alpha x).
\end{aligned}$$

Hence f_1 belongs to $\mathcal{F}(B)$.

The remaining corollaries in this section and the first corollary in the next will all be in the setting of classical abstract Wiener space. It is this space (and variations of it) that has been used in the applications of the Fresnel integral to nonrelativistic quantum mechanics. The case of multiple space dimensions can be dealt with as we will illustrate in the last result of this section. However, we concentrate, for the sake of simplicity, on the case of one space dimension, and we now introduce the appropriate abstract Wiener space (B_0, H_0, ν_0).

Fix $t > 0$ and let $H_0 = H_0[0,t]$ be the space of real-valued functions γ on $[0,t]$ which are absolutely continuous, which vanish at 0, and whose derivative $D\gamma$ is in $L^2[0,t]$. The inner product on H_0 is given by

$$\langle \gamma, \beta \rangle_{H_0} = \int_0^t (D\gamma)(s)(D\beta)(s)ds. \tag{3.2}$$

It is well known that H_0 is a separable Hilbert space over $\mathbf{R}$.

It will be helpful to introduce the family of functions $\{\gamma_\tau \,|\, 0 \le \tau \le t\}$ from H_0:

$$\gamma_\tau(s) = \begin{cases} s, & 0 \le s \le \tau \\ \tau, & \tau \le s \le t. \end{cases} \tag{3.3}$$

These functions have the reproducing property,

$$\langle \gamma, \gamma_\tau \rangle_{H_0} = \gamma(\tau) \text{ for all } \gamma \in H_0. \tag{3.4}$$

If we let B_0 be the space $C_0[0,t]$ of all continuous functions x on $[0,t]$ which vanish at 0 and equip B_0 with the sup norm, then (B_0, H_0, ν_0) is well known [18] to be an abstract Wiener space where ν_0 is the classical Wiener measure. Note that $\gamma_\tau(s) = \min(s,\tau)$, the covariance function associated with the classical Wiener process. In fact, here and in all of the corollaries to follow, the generating Hilbert space is the reproducing kernel Hilbert space, RKHS, of the process. See [12] for a discussion of the role of RKHS's in the study of Gaussian processes. We mention that in [12], the parameter space of a RKHS is allowed to be a complete separable metric space. However, in our corollaries, the parameter space will be either $[0,t]$ or $[0,t]^n$.

Before proceeding to the corollaries, we mention that *all* the results of [6] are implied by Theorem 2. This may not be immediately transparent to the reader because the emphasis in [6] was on several space dimensions rather than on one and also because we have chosen not to give all the corollaries in [6] in the present paper. More importantly, the Hilbert space $L^2[0,t]$ was used in [6] rather than H_0. However, when one takes into account the fact that the differentiation map D is a unitary operator identifying the Hilbert spaces H_0 and $L^2[0,t]$, one see that this difference is just a matter of appearance. We should also mention that the notation $S(1)$ was used in [6] rather than $\mathcal{F}(B_0)$.

REMARK. When H is a RKHS, the reproducing property $\langle\gamma,\gamma_\tau\rangle_H = \gamma(\tau)$ often carries over to the 'stochastic inner product' in the sense that

$$(x,\gamma_\tau)^{\sim} = x(\tau) \text{ for } s-\text{a.e. } x \in B. \tag{3.5}$$

Even in cases where (3.5) fails (or is not known) to hold, it is reasonable to regard $(x,\gamma_\tau)^{\sim}$ as a replacement for the point evaluation $x(\tau)$.

In the quantum mechanical setting, the function ψ in the corollary immediately below is interpreted as the initial state of the quantum system.

COROLLARY 3. *Let $\psi = \hat{\mu}$ where μ is in $M(\mathbf{R})$ and define $f_2 : B_0 \to \mathbf{C}$ by the formula*

$$f_2(x) := \psi((x,\gamma_t)^{\sim}) = \psi(x(t)) \tag{3.6}$$

where γ_t is given by (3.3). Then f_2 belongs to $\mathcal{F}(B_0)$.

Proof. Apply Theorem 2 after making the following choices: $B = B_0$, $(Y,\mathcal{Y},\eta) = ([0,t],\mathcal{B}([0,t]),\eta)$ where η is any probability measure, $d=1$, $\phi_1(\tau) \equiv \gamma_t$ for $0 \le \tau \le t$ and $\theta(\tau;\cdot) = \hat{\mu}(\cdot)$ for $0 \le \tau \le t$. With these choices, the right-hand side of (2.3) becomes

$$\int_0^t \hat{\mu}((x,\gamma_t)^{\sim})d\eta(\tau) = \psi((x,\gamma_t)^{\sim}) = \psi(x(t)) = f_2(x)$$

and so we have the desired result.

