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HYPERSURFACES WITH HARMONIC WEYL TENSOR OF A REAL SPACE FORM

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Introduction

It has been shown in Besse [1] and Gray [2] that there are a few class of Riemannian manifolds which are some generalizations of Einstein manifolds and are characterized by tensorial conditions. Let $(\tilde{M}, g)$ be an $n(\geq 3)$-dimensional Riemannian manifold and $A^p(T^*\tilde{M})$ be a space consisting of p-forms with values in the cotangent bundle $T^*\tilde{M}$. Let Ric be the Ricci tensor defined by $Ric(X,Y) = tr\{Z \to R(Z,X)Y\}$ for any vector fields X, Y and Z on $\tilde{M}$, where R denotes the Riemannian curvature tensor of $\tilde{M}$. The Riemannian manifold $\tilde{M}$ is said to be of *harmonic curvature* (resp. *Harmonic Weyl tensor*), if Ric (resp. $Ric - (2n-2)^{-1}rg$) is a Codazzi tensor, that is, if $dRic = 0$ (resp. $d\{Ric - (2n-2)^{-1}rg\} = 0$), where the Ricci tensor Ric is regarded as a 1-form with values in $T^*\tilde{M}$, r is the scalar curvature of $\tilde{M}$ and d denotes the exterior differential of the bundle $A^1(T^*\tilde{M})$.

On the other hand, ∇Ric is a section of the subbundle of $\otimes^3 T^*\tilde{M}$ denoted by Q if the Ricci tensor satisfies the following condition

$$(*) \qquad 2(n-1)(n+2)\nabla(Ric - (2n-2)^{-1}rg) = (n-2)dr \cdot g,$$

where $\cdot$ denotes the symmetric product of symmetric tensors. It is seen in Besse [1] that the Riemannian manifold satisfying $\nabla Ric \in C^\infty(Q)$ is of harmonic Weyl tensor but not harmonic curvature and there is an example of such a manifold as follows ; $(R \times M_1, dt^2 + f(t)^{-2}g_1)$, where (M_1, g_1) is an Einstein manifold with negative scalar curvature r_1 and the function f is a positive solution of the differential equation

$$f'' - 2(n-1)^{-1}(n-2)^{-1}r_1 f^3 = af$$

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for any positive constant a.

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature c, which is called a *real space form*. The purpose of this paper is to prove

THEOREM. *There are infinitely many hypersurfaces satisfying* ∇ *Ric* $\in C^\infty(Q)$ *of* $M^{n+1}(c)$, $n \geq 3$.

1. Preliminaries

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature c, which is called a real space form. Let M be a connected hypersurface of $M^{n+1}(c)$. We choose an orthonormal local frame $\{E_j, E_{n+1}\}$, $j = 1, \cdots, n$, on $M^{n+1}(c)$ in such a way that, restricted to M, the vectors E_j's are tangent of M and hence the other E_{n+1} is normal to M. With respect to this field of frames on $M^{n+1}(c)$, let $\{w_j, w_{n+1}\}$ and $\{w_{ij}, w_{jn+1}\}$ be the field of the dual frames and the connection forms on $M^{n+1}(c)$, respectively. By restricting these forms to M, they are denoted by the same symbols. Then we have

$$w_{n+1} = 0.$$

By the structure equations on $M^{n+1}(c)$ and the Cartan lemma, the above equation implies

$$w_{n+1i} = \sum_j h_{ij} w_j, \quad h_{ij} = h_{ji}. \tag{1.1}$$

The quadratic form $\sum_{i,j} h_{ij} w_i \otimes w_j$ is called the *second fundamental form* of M. Futhermore it follows from the structure equations on $M^{n+1}(c)$ that the structure equations on the hypersurface M are given by

$$dw_i + \sum_j w_{ij} \wedge w_j = 0, \quad w_{ij} + w_{ji} = 0, \tag{1.2}$$

$$dw_{ij} + \sum_k w_{ik} \wedge w_{kj} = \Omega_{ij},$$

$$\Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} w_k \wedge w_l, \tag{1.3}$$

where Ω_{ij} and R_{ijkl} denote the curvature form and the components of the Riemannian curvature tensor R on the hypersurface M, respectively. Since $M^{n+1}(c)$ is of constant curvature c, we have the Gauss equation

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + h_{il}h_{jk} - h_{ik}h_{jl}. \tag{1.4}$$

The components, S_{ij} of the Ricci tensor Ric defined by $Ric(X,Y) = tr\{Z \to R(Z,X)Y\}$ and the scalar curvature r can be expressed as follows :

$$S_{ij} = (n-1)c\delta_{ij} + hh_{ij} - \sum_k h_{ik}h_{kj}, \tag{1.5}$$

$$r = n(n-1)c + h^2 - \sum_{i,j} h_{ij}h_{ij}, \tag{1.6}$$

where h denotes the trace of h_{ij}.

