

## AN APPLICATION OF THE HOPF BIFURCATION THEOREM IN ECONOMICS

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### 1. Introduction

Mathematical economics has paid a great deal of attention to explicit models, often studied without regard to their predictions' remaining true under small perturbations of the model. This may be seen in particular in the study of oscillation such as the expanding business cycle, which has a large literature (see [1]). We consider here a recent paper by Eckalbar [3], which shows that under certain assumptions the warehouse stocks which firms keep will oscillate for some parameter values of the model given. As the parameters change from values which give stocks a stable equilibrium to values for which they oscillate, the change resembles—but is not—a typical Hopf bifurcation. We show here that with an arbitrarily small perturbation the change does become a Hopf bifurcation, of the 'standard' or 'dual' type depending chiefly on the sign of a particular third-order term. For the second case, for some parameter values the system can be either oscillatory or static, depending on previous history; impossible in the original model. The result is a description of the dynamic behaviour that can *stably* be exhibited by systems approximated by Eckalbar's original assumptions, which in certain important respects the behaviour of the original model cannot.

We use the Hopf Bifurcation Theorem and Poincaré-Bendixson Theorem in the forms proved in [2] and in [5], respectively, to guarantee the existence of a cycle, and compute its form to sufficient precision to evaluate the sign of the *Floquet exponents* that determine its stability. We use the computational machinery of Floquet theory as expounded in [6]. (An equally good source on the Hopf theorem is [7].)

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The original model is derived in detail by Eckalbar [3], for which [10] provides useful background. The derivation involves five dynamic variables: production  $Q \leq \bar{Q}$ , labour  $L \leq \bar{L}$ , actual sales  $S$ , expected sales  $S^e$ , and stocks  $V$ , with  $\bar{Q}$ ,  $\bar{L}$  constant. Assume  $Q$  is given by  $dL$ , for some productivity coefficient  $d$ . By the *Keynesian consumption map* in [10], we have  $S = a + cQ$ , where  $a$  is the minimal demand and  $c$  marginal propensity to consumption,  $0 < c < 1$ . The desired production is given by the expected sales  $S^e$  plus replacement (desired minus existing) stock,

$$(1) \quad S^e + (kS^e - V) = (1 + k)S^e - V,$$

which in a discrete-time model gives exactly the desired stock  $kS^e$  at end of period if the expectation of sales  $S^e$  is borne out. The actual, labour-limited production is given by

$$(2) \quad Q = \min [(1 + k)S^e - V, \bar{Q}].$$

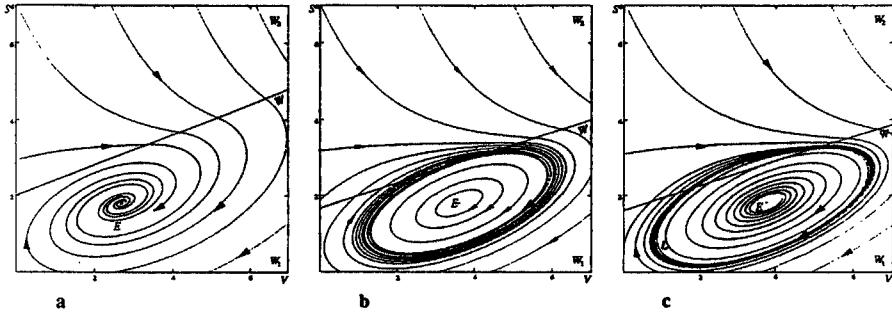
These equations confine us to a creased plane  $P$  in  $(V, S^e, S, Q, L)$ -space, on which we may use  $V$  and  $S^e$  as coordinates since their values determine  $Q$ , and hence determine also  $S = a + cQ$  and  $L = Q/d$ . Assuming that (1) remains appropriate when  $S^e$  becomes an expected sales *rate* in continuous time, conservation of goods gives the rate of stock change, 'warehouse in' less 'warehouse out', as

$$(3) \quad \frac{dV}{dt} = Q - S = (1 - c) \min [(1 + k)S^e - V, \bar{Q}] - a.$$

Expected sales rate is assumed to approach current actual sales at a rate given by their difference,

$$(4) \quad \frac{dS^e}{dt} = S - S^e = a + c \min [(1 + k)S^e - V, \bar{Q}] - S^e,$$

which would give an exponential approach to correct expectations if  $S$  were constant. (Actually since  $Q$  cannot be negative,  $Q$  must be the maximum of 0 and the right side of (2), but we are interested in the local behaviour of this model, so let  $Q$  be given by (2); see [8] for global aspects.)