COROLLARY 4. *Let $\theta = \hat{\mu}$ where μ is in $M(\mathbf{R})$ and define $f_3 : B_0 \to \mathbf{C}$ by the formula*

$$f_3(x) := \int_0^t \theta((x, \gamma_\tau)\tilde{})d\tau = \int_0^t \theta(x(\tau))d\tau \tag{3.7}$$

where γ_τ is given by (3.3). Then f_3 belongs to $\mathcal{F}(B_0)$.

Proof. Take $B, Y, \mathcal{Y}$, and d as in the proof of Corollary 3. Let $\theta = \hat{\mu}$ and let η be Lebesgue measure on $[0, t]$. Finally, let $\phi_1(\tau) = \gamma_\tau$ for $0 \leq \tau \leq t$. Then the right-hand side of (2.3) becomes

$$\int_0^t \theta(\tau; (x, \phi_1(\tau))\tilde{})d\tau = \int_0^t \theta((x, \gamma_\tau)\tilde{})d\tau = f_3(x),$$

and so we have the desired result.

The function θ in Corollary 4 is interpreted as the potential energy in the quantum mechanical setting. The most common function of concern in nonrelativistic quantum mechanics sends x in B_0 to

$$\exp\{\int_0^t \theta(x(\tau))d\tau\}\psi(x(t)).$$

Corollaries 3 and 4 combine with the fact that $\mathcal{F}(B_0)$ is a Banach algebra to give us conditions on θ and ψ under which the function above is in $\mathcal{F}(B_0)$.

In Corollary 4, the potential is time-independent. However, time-dependent potentials are also of interest and are the subject of the next corollary. In addition, we replace Lebesgue measure on $[0, t]$ with a Borel measure η.

COROLLARY 5. *Let η be a Borel measure on $[0, t]$ and let $\theta : [0, t] \times \mathbf{R} \to \mathbf{C}$ be given by $\theta(\tau, \cdot) = \hat{\mu}_\tau(\cdot)$ where μ_τ is in $M(\mathbf{R})$ for every τ in $[0, t]$ and where the family $\{\mu_\tau \,|\, 0 \leq \tau \leq t\}$ satisfies:* (i) *$\mu_\tau(E)$ is a Borel measurable function of τ for every E in $\mathcal{B}(\mathbf{R})$, and* (ii) *$\|\mu_\tau\|$ is integrable over $[0, t]$ with respect to the measure η. Define $f_4 : B_0 \to \mathbf{C}$ by*

$$f_4(x) := \int_0^t \theta(\tau; x(\tau))d\eta(\tau). \tag{3.8}$$

Then f_4 belongs to $\mathcal{F}(B_0)$.

Proof. Take $B = B_0$, $Y = [0,t]$, $\mathcal{Y} = \mathcal{B}([0,t])$ and $d = 1$ as above, and let $\phi_1(\tau) = \gamma_\tau$; let η and θ be as in the statement of this corollary. Now apply Theorem 2 as usual.

REMARK. Of course, Lebesgue measure on $[0,t]$ is one of the choices of η in Corollary 5.

'Potentials' involving a double (or multiple) dependence on time are of interest. One way such potentials arise is in interacting systems when the coordinates associated with one particle are integrated out. Indeed, this possibility was discussed in Feynman's original paper [8, Section 13]. Situations involving more than one parameter seem to be appearing with increasing frequency in the recent physical literature. See [15] for an example of this and for further references. Our next corollary is in the setting of n 'time' parameters.

COROLLARY 6. *Let $\theta : [0,t]^n \times \mathbf{R}^n \to \mathbf{C}$ be given by $\theta((\tau_1,\cdots,\tau_n); (\cdot,\cdots,\cdot)) = \hat{\mu}_{(\tau_1,\cdots,\tau_n)}(\cdot,\cdots,\cdot)$, where $\mu_{(\tau_1,\cdots,\tau_n)}$ is in $M(\mathbf{R}^n)$ for every $(\tau_1,\cdots,\tau_n)$ in $[0,t]^n$ and where the family $\{\mu_{(\tau_1,\cdots,\tau_n)} \mid (\tau_1,\cdots,\tau_n) \in [0,t]^n\}$ satisfies:* (i) *$\mu_{(\tau_1,\cdots,\tau_n)}(E)$ is a Borel measurable function of $(\tau_1,\cdots,\tau_n)$ for every E in $\mathcal{B}(\mathbf{R}^n)$, and* (ii) *$\|\mu_{(\tau_1,\cdots,\tau_n)}\|$ is integrable over $[0,t]^n$ with respect to $|\eta|$ where η is a Borel measure on $[0,t]^n$. Define $f_5 : B_0 \to \mathbf{C}$ by*