Now, let (M,g) be an $n(\geq 3)$-dimensional Riemannian manifold and $H = H(M,g)$ be the vector subbundle of $\otimes^3 T^*M$ the fiber of which, at any point x in M, consists of all trilinear maps ξ of T_xM into R such that $\xi(X,Y,Z) = \xi(X,Z,Y)$ and $\sum_{j=1}^n \{\xi(E_j,E_j,X) - \xi(X,E_j,E_j)/2\} = 0$ for any vectors X, Y and Z at x and any orthonormal basis for T_xM. Since it follows from the second Bianchi identity that $dr = 2\,div\,Ric$, the covariant derivative ∇Ric of the Ricci tensor Ric is a section of H, where ∇ denotes the Riemannian connection of (M,g). There is a naturally defined inner product on $H = H(M,g)$ given by $\langle \xi, \eta \rangle = \sum_{i,j,k} \xi(E_i,E_j,E_k)\eta(E_i,E_j,E_k)$.

Given a Riemannian manifold, one has the following natural bundle homomorphism associated with $\otimes^3 T^*M$: the *contraction* $\gamma : \otimes^3 T^*M \to T^*M$, the *partial alternation* $\alpha : \otimes^3 T^*M \to \wedge^2 M \otimes T^*M$, the *partial symmetrization* $\sigma : \otimes^3 T^*M \to \otimes^3 T^*M$ and the mapping $\phi : T^*M \to H(M,g)$ given by

$$\begin{aligned}
\gamma(\xi)X &= \sum_{j=1}^n \xi(E_j,E_j,X),\\
(\alpha(\xi))(X,Y,Z) &= \{\xi(X,Y,Z) - \xi(Y,X,Z)\}/2,\\
(\sigma(\xi))(X,Y,Z) &= \{\xi(X,Y,Z) + \xi(Y,Z,X) + \xi(Z,X,Y)\}/3,\\
(\phi(\omega))(X,Y,Z) &= g(X,Y)\omega(Z) + g(X,Z)\omega(Y)\\
&\quad + 2n(n-2)^{-2}g(Y,Z)\omega(X)
\end{aligned}$$

for $\xi \in \otimes^3 T^*M, X, Y, Z \in T_xM,\ \omega \in T_x^*M$ and any orthonormal basis $\{E_j\}$ for $T_xM, x \in M$. The subbundle Q, S and A of H are defined by

$$Q = Im\,\phi,\ S = H \cap Ker\,\alpha \subset Ker\,\gamma,\ A = H \cap Ker\,\sigma \subset Ker\,\gamma.$$

It is seen in Besse [1] that the subbundle Q coincides with the orthogonal complement of $H \cap Ker\,\gamma$ and moreover we have

$$H = Q \oplus S \oplus A,$$

which is an orthogonal decomposition of H into a direct sum of invariant subbundles.

By W the Weyl curvature tensor is denoted. The Riemannian manifold (M, g) is said to be of *harmonic curvature* (resp. *harmonic Weyl tensor*), if $d^*R = 0$ (resp. $d^*W = 0$), where R and W are regarded as 2-forms with values in the bundle $\wedge^2 T^*M$, where d^* denotes the codifferential on $A^2(\wedge^2 T^*M)$. A symmetric tensor field T on (M, g) is called a *Codazzi tensor* if $dT = 0$, i.e., if T satisfies the *Codazzi equation* $\nabla_X T(Y, Z) = \nabla_Y T(X, Z)$ for any vector fields X, Y and Z. The natural linear conditions that can be imposed on $\nabla\,Ric$ for the Riemannian manifold M can be characterized as follows:

PROPOSITION 1.1 (BESSE [1]). *(1) $\nabla\,Ric \in C^\infty(S)$ is equivalent to each of the following conditions:*