**Figure 1.** The system (3,4) with parameter values  $(a, c, \bar{Q}) = (0.9, 0.5, 5)$  and  $k$  as 1.5 in  $a$ , 2.0 in  $b$ , and 2.1 in  $c$ . In  $a$  the equilibrium  $E$  is globally attractive, while in  $b$  every orbit near  $E$  is a closed cycle, since  $kc = 2(1 - c)$  and  $\text{trace}(J_1) = 2c + kc - 2 = 0$ . Orbits starting far from  $E$  shrink toward the outermost of these cycles. In  $c$  the equilibrium  $E$  is unstable; trajectories spiral out to approach a limit cycle  $L$  that crosses the boundary  $W$  between  $W_1$  below it and  $W_2$  above.

Since the functions  $Q-S$  and  $S-S^e$  are continuous on  $(V, S^e, S, Q, L)$ -space, they restrict to continuous functions on the creased plane  $P$ , and in fact satisfy a Lipschitz condition; for points  $(V_1, S_1^e, S_1, Q_1, L_1)$  and  $(V_2, S_2^e, S_2, Q_2, L_2)$ ,

$$(5) \quad \begin{aligned} \left| \frac{dV}{dt} \Big|_{(V_1, S_1^e)} - \frac{dV}{dt} \Big|_{(V_2, S_2^e)} \right| &\leq (1+k) \|(V_1, S_1^e) - (V_2, S_2^e)\| \\ \left| \frac{dS^e}{dt} \Big|_{(V_1, S_1^e)} - \frac{dS^e}{dt} \Big|_{(V_2, S_2^e)} \right| &\leq (1+k) \|(V_1, S_1^e) - (V_2, S_2^e)\| \end{aligned}$$

as functions of  $(V, S^e)$ , using for convenience the norm  $\|(V, S^e)\| = |V| + |S^e|$ . Therefore within the positive quadrant (negative stocks and sales not being admissible in the model), the standard results on existence and uniqueness of solutions [9] of ODEs apply to the system (3, 4), with their consequences such as the Poincaré-Bendixson theorem.

The linearity of  $Q - S$  and  $S - S^e$  on  $(V, S^e, S, Q, L)$ -space restricts to linearity on either side of the crease in  $P$ , so that in  $(V, S^e)$  coordinates we have regimes  $W_1 = \{(V, S^e) | (1+k)S^e - V < \bar{Q}\}$ ,  $W_2 = \{(V, S^e) | (1+k)S^e - V > \bar{Q}\}$  separated by the line  $W = \{(V, S^e) | (1+k)S^e - V = \bar{Q}\}$ , in which the system can be expressed as

$$(6) \quad \frac{d}{dt} \begin{bmatrix} V \\ S^e \end{bmatrix} = J_1 \begin{bmatrix} V \\ S^e \end{bmatrix} + H_1 = \begin{bmatrix} c-1 & (1+k)(1-c) \\ -c & c(1+k)-1 \end{bmatrix} \begin{bmatrix} V \\ S^e \end{bmatrix} + \begin{bmatrix} -a \\ a \end{bmatrix}$$

and

$$(7) \quad \frac{d}{dt} \begin{bmatrix} V \\ S^e \end{bmatrix} = J_2 \begin{bmatrix} V \\ S^e \end{bmatrix} + H_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V \\ S^e \end{bmatrix} + \begin{bmatrix} (1-c)\bar{Q} - a \\ c\bar{Q} + a \end{bmatrix},$$

respectively.

Sample phase portraits are shown in Fig.1. Equation (6) has an equilibrium in  $W_1$ , given by

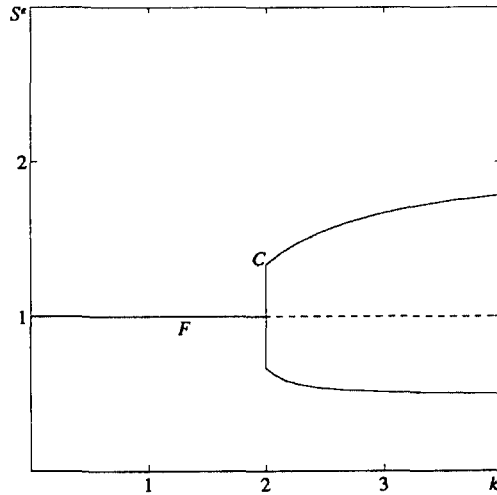
$$(8) \quad E_0 = \begin{bmatrix} V \\ S^e \end{bmatrix} = J_1^{-1}(-H_1) = \frac{a}{1-c} \begin{bmatrix} k \\ 1 \end{bmatrix},$$

as long as  $a < (1-c)\bar{Q}$ , which we follow [3] in supposing. (For more detailed analysis of the piecewise linear model, see [3,8].)