(3.9)
$$f_5(x) := \int_0^t \cdots \int_0^t \theta((\tau_1,\cdots,\tau_n);(x(\tau_1),\cdots,x(\tau_n))) \, d\eta(\tau_1,\cdots,\tau_n).$$

Then f_5 belongs to $\mathcal{F}(B_0)$.

Proof. Apply Theorem 2 with $Y = [0,t]^n$, $\mathcal{Y} = \mathcal{B}([0,t]^n)$, $d = n$, $\phi_j(\tau_1,\cdots,\tau_n) = \gamma_{\tau_j}$ for $j = 1,\cdots,n$. The other choices are clear.

In our final corollary in this section, we illustrate how the case of multiple space dimensions can be dealt with. Let n be a positive integer great than 1. Here we will take $B = B_0^n = C_0^n[0,t]$ and let ν be the product, ν_0^n, of n copies of Wiener measure. Thus (B,ν) is the classical

Wiener space of paths in $\mathbf{R}^n$. The Hilbert space H will be taken as H_0^n, the product of n copies of H_0, with the inner product given by

$$(3.10)\qquad \langle(\gamma_1,\cdots,\gamma_n),(\beta_1,\cdots,\beta_n)\rangle_{H_0^n}=\sum_{j=1}^{n}\langle\gamma_j,\beta_j\rangle_{H_0}.$$

(B_0^n, H_0^n, ν_0^n) is an abstract Wiener space; in fact, it is the classical abstract Wiener space in n dimensions. It turns out that the stochastic inner product acting on $x=(x_1,\cdots,x_n)$ in B_0^n and $\gamma=(\gamma_1,\cdots,\gamma_n)$ in H_0^n is given by

$$(3.11)\qquad (x,\gamma)^{\sim}=\sum_{j=1}^{n}(x_j,\gamma_j)^{\sim}.$$

The corollary to follow is an n space dimension version of Corollary 4.

COROLLARY 7. *Let (B_0^n, H_0^n, ν_0^n) be as above and take $\theta=\hat{\mu}$ where μ is in $M(\mathbf{R}^n)$. Then f_6 defined by the formula*

$$(3.12)\qquad f_6(x):=\int_0^t \theta((x_1(\tau),\cdots,x_n(\tau))d\tau$$

belongs to $\mathcal{F}(B_0^n)$.

Proof. We take $B=B_0^n$, $H=H_0^n$, $\nu=\nu_0^n$, $Y=[0,t]$, and $\mathcal{Y}=\mathcal{B}([0,t])$. Further, let η be Lebesgue measure and $\theta(\tau,\cdot)=\hat{\mu}(\cdot)$. Finally, take $d=n$ and let $\phi_j(\tau)$ $(j=1,\cdots,n)$ be the function from $[0,t]$ to H_0^n which is 0 except in the jth component where it is γ_τ as defined in (3.3). An application of Theorem 2 now assures us that f_6 is in $\mathcal{F}(B_0^n)$.

We remind the reader that all of the corollaries from [6] which assure us that functions are in $S(n)(=\mathcal{F}(B_0^n))$ are consequences of Theorem 2 of this paper.

4. Further corollaries of Theorem 2

Our first corollary in this section is still in the setting of the classical abstract Wiener space (B_0, H_0, ν_0) but is of a quite different character than any of the corollaries in Section 3 or in either of the earlier papers [4,6]. Chang, Johnson, and Skoug concentrated in [6] on corollaries where the first action on x in $B_0 = C_0[0,t]$ involved the evaluation of x at a point in $[0,t]$. Corollary 8 below demonstrates that there are other possibilities even in the setting of [6]. The inner function acting on x in formula (4.3) below can be viewed as an integral operator with kernel g. In the special case where $g(s,y) = g(s)$ is independent of y (4.3) takes the form

$$f(x) = \int_Y \theta(y; \int_0^t g(s)x(s)ds)d\eta(y). \tag{4.1}$$

The choice $g \equiv 1$ produces the function

$$f(x) = \int_Y \theta(y; \int_0^t x(s)ds)d\eta(y), \tag{4.2}$$

whereas, if g is nonnegative and has L^1-norm equal to one, then the inner integral in (4.1) is a weighted average of x.