(a) *$d\,Ric = 0$, i.e., Ric is a Codazzi tensor,*
(b) *$d^*R = 0$, i.e., (M, g) has harmonic curvature.*

(2) $\nabla\,Ric \in C^\infty(Q \oplus S)$ is equivalent to each of the following conditions:

(a) *$d(Ric - (2n-2)^{-2} rg) = 0$, i.e., $Ric - (2n-2)^{-2} rg$ is a Codazzi tensor,*
(b) *$n \geq 4$, $d^*W = 0$, i.e., (M, g) has harmonic Weyl tensor,*
(c) *$n \geq 3$; (M, g) is conformally flat and has constant scalar curvature.*

(3) $\nabla\,Ric \in C^\infty(Q)$ if and only if

$$(1.7) \qquad 2(n-1)(n+2)\nabla(Ric - (2n-2)^{-2} rg) = (n-2)dr \cdot g,$$

where $\cdot$ denotes the symmetric product of symmetric tensor.

For the hypersurface M of $M^{n+1}(c)$, the components h_{ijk} and S_{ijk} of the covariant derivative of the second fundamental form and $\nabla\, Ric$ respectively are given by

$$\begin{aligned}&\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj},\\ (1.8)\qquad &\sum_k S_{ijk}\omega_k = dS_{ij} - \sum_k S_{kj}\omega_{ki} - \sum_k S_{ik}\omega_{kj}.\end{aligned}$$

Differentiating (1.1) exteriorly, one has the Codazzi equation on the hypersurface M

$$(1.9)\qquad h_{ijk} = h_{ikj},$$

because the ambient space is of constant curvature, and by differentiating (1.6) exteriorly, the covariant derivative S_{ijk} of the Ricci tensor Ric satisfies

$$(1.10)\quad \sum_k S_{ijk}\omega_k = \sum_k (h_k h_{ij} + hh_{ijk} - \sum_l h_{ilk}h_{lj} - \sum_l h_{il}h_{ljk}\omega_k,$$

where $dh = \sum_k h_k\omega_k$ and hence

$$(1.11)\qquad \sum_{j,k} S_{ijk}\omega_k \wedge \omega_j = \sum_{j,k}(h_k h_{ij} - \sum_l h_{ilk}h_{lj})\omega_k \wedge \omega_j.$$

Then the condition that Ric or $Ric-(2n-2)^{-1}rg$ is the Codazzi tensor is equaivalent to

$$(1.12)\qquad S_{ijk} = S_{ikj} \text{ or } S_{ijk} - S_{ikj} = (2n-2)^{-1}(r_k\delta_{ij} - r_j\delta_{ik}),$$

where $dr = \sum_k r_k\omega_k$. On the other hand, the condition $\nabla\, Ric \in C^\infty(Q)$ is by (1.7) equivalent to

$$(1.13)\qquad 2(n-1)(n+2)S_{ijk} = 2nr_k\delta_{ij} + (n-2)(r_j\delta_{ik} + r_i\delta_{jk}).$$

2. Hypersurfaces with constant mean curvature

Let M be an $n(\geq 3)$-dimensional hypersurface of $M^{n+1}(c)$. The second fundamental form may be diagonalized so that $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_j \lambda_j \omega_j \otimes \omega_j$, where λ_j's are principal curvatures for M. The principal curvature is said to be *simple* at a point x if the multiplicity at x is equal to 1.

LEMMA 2.1. *Let M be a hypersurface with constant mean curvature of a real space form $M^{n+1}(c)$. If the Ricci tensor Ric satisfies $\nabla\, Ric \in C^\infty(Q \oplus S)$ and if each principal curvature is not simple, then M has harmonic curvature.*

Proof. In order to prove this proposition, it suffices to show that the scalar curvature r is constant. Since we have $h_{ij} = \lambda_i \delta_{ij}$ at a point x in M, the second equation of (1.12) is reduced to

$$(2.1) \qquad (\lambda_k - \lambda_j)h_{ijk} = (2n-2)^{-1}(r_k \delta_{ij} - r_j \delta_{ik})$$

because of (1.11).

Now, by using the notations $[i] = \{j : \lambda_j = \lambda_i\}$, the assumption of no simple principal curvatures implies that the number of index in $[i]$ for any index i is equal to or greater than 2. Thus there is an index k in $[j]$ different from j for any index j. Then we have by (2.1)

$$r_k \delta_{ij} - r_j \delta_{ik} = 0 \quad \text{for any index} \quad i.$$

It turns out that $r_j = 0$ for any index j, which yields that $dr = 0$. This means that r is constant on M.

