J. Korean Math. Soc. 28(1991), No. 2, pp. 207–213

ON WHOLE REGULAR GERMS FOR p–ADIC $S_{p_4}(F)$

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Entire regular germs associated with any maximal torus contained in p-adic $S_{p_4}(F)$ are desirable with the appropriate conditions after we have found the regular germs associated with a particular Cartan subgroup in p-adic $S_{p_4}(F)$. This generalization is done in this paper.

0. Introduction

Ever since J.A. Shalika defined the germs in his thesis [SHJ], many mathematicians such as Harish Chandra, R. Howe, R. Langlands, J. Rogawski, J. Repka and others have contributed to the establishment of the germs for p-adic linear algebraic groups. In particular, it is said that R. Langlands and D. Shelstad set up in principle some theorems about the germs around 1986, but didn't tell the exact conditions under which those are valid or nonzeros.

In the meantime, the regular and subregular germs for p-adic $GL(n)$ and $SL(n)$ were found by J. Repka from 1983 to 1985. Recently the regular germs for p-adic $S_{p_4}(F)$ were found by the author. But these were associated to a particular elliptic torus. So the author intended to complete the establishment of the regular germs associated to any maximal torus in p-adic $S_{p_4}(F)$.

1. Prerequisite propositions

Most notations and conventions are those shown up in [KY]. Tracing this, we shall properly modify important statements which should be mended in time of need. Some of the propositions therein are mentioned without proofs.

PROPOSITION (1.0). *Every unipotent orbit in the semi-simple algebraic group $G = S_{p_4}(F)$ over a p-adic field F with odd residual*

Received June 29, 1990. Revised February 28, 1991.

characteristic intersects the set of all elements of the form

$$\begin{bmatrix} 1 & x & \alpha & \beta \\ 0 & 1 & \beta-\gamma x & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{bmatrix}$$

with any α, β, γ and $x \in F$. Here the action is just the conjugation.

PROPOSITION (1.1). *The regular unipotent orbits are represented by the matrices of the form*

$$\begin{bmatrix} 1 & 1 & 0 & \delta \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

with $\delta \in F^x/(F^x)^2$.

Put $u(\bar{a}) = \begin{bmatrix} 1 & 1 & 0 & \bar{a} \\ 0 & 1 & 0 & \bar{a} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ *and let $S(1) = \{g \in K = S_{p_4}(O) : g \equiv u(1) \mod \tilde{P} = P^r$ for a fixed $r \in Z^+\}$. Here O is the ring of integers in F with the maximal ideal P such that $|O/P| = q$.*

Set $S(\bar{a}) = d(\sqrt{\bar{a}})S(1)d(\sqrt{\bar{a}})^{-1} \ni u(\bar{a})$, where $d(\sqrt{\bar{a}})$ is the diagonal matrix $\mathrm{diag}(\sqrt{\bar{a}}, \sqrt{\bar{a}}, \sqrt{\bar{a}}^{-1}, \sqrt{\bar{a}}^{-1})$ with $\bar{a} \in F^x/(F^x)^2$.

Now every element of $S(1)$ is G-conjugate to the form

$$\begin{bmatrix} (1-p_{43})^{-1} & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -p_{41} & 0 & 1-p_{43} & 0 \\ p_{41} & p_{44} & -1+p_{43} & 1+p_{44} \end{bmatrix} \tag{1.2}$$

by 7 conjugation maps, where p_{41}, p_{43} and p_{44} are arbitrary in $\tilde{P}$. Let S_3 be the set of all elements of this form and let $\hat{P}$ be the composite map of these conjugation maps, whose Jacobian has modulus 1. This shall be used later to yield Proposition (4.0). (cf. 5 [KY]).

PROPOSITION (1.3). *The unipotent orbit* $u(\bar{b})$ *in* G *intersects* $S(\bar{a})$ *iff* $\bar{b}/\bar{a} \in (F^x)^2$.

As for the relation of $u(\bar{a})$ with $S(\bar{a})$, we have at hand

PROPOSITION (1.4). *The only unipotent orbit of* $u(\bar{a})$ *intersects* $S(\bar{a})$ *out of all unipotent orbits, i.e., the only unipotent conjugacy class intersecting* $S(\bar{a})$ *is that of* $u(\bar{a})$.

2. Shalika's formula

Let T be the set of all matrices of the form

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ b\theta_1 & 0 & a & 0 \\ 0 & \beta\theta_2 & 0 & \alpha \end{bmatrix}$$

in $G = S_{p_4}(F)$, where $\theta_j \in F^x\backslash(F^x)^2$ and $a^2 - b^2\theta_1 = 1$, $\alpha^2 - \beta^2\theta_2 = 1$. Then we easily see that T is an elliptic torus as a Cartan subgroup.