COROLLARY 8. *Let $(B, H, \nu) = (C_0[0,t],\ H_0[0,t],\ \nu_0)$ be the classical example of an abstract Wiener space and let $(Y, \mathcal{Y}, \eta)$ and $\theta : Y \times \mathbf{R} \to \mathbf{C}$ satisfy hypotheses* (2) *and* (4), *respectively, of Theorem 2. Finally, let $g : [0,t] \times Y \to \mathbf{R}$ be $\mathcal{B}([0,t]) \times \mathcal{Y}$-measurable and be such that $g(\cdot, y) \in L^1[0,t]$ for every $y \in Y$. Then the function $f : C_0[0,t] \to \mathbf{C}$ defined by*

$$f(x) := \int_Y \theta(y; \int_0^t g(s,y)x(s)ds)d\eta(y) \tag{4.3}$$

belongs to $\mathcal{F}(C_0[0,t])(= \mathcal{F}(B))$.

Proof. Given g satisfying the hypotheses, let

$$\psi(s,y) := -\int_0^s g(\tau,y)d\tau + \int_0^t g(\tau,y)d\tau, \tag{4.4}$$

where $(s,y) \in [0,t] \times Y$. As a function of $s, \psi(s,y)$ is absolutely continuous on $[0,t]$ and $\frac{\partial\psi(s,y)}{\partial s} = -g(s,y)$ for Lebesgue - a.e. s in $[0,t]$. Also $\psi(t,y) = 0$ for every y in Y. Now let

$$\beta(s,y) := \int_0^s \psi(\tau,y)d\tau, (s,y) \in [0,t] \times Y. \tag{4.5}$$

Note that $\frac{\partial\beta(s,y)}{\partial s} = \psi(s,y)$ is in $L^2[0,t]$ as a function of s for every $y \in Y$. Also, $\beta(0,y) = 0$ so that $\beta(\cdot,y)$ is in $H_0[0,t]$ for every y in Y.

We are now prepared to define $\phi : Y \to H_0[0,t])$. Simply take

$$\phi(y) := \beta(\cdot,y). \tag{4.6}$$

We need to show first that ϕ is $\mathcal{Y} - \mathcal{B}(H_0[0,t])$-measurable. Since $H = H_0[0,t]$ is separable, it suffices [9, Corollary 2 on p.73] to show that for every $\gamma \in H_0$, $\langle\phi(y),\gamma\rangle_{H_0}$ is measurable as a function of y. However,

$$\begin{aligned}\langle\phi(y),\gamma\rangle_{H_0} &= \langle\beta(\cdot,y),\gamma(\cdot)\rangle_{H_0} \\ &= \left\langle\frac{\partial\beta(\cdot,y)}{\partial s},\ (D\gamma)(\cdot)\right\rangle_{L^2[0,t]} \\ &= \langle\psi(\cdot,y),(D\gamma)(\cdot)\rangle_{L^2[0,t]} \\ &= \int_0^t \psi(s,y)(D\gamma)(s)ds,\end{aligned} \tag{4.7}$$

and the last expression in (4.7) is measurable as a function of y since $g(s,y)$ and hence $\psi(s,y)$ are jointly measurable and $\psi(\cdot,y)$ and $(D\gamma)(\cdot)$ are both in $L^2[0,t]$.

We have just seen that hypothesis (3) of Theorem 2 holds, and so it follows that the function

$$x \mapsto \int_Y \theta(y;(x,\phi(y))^{\sim})d\eta(y)$$

belongs to $\mathcal{F}(C_0[0,t])$. To finish the proof, it suffices to show that

$$(x,\phi(y))^{\sim} = \int_0^t g(s,y)x(s)ds. \tag{4.8}$$

First we will do the formal calculation which yields (4.8) and then we will comment on the validity of some of the steps.