REMARK. According to a theorem of Nishikawa and Maeda [3], a hypersurface M of a conformally flat Riemannian manifold is conformally flat if and only if any point in M is umbilic or it has two distinct principal curvatures one of which is simple.

On the other hand, Otsuki [4] showed that there are infinitely many minimal hypersurfaces with the same situation as the latter case for principal curvatures of a real space form, which implies that they are conformally flat. Since the harmonic Weyl tensor is naturally drawn from the conformal flatness, it means that there are many minimal hypersurfaces with harmonic Weyl tensor but not harmonic curvature.

3. Hypersurfaces with harmonic Weyl tensor

In this section we are concerned with hypersurfaces with harmonic Weyl tensor but not harmonic curvature of $M^{n+1}(c)$. Let M be a hypersurface with harmonic Weyl tensor of $M^{n+1}(c)$. Then the Ricci tensor of M satisfies (1.13). By virtue of (1.10) we have

$$2(n-1)(n+2)\{h_k h_{ij} + h h_{ijk} - \sum_l (h_{ilk} h_{lj} + h_{il} h_{ljk})\}$$
$$= 2n r_k \delta_{ij} + (n-2)(r_j \delta_{ik} + r_i \delta_{jk}).$$

Accordingly we get

$$(3.1) \quad 2(n-1)(n+2)\{\lambda_j h_k \delta_{ij} + (h - \lambda_i - \lambda_j) h_{ijk}\}$$
$$= 2n r_k \delta_{ij} + (n-2)(r_j \delta_{ik} + r_i \delta_{jk}),$$

from which the following equation is derived:

$$(3.2) \quad 2(n-1)\{\lambda_j h_k \delta_{ij} + \lambda_k h_j \delta_{ik} + (\lambda_k - \lambda_j) h_{ijk}\} = r_k \delta_{ij} - r_j \delta_{ik}.$$

Taking account of Lemma 2.1 and the above equation (3.2), one proves the following

THEOREM 3.1. *Let M be a hypersurface of $M^{n+1}(c)$ satisfying $\nabla Ric \in C^\infty(Q \oplus S)$, $n \geq 3$. If M is not of harmonic curvature, then there are at least one simple principal curvature.*

Proof. Suppose that each principal curvature of M is not simple, that is, the number of indices containing in $[j]$ for any index j is greater than 1. So, there is an index k in $[j]$ such that $j \neq k$. Putting $i = k$ in (3.2), we get

$$(3.3) \qquad r_j = 2(n-1)\lambda_j h_j \text{ for any index } j.$$

Substituting this relationship into (3.2), one has

$$(\lambda_j - \lambda_k)(h_k \delta_{ij} + h_j \delta_{ik} - h_{ijk}) = 0$$

for any indices i, j and k.

Let M_0 be the subset which consists of not umbilic points in M, i.e., points x in M at which they satisfy $h_{ij}(x) \neq \lambda(x)\delta_{ij}$. Suppose that the open set M_0 is not empty. Then there is an index k which is not contained in $[j]$ such that

$$(3.4) \qquad h_{ijk} = h_k\delta_{ij} + h_j\delta_{ik} \text{ for any index } i.$$

Substituting (3.3) and (3.4) into (3.1), we get

$$\{2n\lambda_k - (n+2)(h-\lambda_i)\}h_k\delta_{ij} - \{2n\lambda_j - (n+2)(h-\lambda_i)\}h_j\delta_{ik} = 0$$

for any index i and any index k not containing in $[j]$, from which the following equations are derived:

$$(3.5) \qquad \begin{aligned} \{2n\lambda_k - (n+2)(h-\lambda_j)\}h_k = 0, \\ \{2n\lambda_j - (n+2)(h-\lambda_k)\}h_j = 0. \end{aligned}$$

For any fixed index $k \in [j]$, let M_{jk} be the subset of M_0 which consists of points x in M_0 such that $h_jh_k(x) \neq 0$. Suppose that the open set M_{jk} is not empty. Then it follows from (3.5) that $(3n+2)\lambda_k = (n+2)h$ and $(3n+2)\lambda_j = (n+2)h$, a contradiction. Hence the set M_{jk} is empty.