According to J.A. Shalika, there is an asymptotic expansion

$$\int_{T\backslash G} f(t^g)d\dot{g} = \sum_{j=1}^{n} \Gamma_j(t) \int_{Z(u_j)\backslash G} f(u_j^g)d\dot{g} \tag{2.0}$$

for any $f \in C_c^\infty(G)$, a regular element $t \in T'$ sufficiently close to the identity, and a finite set $\{u_j\}$ of representatives of the unipotent orbits. Of course, T may be any maximal torus in G. Here the functions Γ_j called Shalika's germs do not depend on any locally constant function with compact support, but depend on $t \in T'$ (the set of all regular elements in T).

We are concerned about computing the functions $\Gamma_{\bar{a}}(t)$ corresponding to the element $u(\bar{a})$, given any other Cartan subgroup $°T$ of G. Putting $f = \chi_{S(\bar{a})}$, the characteristic function of the set $S(\bar{a})$, will facilitate for us to compute the germs concerned. For the integrals except for the one corresponding to $u(\bar{a})$ all vanish in this situation thanks to the proposition (1.4).

3. Previous conditions

It is known that all Cartan F-subgroups of G are conjugate over the algebraically closed field $\bar{F}$. So there exists $h \in S_{p_4}(\bar{F})$ satisfying $^{\circ}T = T^h \cap G$, where T is an F-maximal torus in $S_{p_4}(\bar{F})$. For example, h might be $\operatorname{diag}(\sqrt{a}, \sqrt{a}, \sqrt{a}^{-1}, \sqrt{a}^{-1}) = d(\sqrt{a})$ with $a \in F^x$.

Now we are going to determine whether we may find $g \in G$ satisfying $^{\circ}t^{gd(\sqrt{\bar{a}})} = s_3$ with $s_3 \in S_3$, $^{\circ}t \in {}^{\circ}T'$ (the set of all regular elements in $^{\circ}T$). But we have seen in [KY] that if $\bar{a}b/2P(\alpha - a) \in N_F^{E^{\theta_1}}((E^{\theta_1})^x)$ and $\bar{a}\beta/2Q(a-\alpha) \in N_F^{E^{\theta_2}}((E^{\theta_2})^x)$ with $P = (1-p_{43})^2 - 2a(1-p_{43})+1$, $Q = (1-p_{43})^2 - 2\alpha(1-p_{43})+1$, then there exists $g \in G$ satisfying $t^g = s$ for every $s \in S(\bar{a})$ satisfying $p_{41} \neq 0$ in the form (1.2) and satisfying $p_{41} = (1 - p_{43} + \frac{1}{1-p_{43}})(2+p_{44}) - 4a\alpha$, $p_{44} = 2(a+\alpha) - 4 + p_{43} - \frac{p_{43}}{1-p_{43}}$. The converse is also true. Moreover the condition $p_{41} \neq 0$ may be ignored in the sense of measure.

Hence the impending question may now be changed to the form whether we may find $x \in G$ such that $^{\circ}t^x = t^{hx} = s$ for each $s \in S(\bar{a})$ or not.

4. Generalized conditions

As we have mentioned, any Cartan subgroup or maximal torus in the case of $G = S_{p_4}(F)$ is conjugate by some element $h \in S_{p_4}(\overline{F})$ to the given elliptic torus T as in [KY], where we observed that for a regular element $t \in T$ it does not concern with a chosen $s_3 \in S_3$ (except possibly for the set of measure zero) subject to $\operatorname{char}(t) = \operatorname{char}(s_3)$ whether it belongs to $t^{G\, d(\sqrt{\bar{a}})} = \{t^{gd(\sqrt{\bar{a}})} : g \in G\}$ or not.

Analogously by using this fact, we may conclude that it does not concern with a chosen $s_3 \in S_3$ (except possibly for the set of measure zero) subject to $\operatorname{char}(t^h) = \operatorname{char}(s_3)$ whether or not $s_3 \in (t^h)^{Gd(\sqrt{\bar{a}})} = \{t^{hgd(\sqrt{\bar{a}})} : g \in G\}$. So we may put $p_{43} = 0$ in S_3 from the first, since $p_{43} = 0$ is excluded from the exceptional set.