(4.9)

$$\begin{aligned}(x,\phi(y))\tilde{} &= (x,\beta(\cdot,y))\tilde{} = \int_0^t \frac{\partial\beta(s,y)}{\partial s}\tilde{d}x(s)\\ &= \int_0^t \psi(s,y)\tilde{d}x(s) = \int_0^t \psi(s,y)dx(s)\\ &= \psi(t,y)x(t) - \psi(0,y)x(0) - \int_0^t x(s)d\psi(s,y)\\ &= -\int_0^t x(s)\frac{\partial\psi(s,y)}{\partial s}ds\\ &= \int_0^t g(s,y)x(s)ds.\end{aligned}$$

The integral on the right-hand side of the fourth equality in (4.9) is a Riemann-Stieltjes integral which exists since $\psi(\cdot,y)$ is absolutely continuous and so of bounded variation on $[0,t]$. It is known that such a Riemann-Stieltjes integral is equal s - a.e. to the stochastic (i.e., Paley-Wiener-Zygmund) integral [6] on the left-hand side of the fourth equality. The fifth equality comes from the integration by parts formula for Riemann-Stieltjes integrals and the last equality holds since $\frac{\partial\psi}{\partial s} = -g(s,y)$.

We now leave the setting of classical Wiener space B_0(or B_0^n). Our next corollary is the analogue of Corollary 5 for 'n-parameter Wiener space' $B_{n,0}$. We simply state the facts about n-parameter Wiener space which we will need. $B_{n,0} = C_0[0,t]^n$ consists of all $\mathbf{R}$-valued continuous functions x on $[0,t]^n$ such that $x(\tau_1,\cdots,\tau_n) = 0$ whenever at least one of $\tau_1,\cdots,\tau_n$ equals 0. We consider $B_{n,0}$ as equipped with the sup norm under which it is a Banach space. Let $\nu_{n,0}$ denote n-parameter Wiener measure on $(B_{n,0},\mathcal{B}(B_{n,0}))$.

The 2-parameter Wiener space $(C_0[0,t]^2,\ \mathcal{B}(C_0[0,t]^2),\nu_{2,0})$ is often called Yeh-Wiener space, and the sample functions x in $C_0[0,t]^2$ are often referred to as Brownian surfaces or Brownian sheets.

The covariance function of the n-parameter Wiener process is the function sending $((\tau_1, \cdots, \tau_n), (s_1, \cdots, s_n)) \in [0,t]^n \times [0,t]^n$ into

$$\prod_{j=1}^{n} \min(\tau_j, s_j).$$

We now describe the RKHS of this process. Let $H_{n,0} = H_{n,0}[0,t]^n$ denote the set of all functions $\gamma : [0,t]^n \to \mathbf{R}$ for which there exists g in $L^2[0,t]^n$ such that for all $(s_1, \cdots, s_n) \in [0,t]^n$,

$$\gamma(s_1, \cdots, s_n) = \int_0^{s_1} \cdots \int_0^{s_n} g(\tau_1, \cdots, \tau_n) d\tau_n \cdots d\tau_1. \tag{4.10}$$

It is apparent that $\gamma(0, s_2, \cdots, s_n) = \gamma(s_1, 0, s_3, \cdots, s_n) = \cdots = \gamma(s_1, \cdots, s_{n-1}, 0) = 0$ for all $(s_1, \cdots, s_n)$ in $[0,t]^n$ and it is not hard to show that $\frac{\partial^n \gamma}{\partial s_1 \cdots \partial s_n}$ exists an equals $g(s_1, \cdots, s_n)$ for almost every $(s_1, \cdots, s_n)$ in $[0,t]^n$. The inner product on $H_{n,0}$ is defined by

(4.11)

$$\langle \gamma, \beta \rangle_{H_{n,0}} := \int_0^t \cdots \int_0^t \left[\frac{\partial^n \gamma}{\partial s_1 \cdots \partial s_n} \right] \left[\frac{\partial^n \beta}{\partial s_1 \cdots \partial s_n} \right] ds_1 \cdots ds_n.$$

Then $H_{n,0}$, equipped with this inner product, is a separable Hilbert space over $\mathbf{R}$. The family of functions $\left\{ \gamma_{(\tau_1, \cdots, \tau_n)} : (\tau_1, \cdots, \tau_n) \in [0,t]^n \right\}$ from $H_{n,0}$ defined by

$$\gamma_{(\tau_1, \cdots, \tau_n)}(s_1, \cdots, s_n) := \prod_{j=1}^{n} \min(s_j, \tau_j) \tag{4.12}$$

has the reproducing property

$$\langle \gamma, \gamma_{(\tau_1, \cdots, \tau_n)} \rangle_{H_{n,0}} = \gamma(\tau_1, \cdots, \tau_n) \text{ for all } \gamma \in H_{n,0}. \tag{4.13}$$