Next, suppose that there are an index j and a point x in M_0 such that $h_j(x) \neq 0$. For the fixed j, let M_j be the subset of M_0 which consists of points x such that $h_j(x) \neq 0$. Then the subset is not empty and for any index $k \in [j]$ and for any point x in M_j, we have $h_k(x) = 0$, because M_{jk} is empty. Combining this together with (3.5), one has

$$(3.6) \qquad (n+2)\lambda_k = (n+2)h - 2n\lambda_j,$$

which means that the number of distinct principal curvatures is equal to 2 in M_j, say λ and μ. By s and t the multiplicities of λ and μ are denoted respectively. Then (3.6) is reduced to

$$(3.7) \qquad \mu = b\lambda,\ b = \{(n+2)s - 2n\}/(n+2)(1-t) \neq 1,$$

because of $h = s\lambda + t\mu$, $s, t \geq 2$ and $s + t = n \geq 3$. Since the scalar curvature r is given by $r = n(n-1)c + (s\lambda + t\mu)^2 - (s\lambda^2 + t\mu^2)$, we have by (3.7)

$$r = n(n-1)c + \{(s+bt)^2 - (s+b^2t)\}\lambda^2.$$

Since the principal curvature λ is smooth on the set M_j and the coefficient of the above equation is constant on M_j, we have $r_j = 2\{(s+bt)^2-(s+b^2t)\}\lambda\lambda_{,j}$, where $d\lambda = \sum_j \lambda_{,j}\omega_j$. Also, since (3.3) is equivalent to $r_j = 2(s+t-1)(s+bt)\lambda\lambda_{,j}$, we have

$$[\{(s+bt)^2-(s+b^2t)\}-(s+t-1)(s+bt)]\lambda\lambda_{,j}=0.$$

On the subset M_j it is easily seen that $\lambda\lambda_{,j} \neq 0$ and hence it follows from the above equation that we have $t=1$, which is a contradiction. This yields that the set M_j is empty, which means that the function h must be constant on M_0.

Suppose lastly that the set $M - M_0$ has a non-empty interior. Then any point in $M - M_0$ is umbilic and therefore h is constant on the interior of the set. This means that the mean curvature is constant on M, because of the continuity of h and the constantness on M_0. In the case of the interior of $M - M_0$ is empty, the same conclusion is given.

Thus, by means of Lemma 2.1, the hypersurface M has harmonic curvature.

4. Examples

This section is devoted to the investigation about examples of hypersurfaces satisfying $\nabla Ric \in C^\infty(Q)$ of a real space form $M^{n+1}(c)$. In this case, the hypersurface has harmonic Weyl tensor but not harmonic curvature. Accordingly, by taking account of Theorem 3.1, it is seen that at least one principal curvature ought to be simple. Moreover, since the hypersurfaces with distinct two principal curvatures one of which is simple of $M^{n+1}(c)$ are conformally flat, we may consider such hypersurfaces.

Let M be a hypersurface of $M^{n+1}(c)$, $n \geq 3$, and assume that the principal curvatures λ_j's on M satisfy

$$\begin{aligned} &\lambda_1 = \cdots = \lambda_{n-1} = \lambda \neq 0, \\ &\lambda_n = \mu, \end{aligned} \tag{4.1}$$

such that $\lambda \neq \mu$. Without loss of generality, we may suppose that $\lambda > 0$. As is seen in Otsuki [4], the distribution of the space of principal vectors at any point corresponding to the principal curvature λ is completely integrable, because the multiplicity of each principal curvature is constant. Now, since λ and μ are smooth functions on M, we have, by the definition of the covariant derivative h_{ijk},

$$d\lambda = h_{aaa}\omega_a + \sum_{a\neq b} h_{aab}\omega_b + h_{aan}\omega_n, \tag{4.2}$$

where indices $a, b, \cdots$ run over the range $\{1, \cdots, n-1\}$. Because of $\omega_{n+1a} = \lambda\omega_a$, we have

$$\begin{aligned} d\omega_{n+1a} &= d\lambda \wedge \omega_a + \lambda d\omega_a \\ &= d\lambda \wedge \omega_a + \lambda(-\sum_b \omega_{ab} \wedge \omega_b), \end{aligned}$$

while the restriction of the structure equation for the ambient space to the hypersurface M yields

$$\begin{aligned} d\omega_{n+1a} &= -\sum_k \omega_{n+1k} \wedge \omega_{ka} \\ &= -\lambda \sum_b \omega_b \wedge \omega_{ba} - \mu\omega_n \wedge \omega_{na}. \end{aligned}$$