Bifurcation between the stable point in Fig.1a and the stable cycle in Fig.1c occurs via the situation in Fig.1b, where a continuum of closed cycles coexist. (Fig.2 shows the  $k$ -dependence of the attractors for a particular choice of  $c$ .) This is structurally unstable, and becomes false if we add an arbitrarily small non-linear term. This coexistence of an infinity of cycles at one  $(c, k)$ , and the jump from the existence of a stable point to the existence of a stable cycle, is false if we add a small non-linear term.

In §3 we consider dynamics near  $E_0$  if a non-linear term is added, so we restrict our concern here to the region  $W_1$  containing  $E_0$ . To do so, let us transform equations (6) into a simpler form. The eigenvalues of  $J_1$  are  $\frac{1}{2}(2c+kc-2 \pm \sqrt{c^2(k+2)^2 - 4c(k+1)})$ . If  $k \geq \frac{2}{c}(1-c + \sqrt{1-c})$ , the eigenvalues are real and positive, so  $E_0$  is a repeller. Let us consider  $k < \frac{2}{c}(1-c + \sqrt{1-c})$ . Then we have eigenvalues

$$(9) \quad \nu(k) = \frac{1}{2} \left( 2c + kc - 2 + i\sqrt{4c(k+1) - c^2(k+2)^2} \right)$$



**Figure 2.** With the values  $a = c = 0.5$ ,  $\bar{Q} = 3.0$  fixed in the original model, we vary  $k$  from 0 to 4. For each value of  $k$ , the  $S^e$  values shown are those where the line  $V = ak/(1 - c)$  in  $(V, S^e)$ -space meets  $E = (ak/(1 - c), a/(1 - c)) = (k, 1)$  with these  $a, c$  and  $\bar{Q}$ , or a cycle. The line  $S^e = 1$  thus corresponds to  $E$  for each  $k$ , and is shown dotted for  $k > 2(1 - c)/c = 2$  to indicate its instability. For the critical value  $k = 2$ , an infinity of cycles coexist.

and its conjugate, with real part  $\xi(k) = (kc + 2c - 2)/2$  and imaginary part  $\eta(k) = \sqrt{4c(k + 1) - c^2(k + 2)^2}/2$ . Defining a new parameter  $\mu = k - \frac{2(1-c)}{c}$ , and setting  $r = \sqrt{4 - 4c - \mu^2 c^2}$  and  $w = \mu c + 2 - 2c$  for short, we have

$$(10) \quad \xi(\mu) = \frac{\mu c}{2}, \quad \eta(\mu) = \frac{r}{2}.$$

The eigenvalues and - vectors at  $E_0$  are  $\nu(\mu) = (\mu c + ir)/2$  and  $\zeta(\mu) = (\frac{w-ir}{2c}, 1)$  respectively, and their conjugates. Making the substitution

$$(11) \quad \begin{bmatrix} V - \frac{ak}{1-c} \\ S^e - \frac{a}{1-c} \end{bmatrix} = \begin{bmatrix} \frac{r}{2c} & \frac{w}{2c} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

from (6) we get a simple ODE

$$(12) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\mu c}{2} & \frac{r}{2} \\ -\frac{r}{2} & \frac{\mu c}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with only  $\mu$  as a parameter ( $r$  being mere shorthand, fixed by  $\mu$ ).

## 2. The local effect of perturbations

We wish to consider the effects of perturbation on cycles near  $E_0$ . Topological arguments (as in the usual proofs of the Hopf Bifurcation Theorem) show that if we perturb the system slightly, both the curve  $F$  of equilibrium points in Fig.2 and the curve  $C$  corresponding to cycles persist, close to their original positions. However, the existence of infinitely many cycles for a single value  $k = 2(1 - c)/c$  is infinitely unstable; a perturbation could put arbitrarily many wiggles in the vertical part of  $C$ , leaving any finite number of nested attracting and repelling cycles. Zhang [11] has discussed one perturbation of the model that does this, by changing the target stock level from  $kS^e$  to a strictly convex or concave increasing function  $f(S^e)$ , and hence changing the equilibrium; we take a perturbation that leaves the equilibrium itself unchanged, while altering the Taylor expansion there by a clearly small amount.