In fact, the triple $(B_{n,0}, H_{n,0}, \nu_{n,0})$ is an abstract Wiener space [7, 16] and the reproducing property carries over to $B_{n,0} \times H_{n,0}$ in the sense that

$$(x, \gamma_{(\tau_1, \cdots, \tau_n)})^{\sim} = x(\tau_1, \cdots, \tau_n) \text{ for } s\text{-a.e. } x \text{ in } B_{n,0}. \tag{4.14}$$

COROLLARY 9. *Let $(B_{n,0}, H_{n,0}, \nu_{n,0})$ be the abstract Wiener space associated with the n-parameter Wiener process as described above. Let $\theta : [0,t]^n \times \mathbf{R} \to \mathbf{C}$ be given by $\theta((\tau_1, \cdots, \tau_n); \cdot) = \hat{\mu}_{(\tau_1,\cdots,\tau_n)}(\cdot)$ where $\mu_{(\tau_1,\cdots,\tau_n)}$ is in $M(\mathbf{R})$ for every $(\tau_1, \cdots, \tau_n) \in [0,t]^n$ and where the family $\{\mu_{(\tau_1,\cdots,\tau_n)} : (\tau_1, \cdots, \tau_n) \in [0,t]^n\}$ satisfies : (i) $\mu_{(\tau_1,\cdots,\tau_n)}(E)$ is a Borel measurable function of $(\tau_1, \cdots, \tau_n)$ for every E in $\mathcal{B}(\mathbf{R})$, and (ii) $\|\mu_{(\tau_1,\cdots,\tau_n)}\| \in L^1([0,t]^n, |\eta|)$ where η is a Borel measure on $[0,t]^n$. Define $f_7 : B_{n,0} \to \mathbf{C}$ by*

$$(4.15)\quad f_7(x) := \int_0^t \cdots \int_0^t \theta((\tau_1, \cdots, \tau_n); x(\tau_1, \cdots, \tau_n)) d\eta(\tau_1, \cdots, \tau_n).$$

Then f_7 belongs to $\mathcal{F}(B_{n,0}) = \mathcal{F}(C_0[0,t]^n)$.

Proof. Apply Theorem 2 with the abstract Wiener space taken as above and with $Y = [0,t]^n$, $\mathcal{Y} = \mathcal{B}([0,t]^n)$, $d = 1$, and $\phi_1(\tau_1, \cdots, \tau_n) = \gamma_{(\tau_1,\cdots,\tau_n)}$ where $\gamma_{(\tau_1,\cdots,\tau_n)}$ is given by (4.12).

Our next to last corollary will be the analogue of Corollary 5 for the Brownian bridge stochastic process. Let $B_{0,0} = C_{0,0}[0,t]$ be the space of all $\mathbf{R}$ - valued continuous function x on $[0,t]$ such that $x(0) = x(t) = 0$. Under the sup norm, $B_{0,0}$ is a Banach space. Let $\nu_{0,0}$ be the probability measure on $(B_{0,0}, \mathcal{B}(B_{0,0}))$ corresponding to the Brownian bridge process. The covariance function of this process [17, p.753] maps $(\tau, s) \in [0,t] \times [0,t]$ into

$$\min(\tau, s) - s\tau/t.$$

The associated RKHS $H_{0,0} = H_{0,0}[0,t]$ consists of all $\mathbf{R}$-valued absolutely continuous functions γ on $[0,t]$ such that $\gamma(0) = \gamma(t) = 0$ and $D\gamma \in L^2[0,t]$. The inner product on $H_{0,0}$ is given by

$$(4.16)\qquad \langle \gamma, \beta \rangle_{H_{0,0}} = \int_0^t (D\gamma)(s)(D\beta)(s)ds$$

and $H_{0,0}$ is a separable Hilbert space over $\mathbf{R}$ under this inner product. The family of functions $\{\gamma_\tau : \tau \in [0,t]\}$ from $H_{0,0}$ defined by

$$(4.17)\qquad \gamma_\tau(s) = \min(\tau, s) - s\tau/t$$

has the reproducing property

$$(4.18) \qquad \langle \gamma, \gamma_\tau \rangle_{H_{0,0}} = \gamma(\tau) \text{ for all } \gamma \in H_{0,0}.$$

Further, the triple $(B_{0,0}, H_{0,0}, \nu_{0,0})$ is an abstract Wiener space [17] and we have

$$(4.19) \qquad (x, \gamma_\tau)\tilde{} = x(\tau) \text{ for } s\text{ -a.e. } x \text{ in } B_{0,0}.$$