Combining together with above two equations, we have

$$\sum_b \lambda_{,b}\omega_b \wedge \omega_a + \{(\mu - \lambda)\omega_{an} - \lambda_{,n}\,\omega_a\} \wedge \omega_n = 0 \tag{4.3}$$

for any index a, where $d\lambda = \sum_b \lambda_{,b}\,\omega_b + \lambda_{,n}\,\omega_n$. This implies

$$\begin{aligned} &\lambda_{,a} = 0, \\ &(\mu - \lambda)\omega_{an} - \lambda_{,n}\,\omega_a = \sigma_a\omega_n \end{aligned} \tag{4.4}$$

for any index a, where σ_a is a function on M. From (4.2) and the first equation of (4.4) it follows that we have

$$h_{aaa}\omega_a + \sum_{a\neq b} h_{aab}\omega_b + h_{aan}\omega_n = \lambda_{,n}\,\omega_n,$$

and hence

$$h_{aaa} = 0, \quad h_{aab} = 0 \ (b \neq 0), \quad h_{aan} = \lambda_{,n} \,. \tag{4.5}$$

Similarly, for the other principal curvature μ one has

$$d\mu = \sum_b h_{nnb}\omega_b + h_{nnn}\omega_n.$$

Because of $\omega_{n+1n} = \mu\omega_n$, by the same argument as that of λ we have

$$d\omega_{n+1n} = -\lambda \sum_b \omega_{nb} \wedge \omega_b = d\mu \wedge \omega_n - \mu \sum_b \omega_{nb} \wedge \omega_b,$$

and hence

$$d\mu \wedge \omega_n + (\lambda - \mu) \sum_b \omega_{nb} \wedge \omega_b = 0.$$

We set $d\mu = \sum_b \mu_{,b}\,\omega_b + \mu_{,n}\,\omega_n$. This together with (4.4) implies

$$\mu_{,a} = \sigma_a \quad \text{for any index} \quad a. \tag{4.6}$$

By definition, we have

$$h_{nna} = \mu_{,a} \,, \quad h_{nnn} = \mu_{,n} \,. \tag{4.7}$$

On the other hand, for distinct indices a and b, one has

$$h_{abk} = 0 \quad \text{for any index} \quad k. \tag{4.8}$$

In particular, let M be a hypersurface satisfying $\nabla\, Ric \in C^\infty(Q)$ of $M^{n+1}(c)$. Then the principal curvature λ_j satisfies

$$\begin{aligned}(n-1)(n-2)\{h_k\lambda_j\delta_{ij} + (h - \lambda_i - \lambda_j)h_{ijk}\} \\ = 2n(hh_k - \sum_l \lambda_l h_{kll})\delta_{ij} \\ + (n-2)\{(hh_j - \sum_l \lambda_l h_{jll})\delta_{ik} \\ + (hh_i - \sum_l \lambda_l h_{ill})\delta_{jk}\},\end{aligned} \tag{4.9}$$

because of (3.1). Since the function h is given by $h=(n-1)\lambda+\mu$ and $dh=\sum_k h_k\omega_k$, we have

$$h_k=(n-1)\lambda_{,k}+\mu_{,k}$$

for any index k. Moreover, by (4.4), (4.5) and (4.7) we have

$$h_a=\mu_{,a}=h_{ann} \quad \text{for any index} \quad a. \tag{4.10}$$

Considering the case where $j=a$ and $k=n$ in (4.9) and using (4.5), one gets

$$\begin{aligned}(n-1)(n+2)\{\lambda h_n\delta_{ia}+(h-\lambda-\lambda_i)h_{ani}\}\\ =2n(hh_n-\lambda h_{aan}-\lambda\sum_{b\neq a}h_{bbn}-\mu h_{nnn})\delta_{ia}\\ +(n-2)(hh_a-\mu h_{ann})\delta_{in}\end{aligned}$$

for any indices a and i. This means that if follows from the above equation that

$$\begin{aligned}&\{(n-1)(n+2)\lambda+\mu\}h_{ann}=hh_a,\\ &(n-1)(n+2)\{\lambda h_n+(h-2\lambda)h_{aan}\}\\ &\quad=2n(hh_n-\lambda h_{aan}-\lambda\sum_{a\neq b}h_{bba}-\mu h_{nnn}).\end{aligned} \tag{4.11}$$