We retain  $kS^e$  as the target stock level, but the stock adjustment production  $kS^e - V$ , which is precisely appropriate for a discrete adjustment aimed at the end of an accounting period, is less obvious when considered as an instantaneous rate. A manager might want to close a wider gap faster, to keep on top of an evolving situation, or be cautious about responding as fast as  $kS^e - V$  suggests, since a large imbalance suggests a changing environment where the expectation  $S^e$  may be held with less conviction. We do assume smoothness, and without loss of generality a linearised response that matches the original model, so that our stock adjustment term  $f$  has the Taylor expansion

$$(13) \quad f(kS^e - V) = (kS^e - V) + \sigma(kS^e - V)^2 + \rho(kS^e - V)^3 + \dots,$$

where higher terms do not affect our analysis. Since in what follows  $\sigma$  and  $\rho$  may be arbitrarily small, even slight failures of linearity have topological effects, so the onus is on an objector to (13) to argue for *exact* linearity. A particular choice of  $f$  would most importantly have to justify the sign of the associated  $\rho$ ; we merely assume  $\rho \neq 0$ , on grounds of generality, and examine the effects of each sign. The interest here is in crossing the set  $ck = 2(1 - c)$ , inside the region where the eigenvalues

of  $E_0$  are complex. We first outline the results, giving more details of the calculations in §4.

The model now becomes

$$(14) \quad \begin{aligned} \frac{dV}{dt} &= (1 - c) \min [S^e + f(kS^e - V), \bar{Q}] - a \\ \frac{dS^e}{dt} &= a + c \min [S^e + f(kS^e - V), \bar{Q}] - S^e \end{aligned}$$

with the same equilibrium  $E_0$  as before, still assumed “in the Keynesian spirit of these models” [3] to lie in the region of less than full employment, below the curve that replaces the straight line  $W$ . The linearisation around  $E_0$  is exactly the old model below  $W$ . Making the complex substitution

$$(15) \quad \begin{bmatrix} V - \frac{ak}{1-c} \\ S^e - \frac{a}{1-c} \end{bmatrix} = \begin{bmatrix} \frac{r+iw}{2} & \frac{r-iw}{2} \\ i & -i \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix}$$

with  $r$  and  $w$  in §2, (14) gives  $dz/dt$  to third order in  $z$  as

$$(16) \quad \nu(z - \frac{\sigma}{4cr} [r(z + \bar{z}) - iw(z - \bar{z})]^2 + \frac{\rho}{8c^2r} [r(z + \bar{z}) - iw(z - \bar{z})]^3) + \dots$$

with only  $\mu$  as a parameter ( $r, w$  and  $\nu$  being fixed as in §2 by  $\mu$ .) We seek periodic solutions to (16). We begin by mapping cycles to parameter values, rather than by fixing  $\mu$  and looking for a cycle. Here we label  $z(t)$  of period  $2\pi/\omega$  by its value of

$$(17) \quad \epsilon \equiv \frac{\omega}{2\pi} \int_0^{2\pi/\omega} e^{-i\omega t} z(t) dt \equiv [z],$$

which recovers  $\epsilon$  in the simple case  $z(t) = \epsilon e^{i\omega t}$ , so  $\epsilon$  is called the **amplitude**. The Hopf Bifurcation Theorem [2,4,5] guarantees a unique cycle near  $(\mu, z) = (0, 0)$  for each small amplitude  $\epsilon$ , so we can write

$$(18) \quad z(t) = u(\omega(\epsilon)t, \epsilon)$$

for the cycle, with  $u(s + 2\pi, \epsilon) \equiv u(s, \epsilon)$ , and  $\mu(\epsilon)$  for the parameter value at which the cycle occurs. We expand

$$(19) \quad u(s, \epsilon) = \sum_{n=1}^{\infty} u_n(s) \epsilon^n, \quad \omega = \sum_{n=0}^{\infty} \omega_n \epsilon^n, \quad \mu = \sum_{n=0}^{\infty} \mu_n \epsilon^n$$

where the  $u_i(\cdot)$  are themselves functions of period  $2\pi$ , while the  $\omega_n$  and  $\mu_n$  are real coefficients. Since  $[u] = \varepsilon$ , we have  $[u_i] = \delta_{1i}$ ; the calculations in §4 exploit this and the periodicity of the  $u_i$ . Substituting (19) in (16) and equating successive coefficients of  $\varepsilon^n$ , gives  $\omega_1 = \mu_1 = 0$  and

$$(20) \quad \mu_2 = \frac{1-c}{3c^3}(-9\rho(1-c)(2-c) - 2\sigma^2\sqrt{1-c}(4c^2 - 17c + 14)).$$