COROLLARY 10. *Let $(B_{0,0}, H_{0,0}, \nu_{0,0})$ be the abstract Wiener space associated with the Brownian bridge process as just described. Let $\theta : [0,t] \times \mathbf{R} \to \mathbf{C}$ be given by $\theta(\tau;\cdot) = \hat{\mu}_\tau(\cdot)$ where μ_τ is in $M(\mathbf{R})$ for every τ in $[0,t]$ and where the family $\{\mu_\tau : \tau \in [0,t]\}$ satisfies:*

(i) *$\mu_\tau(E)$ is a Borel measurable function of τ for every E in $\mathcal{B}(\mathbf{R})$, and*

(ii) *$\|\mu_\tau\| \in L^1([0,t], |\eta|)$ where η is a Borel measure on $[0,t]$. Let $f_8 : B_{0,0} \to \mathbf{C}$ be given by*

$$(4.20) \qquad f_8(x) := \int_0^t \theta(\tau; x(\tau)) d\eta(\tau).$$

Then f_8 is in $\mathcal{F}(B_{0,0}) = \mathcal{F}(C_{0,0}[0,t])$.

Proof. The result is a consequence of Theorem 2 and the proof follows a pattern which should by now be clear.

Our last corollary is based on Example 1.2 in Watanabe's book [20, pp.7–9]. His 'example' is actually a rather large family of examples. Watanabe considers the case of an arbitrary finite number of space dimensions and an arbitrary finite number of 'time' parameters. We will simplify matters by taking both of these numbers equal to one.

Let I be the interval $[0,t]$ and let p be a positive integer. Suppose that K is an $\mathbf{R}$-valued, symmetric, nonnegative definite function (or kernel) on $I \times I$ having $2p$ continuous derivatives and satisfying the following condition: There exists $\delta \in (0,1]$ and $c > 0$ such that

(4.21)

$$|(D_s^p D_\tau^p K)(s,s) + (D_s^p D_\tau^p K)(\tau,\tau) - 2(D_s^p D_\tau^p K)(s,\tau)| \le c|s-\tau|^{2\delta}$$

for all $(s,\tau) \in I \times I$. Now given $0 \le \epsilon \le 1$ and any function x on I possessing p continuous derivatives, let

(4.22)
$$\|x\|_{p,\epsilon} := \sum_{\alpha=0}^{p} \left\{ \max_{s \in I} |(D^\alpha x)(s)| + \sup_{\substack{s,\tau \in I \\ s \neq \tau}} \frac{|(D^\alpha x)(s) - (D^\alpha x)(\tau)|}{|s-\tau|^\epsilon} \right\}$$

and let

$$C^{p,\epsilon} = C^{p,\epsilon}(I) = \{x : \|x\|_{p,\epsilon} < \infty\}. \tag{4.23}$$

Now $(C^{p,\epsilon}, \|\cdot\|_{p,\epsilon})$ is a Banach space. Also, it is known [20, p.8] that for any ϵ, $0 \le \epsilon < \delta$, there exists a mean zero Gaussian measure $\nu_{p,\epsilon}$ on $(C^{p,\epsilon}, \mathcal{B}(C^{p,\epsilon}))$ having $K(s,\tau)$ as its covariance function. Further, the RKHS $H^{p,\epsilon} = H^{p,\epsilon}(I)$ associated with K is such that $(C^{p,\epsilon}, H^{p,\epsilon}, \nu_{p,\epsilon})$ is an abstract Wiener space [20, p.8]. We remark that to obtain the space $H^{p,\epsilon}$, one begins by defining an inner product on the vector space of all finite linear combinations of functions of the form $K(s,\cdot)$; given two such functions, $\sum_{k=1}^{n_1} a_k K(s_k,\cdot)$ and $\sum_{l=1}^{n_2} b_l K(\tau_l,\cdot) = \sum_{l=1}^{n_2} b_l K(\cdot,\tau_l)$, take their inner product to be

$$\sum_{k=1}^{n_1}\sum_{l=1}^{n_2} a_k b_l K(s_k,\tau_l).$$

The Hilbert space $H^{p,\epsilon}$ is the completion of this inner product space.

If we let $\gamma_\tau = K(\cdot,\tau)$ for every $\tau \in I$, then we have (as usual)

$$\langle \gamma, \gamma_\tau \rangle_{H^{p,\epsilon}} = \gamma(\tau) \tag{4.24}$$

for every $\gamma \in H^{p,\epsilon}$. In this setting, we are not sure that the equality $(x,\gamma_\tau)^{\sim} = x(\tau)$ holds s -a.e., and so the expression $(x,\gamma_\tau)^{\sim}$ appears in the statement of the corollary rather than $x(\tau)$. Here, for simplicity, we choose to give an analogue of Corollary 4 although an analogue of Corollary 5 could be given just as was done in Corollary 10.