Accordingly, combining the above equation together with (4.10), we have

$$\lambda h_a=0, \quad \text{that is,}\ h_a=0,\ \sigma_a=0.$$

Thus, by (4.4), we have

$$\begin{aligned}&\omega_{na}=\lambda_{,n}(\lambda-\mu)^{-1}\omega_a,\\ &\mu_{,n}=h_{nnn},\ \lambda_{,a}=\mu_{,a}=0.\end{aligned} \tag{4.12}$$

Consequently, in order for M to satisfy the condition $\nabla Ric\in C^\infty(Q)$, their principal curvatures λ and μ must satisfy (4.11) and (4.12). Moreover, we have $d\omega_n=0$, which shows that we may put

$$\omega_n=dv. \tag{4.13}$$

Thus we have

$$\omega_{na} = \lambda'(\lambda - \mu)^{-1}\omega_a, \tag{4.14}$$

where the prime denotes the derivative with respect to the parameter v. Substituting the above equation into the structure equation

$$d\omega_{na} + \sum_b \omega_{nb} \wedge \omega_{ba} = (c + \lambda\mu)\omega_n \wedge \omega_a,$$

we have

$$d(\lambda'(\lambda - \mu)^{-1}\omega_a) = -\lambda'(\lambda - \mu)^{-1} \sum_b \omega_b \wedge \omega_{ba} + (c + \lambda\mu)\omega_n \wedge \omega_a.$$

Since the left hand side is reduced to

$$\{(\lambda - \mu)^{-1}\lambda'\}'\omega_n \wedge \omega_a + (\lambda - \mu)^{-1}\lambda'(-\sum_b \omega_{ab} \wedge \omega_b + \omega_{an} \wedge \omega_n),$$

we get

$$\{(\lambda - \mu)^{-1}\lambda'\}' - \{(\lambda - \mu)^{-1}\lambda'\}^2 - (c + \lambda\mu) = 0,$$

which is reformed to

$$(\lambda - \mu)\lambda'' - \lambda'(2\lambda' - \mu') - (c + \lambda\mu)(\lambda - \mu)^2 = 0. \tag{4.15}$$

On the other hand, it follows from (4.5), (4.7) and (4.12) that the condition (4.11) is equivalent to

$$4\lambda\lambda' - (\lambda\mu)' = 0, \text{ i.e., } 2\lambda^2 - \lambda\mu = c_1, \tag{4.16}$$

where c_1 is the integral constant, because of $h = (n - 1)\lambda + \mu$.

REMARK. By (4.16) if the mean curvature of M is constant, then so their principal curvatures λ and μ, and hence M is locally a product manifold of two space forms of dimension $n - 1$ and 1. These examples are excluded, because they are trivial.

By (4.15) and (4.16) the differential equation for λ is given as follows:

$$\lambda(\lambda^2 - c_1)\lambda'' - c_1\lambda'^2 + (2\lambda^2 + c - c_1)(\lambda^2 - c_1)^2 = 0, \tag{4.17}$$

where $\lambda^2 \neq c_1$. In particular, suppose that $c_1 = 0$. Then we have

$$2\lambda = \mu, \quad \lambda'^2 + (\lambda^4 + c\lambda^2) = c_2, \tag{4.18}$$

where c_2 is the integral constant. Hence we have many hypersurfaces satisfying $\nabla Ric \in C^\infty(Q)$ of $M^{n+1}(c)$ corresponding to the values of the constant c_2.

In the sequel, we suppose that $c = 1$ and $M^{n+1}(1)$ is an $(n+1)$-dimensional unit sphere S^{n+1} in R^{n+2}. We may consider the frame $(\mathrm{x}, e, \cdots, e_n, e_{n+1})$ in R^{n+2} such that $\mathrm{x} = e_{n+2}$. Then, by (4.12), (4.13) and (4.16), we have

$$\begin{aligned} de_a &= \sum_b \omega_{ba} e_b + \omega_{na} e_n + \omega_{a+1a} e_{n+1} + \omega_{n+2a} e_{n+2} \\ &= \sum_{b \neq a} \omega_{ba} e_b + \{-(\log\lambda)' e_n + \lambda e_{n+1} - e_{n+2}\}\omega_a, \end{aligned}$$