Since  $\mu = \mu_2\varepsilon^2 + O(\varepsilon^3)$ , for  $\mu_2 > 0$  and  $\varepsilon$  small the cycle occurs for  $\mu > 0$ , surrounding unstable  $E_0$  as a stable cycle (as one can show topologically, or by explicit calculation as in §3 of the Floquet exponents [6] which determine stability). Thus as  $k$  or  $c$  increases, carrying  $\mu = 2 + k - 2/c$  through 0, the post-transient behaviour of the model changes continuously from stability at  $E_0$  to stable oscillation, of amplitude growing like  $\sqrt{\mu}$ . If  $\mu_2 < 0$  the cycle exists near  $E_0$  only for  $\mu < 0$ , and is unstable, so that for  $\mu > 0$  where  $E_0$  is unstable there is no attractor nearby; the system must move off to a distant behaviour—in this case a distant cycle, if the perturbation term  $f(kS^e - V) - (kS^e - V)$  is everywhere small. This attracting cycle must thus coexist with  $E_0$  for parameter values for which  $E_0$  is attractive. A typical example is shown in Fig.3.

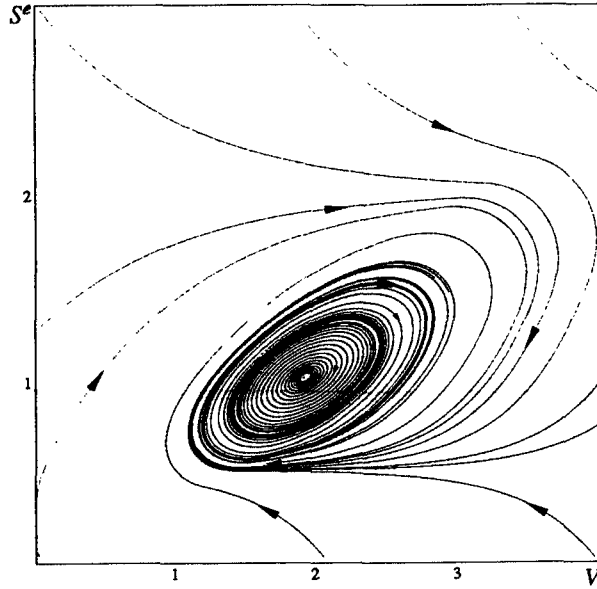
The sign of  $\mu_2$  is usually that of  $\rho$ , since if  $\rho$  and  $\sigma$  are comparably small,  $|\rho| \gg \sigma^2$ ; to first order in  $(\sigma, \rho)$ , positive  $\rho$  gives unstable cycles, negative  $\rho$  stable ones. In the case of equality, higher order terms are needed to determine stability. To illustrate the global behaviour, we numerically compute diagrams (Fig.4) analogous to Fig.2.

### 3. Detailed calculations

As with (12) in §2, (14) is transformed in a neighborhood of  $E_0$  to

$$(21) \quad \begin{aligned} \dot{x}_1 &= \frac{\mu c}{2}x_1 + \frac{r}{2}x_2 - \frac{\mu\sigma}{4r}(rx_1 - wx_2)^2 \\ &\quad + \frac{\mu\rho}{8cr}(rx_1 - wx_2)^3 + \dots \\ \dot{x}_2 &= -\frac{r}{2}x_1 + \frac{\mu c}{2}x_2 + \frac{\sigma}{4c}(rx_1 - wx_2)^2 \\ &\quad - \frac{\rho}{8c^2}(rx_1 - wx_2)^3 + \dots \end{aligned}$$



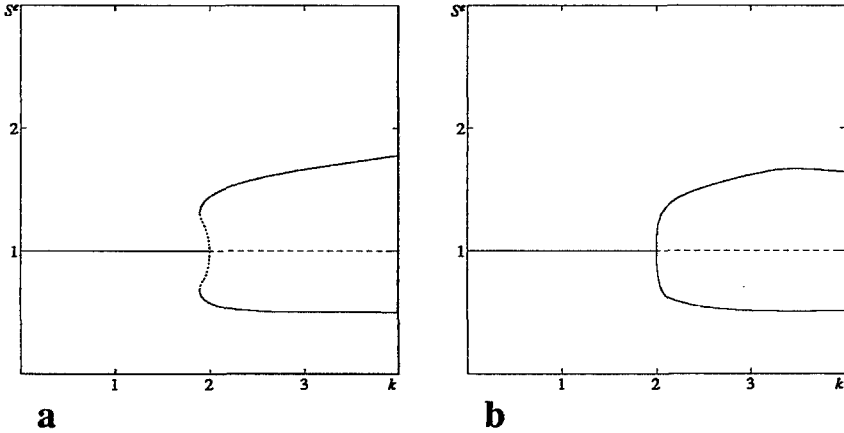


**Figure 3.** The perturbed flow for  $a = c = 0.5$ ,  $k = 1.91$ ,  $\bar{Q} = 3.0$ , with  $\sigma = 0.1$  and  $\rho = 0.5$ . If  $(V, S^e)$  begins near the equilibrium, oscillations will damp out. If it begins outside the inner, unstable cycle, the trajectory will approach the stable cycle.