COROLLARY 11. *Let $(C^{p,\epsilon}, H^{p,\epsilon}, \nu_{p,\epsilon})$ be the abstract Wiener space described above. Let $\theta = \hat{\mu}$ where μ is in $M(\mathbf{R})$. Define $f_9 : C^{p,\epsilon} \to \mathbf{C}$ by the formula*

$$f_9(x) := \int_0^t \theta((x,\gamma_\tau)^{\sim})d\tau. \tag{4.25}$$

Then f_9 belongs to $\mathcal{F}(C^{p,\epsilon})$.

ACKNOWLEDGEMENTS. The first author gratefully acknowledges the support of the Ministry of Education, Korea as well as the hospitality of the University of Nebraska and especially the kind help of Professors G.W. Johnson and D.L. Skoug.

References

1. S. Albeverio and R.Høegh-Krohn, *Mathematical Theory of Feynman Path Integrals*, Springer Lecture Notes in Mathematics, Berlin, **523** (1976).
2. R.H. Cameron and D.A. Storvick, *Some Banach algebras of analytic Feynman integrable functionals*, Analytic Functions, Kozubnik, 1979, Springer Lecture Notes in Mathematics, Berlin, **798** (1980), 18–67.
3. K.S. Chang, G.W. Johnson and D.L. Skoug, *Necessary and sufficient conditions for the Fresnel integrability of certain classes of functions*, J. Korean Math. Soc. **21** (1984), 21–29.
4. ———, *Functions in the Fresnel class*, Proc. Amer. Math. Soc., **100** (1987), 309–318.
5. ———, *Necessary and sufficient conditions for membership in the Banach algebra S for certain classes of functions*, Supplements ai Rendiconti del Cirolo Matematico di Palermo, Serie II - numero **17** (1987), 153–171.
6. ———, *Functions in the Banach algebra $S(\nu)$*, J. Korean Math. Soc., **24** (1987), 151–158.
7. D.M. Chung, *Scale-invariant measurability in abstract Wiener space*, Pacific J. Math., **130** (1987), 27–40.
8. R.P. Feynman, *Space time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys., **20** (1948), 367–387.
9. E. Hille and R.S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloquium Publication **31** (1957).
10. G.W. Johnwon, *The equivalence of two approaches to the Feynman integral*, J. Math. Phys., **23** (1982), 2090–2096.
11. ———, *An unsymmetric Fubini theorem*, Amer. Math. Monthly, **91** (1984), 131–133.
12. G. Kallianpur, *The role of reproducing kernel Hilbert spaces in the study of Gaussian processes*, Advances in Probability, **2** (Peter Ney, editor), Dekker, New York, (1970), 49–83.
13. G. Kallianpur and C. Bromley, *Generalized Feynman integrals using analytic continuation in several complex variables*, Stochastic Analysis and Applications (M.A. Pinsky, editor), Dekker, New York, (1984), 433–450.
14. G. Kallianpur, D. Kannan and R.L. Karandikar, *Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces, and a Cameron-Martin formula*, Ann. Inst. Henri Poincaré, **21** (1985), 323–361.
15. D.C. Khandekar, K.V. Bhagwat and F.W. Wiegel, *End to end distribution functions for a class of polymer models*, In Path Summation: Achievements

and Goals; World Scientific Pub. (1988), Proceedings of the Adriatico Research Conference, Trieste, 410–423.

16. J.Kuelbs, *Abstract Wiener spaces and applications to analysis*, Pacific J. Math., **31** (1969), 433–450.
17. ______, *A strong convergence theorem for Banach space valued random variables*, Ann. Probab., **4** (1976), 744–771.
18. H.H. Kuo, *Gaussian Measures in Banach Spaces*, Springer Lecture Notes in Mathematics, New York, **463** (1975).
19. W. Rudin, *Real and Complex Analysis*, McGraw Hill, New York, (1966).
20. S. Watanabe, *Stochastic Differential Equations and Malliavin Calculus*, Springer -Verlag (Published for the Tata Institute of Fundamental Research), New York, (1984).

Department of Mathematical Education
Kon-kuk University
Seoul 133–701, Korea and visiting the University of Nebraska

Department of Mathematics
University of Nebraska
Lincoln, Nebraska 68588–0323

Department of Mathematics
University of Nebraska
Lincoln, Nebraska 68588–0323