$$\begin{aligned} & d\{-(\log\lambda)' e_n + \lambda e_{n+1} - e_{n+2}\} \\ =& \{-(\log\lambda)'' e_n + \lambda' e_{n+1}\}\omega_n - (\log\lambda)'(\sum_a \omega_{an} e_n + \mu\omega_n e_{n+1} \\ & - \omega_n e_{n+2}) + \lambda(-\lambda\sum_a \omega_a e_a - \mu\omega_n e_n) + (\sum_a \omega_a e_a + \omega_n e_n) \\ =& \{-(\log\lambda)'' - \lambda\mu - 1\}\omega_n e_n + \{\lambda' - \mu(\log\lambda)'\}\omega_n e_{n+1} \\ & + (\log\lambda)'\omega_n e_{n+2} \quad (\text{mod. } e_1, \cdots, e_{n-1}\} \\ =& -(\log\lambda)'\{-(\log\lambda)' e_n + \lambda e_{n+1} - e_{n+2}\} dv, \end{aligned}$$

by means of (4.15). Hence, putting

$$W = e_1 \wedge \cdots \wedge e_{n-1} \wedge \{-(\log\lambda)' e_n + \lambda e_{n+1} - e_{n+2}\}, \tag{4.19}$$

we get

$$dW = -(\log\lambda)' W dv, \tag{4.20}$$

which shows that the n-vector W in R^{n+2} is constant along $M^{n-1}(v)$. Hence there exists an n-dimensional linear subspace $E^n(v)$ in R^{n+2} containing $M^{n-2}(v)$. By (4.20) the n-vector field W depends only on v and by integrating it we get

$$W(v) = \lambda(v_0)W(v_0)/\lambda(v).$$

Hence we have $E^n(v)$ is parallel to $E^n(v_0)$ in R^{n+2}.

Since the sectional curvature of $M^{n-1}(v)$ is given by $(\log\lambda)' + \lambda^2 + 1$, because of

$$\begin{aligned} &d\omega_{ab} + \sum_c \omega_{ac} \wedge \omega_{cb} \\ &= -\,\omega_{an} \wedge \omega_{nb} - \omega_{an+1} \wedge \omega_{n+1b} - \omega_{an+2} \wedge \omega_{n+2b} \\ &= \{(\log\lambda)' + \lambda^2 + 1\}\omega_a \wedge \omega_b, \end{aligned}$$

the center $q = q(v)$ of the $(n-1)$-dimensional sphere $S^{n-1}(v) = E^n(v) \cap S^{n+1}$ is given by

$$(4.21) \qquad q = \{(\log\lambda)'^2 + \lambda^2 + 1\}\{(\log\lambda)' e_n + \lambda e_{n+1} - e_{n+2}\}$$

and lies in a fixed plane E^2 through the origin of R^{n+2} and orthogonal to $E^n(v_0)$. Therefore the point $q = q(v)$ makes a plane curve in E^2. Thus one finds

THEOREM 4.1. *Let M be an $n(\geq 3)$-dimensional hypersurface satisfying the condition $\nabla\, Ric \in C^\infty(Q)$ of $M^{n+1}(c)$. If it has exactly two distinct principal curvatures, one of which is simple and the other λ has no zero points, the following assertions are true:*

(1) *M is a locus of moving $(n-1)$-dimensional submanifold $M^{n-1}(v)$ along which the principal curvature λ is constant and which is umbilic in M and of constant curvature $\{d/dv(\log\lambda)^2 + \lambda^2 + c\}$, where v is the arc length of an orthogonal trajectory of the family $M^{n-1}(v)$, and $\lambda = \lambda(v)$ satisfies the ordinary differential equation (4.18) of order 2.*

(2) *If $M = S^{n+1}(c)$ in R^{n+2}, then $M^{n-1}(v)$ is contained in an $(n-1)$-dimensional sphere $S^{n-1}(v) = E^n(v) \cap S^{n+1}$ of the intersection of S^{n+1} and an n-dimensional linear subspace $E^n(v)$ in R^{n+2} which is paralled to a fixed E^n. The center q moves on a plane curve in a plane R^2 through the origin R^{n+2} and orthogonal to E^n.*

COROLLARY. *There exist infinitely many hypersurfaces satisfying the condition $\nabla\, Ric \in C^\infty(Q)$ of $M^{n+1}(c)$, which is not congruent to each other on it.*

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