The linearisation of (21) has eigenvectors  $(1, \pm i)$ . Write a general real vector  $(x_1, x_2)$  in the form  $z(1, i) + \bar{z}(1, -i)$ , with a complex number  $z$ ; thus we get  $x_1 = z + \bar{z}$ ,  $x_2 = (z - \bar{z})i$ . Substituting this in (21), we obtain the amplitude expression (16) (see [6]); let  $\phi(\mu, z, \bar{z})$  denote (16). We seek the periodic solution guaranteed by the Hopf bifurcation theorem near the equilibrium  $E_0$  at  $\mu = 0$ , in the form

$$(22) \quad z(t) = u(s, \varepsilon), \quad s = \omega(\varepsilon)t, \quad \mu = \mu(\varepsilon), \quad \omega(0) = \omega_0 \equiv \eta(0) \text{ for short,}$$

where  $\varepsilon$  is the amplitude of  $z$  defined by (17). We will write  $u(s) \equiv u(s, \varepsilon)$  and  $\dot{u}(s) \equiv \frac{\partial}{\partial s} u(s, \varepsilon)$  for convenience. Since  $\phi(\mu, u, \bar{u})$  is analytic in  $(\mu, u, \bar{u})$ , the solution must be analytic in  $\varepsilon$ . Thus  $u, \omega$  and  $\mu$  may be expressed as series (19) in  $\varepsilon$ . Since  $\frac{d}{dt} z(t) = \frac{d}{dt} u(s, \varepsilon) = \omega(\varepsilon) \frac{\partial}{\partial s} u(s, \varepsilon)$ ,



**Figure 4.** When  $f(kS^e - V) = (kS^e - V) + \sigma(kS^e - V)^2 + \rho(kS^e - V)^3$  precisely, the bifurcation of the perturbed system is as shown in (a) when  $\sigma = 0.1, \rho = 0.5$  and (b) when  $\sigma = 0.01, \rho = -0.1$ . Fig.5 is the phase portrait for the point  $k = 1.91$  in (a); the line  $V = ak/(1+k)$  through the equilibrium meets the repelling and attracting cycles at the points where the link  $k = 1.91$  meets the curve shown here.

we have

$$(23) \quad \omega \dot{u} = \phi(\mu, u, \bar{u}).$$

We know from the condition  $[u] = \varepsilon$  that

$$(24) \quad [u_i] = \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases}.$$

The coefficients of  $u, \omega$  and  $\mu$  can be found by equating coefficients of  $\varepsilon^n$  in the amplitude equation. We have

$$(25) \quad \begin{aligned} \omega \dot{u} &= \left( \omega_0 + \sum_{n=1}^{\infty} \omega_n \varepsilon^n \right) \left( \sum_{n=1}^{\infty} \dot{u}_n(s) \varepsilon^n \right) \\ &= \omega_0 \dot{u}_1 \varepsilon + (\omega_0 \dot{u}_2 + \omega_1 \dot{u}_1) \varepsilon^2 \\ &\quad + (\omega_0 \dot{u}_3 + \omega_1 \dot{u}_2 + \omega_2 \dot{u}_1) \varepsilon^3 + \dots \end{aligned}$$

With the convenient assistance of **Mathematica**, the right side of (23) can be expressed as follows:

(26)

$$\begin{aligned} \phi(\mu, u, \bar{u}) &= i\sqrt{1-c}u_1\varepsilon + \left( i\sqrt{1-c}u_2 - \frac{2\sigma(1-c)}{c}|u_1|^2 \right. \\ &\quad \left. - \frac{\sigma(1-c)}{2}(u_1 - \bar{u}_1)^2 + \frac{i(1-c)^{\frac{3}{2}}\sigma}{c}(u_1^2 - \bar{u}_1^2) + \frac{c\mu_1}{2}u_1 \right) \varepsilon^2 \\ &\quad + \left( U_1 \left( -\frac{\sigma i(1-c)}{c}U_2 - \frac{\sigma\mu_1\sqrt{1-c}}{2}(u_1 - \bar{u}_1) \right) \right. \\ &\quad \left. + \frac{i\rho(1-c)^{\frac{3}{2}}}{2c^2}U_1^3 + i\sqrt{1-c}u_3 + \frac{\mu_1u_2}{2} - \frac{\sigma\mu_1\sqrt{1-c}}{4c}U_1^2 \right. \\ &\quad \left. + \left( -\frac{ic^2\mu_1^2}{8\sqrt{1-c}} + \frac{c\mu_2}{2} \right) u_1 \right) \varepsilon^3 + O(\varepsilon^4) \end{aligned}$$

where  $U_1 = (u_1 + \bar{u}_1 - i\sqrt{1-c}(u_1 - \bar{u}_1))$ ,  $U_2 = (u_2 + \bar{u}_2 - i\sqrt{1-c}(u_2 - \bar{u}_2))$ .

Equating coefficients, we find at order 1 that  $\dot{u}_1 - iu_1 = 0$ , that is,  $u_1 = e^{is}$  since  $[u_1] = 1$ . At order 2, we have

$$\begin{aligned} \sqrt{1-c}\dot{u}_2 + i\omega_1e^{is} &= i\sqrt{1-c}u_2 - \frac{2\sigma(1-c)}{c} - \frac{\sigma(1-c)}{2}(e^{is} - e^{-is})^2 \\ &\quad + \frac{i\sigma(1-c)^{\frac{3}{2}}}{2}(e^{2is} - e^{-2is}) + \frac{c\mu_1}{2}e^{is}. \end{aligned}$$

An ODE of the form  $\dot{u}(s) - iu(s) = f(s) = f(s + 2\pi)$  is solvable for  $u(s) = u(s + 2\pi)$  if and only if the Fourier expansion of  $f(s)$  has no term proportional to  $e^{is}$ .

*Proof.* If  $f = a_0 + a_1e^{is} + a_2e^{2is} + \dots$ , then  $u(s) = u(0)e^{is} + e^{is} \int_0^s f(t)e^{-it}dt = u(0)e^{is} + e^{is}((e^{-is} - 1)i + a_1s - a_2(e^{is} - 1)i + \dots)$ . Since  $u(s) = u(s + 2\pi)$ ,  $a_12\pi = 0$ , or  $a_1 = 0$ . The converse is trivial.) Therefore,  $\omega_1 = \mu_1 = 0$ , so that  $\dot{u}_2 - iu_2$  becomes

$$\frac{\sqrt{1-c}}{2c}(e^{2is}(2\sigma i\sqrt{1-c} - \sigma c) - e^{-2is}(\sigma c + 2\sigma i\sqrt{1-c}) - 2\sigma(2-c)),$$

yielding  $u_2 = e^{is}(D + \frac{\sigma i\sqrt{1-c}}{2c}(A(e^{is_0} - e^{is}) - \frac{B}{3}(e^{-3is_0} - e^{-3is}) + C(e^{-is} - e^{-is_0})))$ , where  $s_0$  is an initial time (say,  $s_0 = 0$ ),  $A = (2i\sqrt{1-c} - c)$ ,

$B = -\bar{A}$ ,  $C = 2(c - 2)$ , and  $D = u_2(s_0)$ , is the solution. By (24),  $D = \frac{\sigma i \sqrt{1-c}}{2c} (C e^{-is_0} - A e^{is_0} + \frac{B}{3} e^{-3is_0})$ ; this means that  $u_2(s) = \frac{\sigma i \sqrt{1-c}}{2c} (C - A e^{2is} - \frac{\bar{A}}{3} e^{-2is})$ . At order 3,

$$(27) \quad \begin{aligned} & \sqrt{1-c} \dot{u}_3 - i\sqrt{1-c} u_3 + \omega_2 i e^{is} \\ &= \frac{-i\sigma(1-c)}{c} U_1 U_2 + \frac{i\rho(1-c)^{\frac{3}{2}}}{2c^2} U_1^3 + \frac{c\mu_2}{2} u_1. \end{aligned}$$

Defining  $q(s)$  as the RHS of (27),  $\dot{u}_3 - iu_3 = (q(s) - i\omega_2 e^{is})/\sqrt{1-c}$ , with the general solution

$$u_3(s) = u_3(0)e^{is} + i \frac{e^{is}}{\sqrt{1-c}} \int_0^s e^{-it} (q(t) - i\omega_2 e^{it}) dt.$$

With the help of **Mathematica**, we obtain

$$(28) \quad \begin{aligned} & \frac{1}{\sqrt{1-c}} \int_0^s (q(t)e^{-it} - i\omega_2) dt \\ &= 7\sigma^2 s + \frac{14\sigma^2 s}{3c^2} - \frac{31\sigma^2 s}{3c} - \frac{4c\sigma^2 s}{3} + \frac{3i\rho s}{2} + \frac{3i\rho s}{c^2} - \frac{9i}{2c\rho s} \\ &+ \frac{c\mu_2 s}{2\sqrt{1-c}} + \frac{11i\sigma^2 s}{(1-c)^{\frac{3}{2}}} + \frac{2i\sigma s}{c^2(1-c)^{\frac{3}{2}}} - \frac{23i}{3\sigma^2 s c(1-c)^{\frac{3}{2}}} - \frac{7ic\sigma^2 s}{(1-c)^{\frac{3}{2}}} \\ &+ \frac{5i}{(1-c)^{\frac{3}{2}} 3c^2 \sigma^2 s} + \frac{27\rho s}{2(1-c)^{\frac{3}{2}}} + \frac{3\rho s}{c^2(1-c)^{\frac{3}{2}}} \\ &- \frac{21\rho s}{3c(1-c)^{\frac{3}{2}}} - \frac{15c\rho s}{2(1-c)^{\frac{3}{2}}} + \frac{3c^2\rho s}{2(1-c)^{\frac{3}{2}}} \\ &- \frac{is\omega_2}{(1-c)^{\frac{3}{2}}} + \frac{ics\omega_2}{(1-c)^{\frac{3}{2}}} - \frac{i\omega_2 s}{\sqrt{1-c}} \\ &+ (\text{terms with exponentials like } e^{2is}, e^{-2is}, e^{is}, \text{ or } e^{-is}). \end{aligned}$$

Since  $u_3(s) = u_3(s + 2\pi)$ , the expression (28) without the exponential terms must vanish. Since  $c, \mu_2, \omega_2, s, \sigma$ , and  $\rho$  are all real, we have

$$(29) \quad \mu_2 = \frac{1-c}{3c^3} (-2\sigma^2 \sqrt{1-c} (4c^2 - 17c + 14) - 9\rho(1-c)(2-c)).$$

Continuing this procedure, we get

$$(30) \quad \mu(\varepsilon) = \mu_2\varepsilon^2 + O(\varepsilon^3).$$

Now we may factorise the Floquet exponent  $\gamma(\varepsilon)$  (see [6], p.133) as

$$(31) \quad \begin{aligned} \gamma(\varepsilon) &= \mu_\varepsilon(\varepsilon)\hat{\gamma}(\varepsilon) = -\mu_\varepsilon(\varepsilon)(\xi_\mu(0)\varepsilon + O(\varepsilon^2)) \\ &= \frac{1-c}{3c^2}(9\rho(1-c)(2-c) \\ &\quad + 2\sigma^2\sqrt{1-c}(4c^2 - 17c + 14))\varepsilon^2 + O(\varepsilon^3). \end{aligned}$$

Since stability occurs for negative exponents, instability for positive ones, and  $c \leq 1$  at all points where the trace of the linearisation vanishes, we may summarise for small  $\varepsilon$  as follows:

**THEOREM 1.** *The system (14) shows Hopf bifurcation unless  $\rho$  is equal to  $-\frac{2\sigma^2(4c^2-17c+14)}{9\sqrt{1-c}(2-c)}$  exactly. If  $\rho$  is less than this, the limit cycles near  $E_0$  appear for  $k$  greater than the critical value  $k_0 = 2(1-c)/c$ , and are stable; otherwise they appear for  $k$  below  $k_0$  and are unstable. In each case, as  $k$  approaches  $k_0$  from the appropriate side, they shrink to  $E_0$ .*

This confirms the stability description obtained in §3 from the general character of Hopf bifurcation, and tested numerically.

#### 4. Concluding remarks

In Eckalbar's paper, as in Fig.1, if  $k$  is larger than the bifurcation value  $k_0 = 2(1-c)/c$ , then orbits tend to a stable cycle, that is, the economy fluctuates. If  $k = k_0$ , the economy also oscillates and if  $k < k_0$ , the economy tends to stasis.

In our perturbed model, as we see in Fig.3, a stable and an unstable cycle and a stable equilibrium can coexist if  $k$  is less than the bifurcation value and  $k$  is just below that value.

Eckalbar studied productions and inventory fluctuations depending on only sale expectations and stocks as variables and ignored wages and prices, but in reality prices have a large effect, and the wage/price bargaining he assumed instantaneous (so reducing the dimension of the model) can be slow. Our future main object, therefore, is to analyse

the model with price fluctuations, but here we have discussed only the dynamical effects of a small change in Eckalbar's model; a change in dimension is a large change, and could be expected to have large effects. We have displayed qualitative difference with an arbitrarily small perturbation.

The locality of our analysis keeps it away from the switching line, which was Eckalbar's original mechanism to produce bounded oscillation. Some global phenomena involving this line are discussed in [8].

Finally, we like Eckalbar have worked in a continuous time model, though his motivation of the model is in terms of discrete time in units of a planning period. Implementing this underlying model numerically produces apparent chaotic attractors of quite varied geometry, on which we hope to report elsewhere.

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