In summary, we may arrange our results about orbital integrals including our previous conditions as follows :

PROPOSITION (4.0). *Suppose that* $^{\circ}T = T^h \cap G$ *for some* $h \in S_{p_4}(\bar{F})$. *Then the orbital integral is obtained in two cases with the normalized Haar measures as in 6* [KY].

(i) *Let* X *be the matrix*

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 4(a-1)(\alpha-1) & 0 & 1 & 0 \\ -4(a-1)(\alpha-1) & 2(a+\alpha)-4 & -1 & 2(a+\alpha)-3 \end{bmatrix}$$

If $X \in (t^h)^{Gd(\sqrt{\bar{a}})}$ *with* t^h *sufficiently close to the identity, then*

$$\int_{^\circ T\backslash G} \chi_{S(\bar{a})}(^\circ t^g)d\dot{g} = q^{-8r} \times |D(^\circ t)|^{-\frac{1}{2}}, \text{ where } r \in \mathbf{Z}^+;$$

(i)' *In particular, if* $h \in G$, *the above condition is equivalent to* $\frac{\bar{a}b}{(1-a)(\alpha-a)} \in N_F^{E^{\theta_1}}((E^{\theta_1})^x)$ *and* $\frac{\bar{a}\beta}{(1-\alpha)(a-\alpha)} \in N_F^{E^{\theta_2}}((E^{\theta_2})^x)$ *in a more concrete version as in* [KY];

(ii) *Otherwise,* $\int_{^\circ T\backslash G} \chi_{S(\bar{a})}(^\circ t^g)d\dot{g} = 0$.

Proof. For a fixed regular element $^\circ t \in {}^\circ T$, we set up a mapping $c^\circ t : {}^\circ T\backslash G \to G$ via $c^\circ t(g) = g^{-1}{}^\circ tg = {}^\circ t^g$. Put $E(^\circ t) = (c^\circ t)^{-1}(S(\bar{a}))$. The measure of $E(^\circ t)$ just equals $\int_{^\circ T\backslash G} \chi_{S(\bar{a})}(^\circ t^g)d\dot{g}$. Let $\hat{P} : S(1) \to S_3 \times (\tilde{P})^7$ be the composite map defined in [1]. Let $c^{\bar{a}}$ be the map $S(\bar{a}) \to S(1)$ given by $d(\sqrt{a})sd(\sqrt{a})^{-1} \to s$ for an arbitrary $s \in S(1)$. Next define the map $P' : S_3 \times (\tilde{P})^7 \to \tilde{P} \times (\tilde{P})^7$ which acts on the first factor as $(p_{41}, p_{43}, p_{44}) \to p_{43}$ and acts as identity on the 2nd factor. So we have a composite map

$$P' \circ \hat{P} \circ c^{\bar{a}} \circ c^\circ t : E(^\circ t) \rightarrowtail S(\bar{a}) \twoheadrightarrow S(1) \twoheadrightarrow S_3 \times \tilde{P}^7 \twoheadrightarrow \tilde{P} \times (\tilde{P})^7.$$

As in 5, 6 in [KY], this map is bijective except possibly for a set of measure zero and has the modulus of its Jacobian equal to $|D(^\circ t)|^{\frac{1}{2}} = \det[id - Ad(^\circ t)]_{\mathcal{G}/^\circ\mathbf{T}}$, where $\mathcal{G}$ and $^\circ\mathbf{T}$ are the associated Lie algebras of G and $^\circ T$ respectively. So we have the result immediately considering the above remarks preceding this proposition.

PROPOSITION (4.1). *With the normalized Haar measures as in* [KY], *we have*

$$\int_{Z(u(\bar{a}))\backslash G} \chi_{S(\bar{a})}(u(\bar{a})^g)d\dot{g} = q^{-8r}$$

Proof. See (6.2) 6 in [KY].

5. Final results

We have seen that if $\theta_1 \cdot \theta_2 \in (F^x)^2$, then there always exists $t \in T'$ at which all the regular germs vanish. So we may say in the general case of $°T$ that all the regular germs may or may not vanish. Finally combining everything, we see trivially

THEOREM (5.0). *The regular germs $\Gamma_{u(\bar{a})}(°t)$ with $°t = t^h \in °T = T^h \cap G$ associated to the representative unipotent matrices $u(\bar{a})$ are as follows:*

(i) *If $X \in (t^h)^{Gd(\sqrt{\bar{a}})}$ with t^h sufficiently close to the identity, then*

$$\Gamma_{u(\bar{a})}(°t) = |D(°t)|^{-\frac{1}{2}};$$

(i)′ *In particular in the case of $h \in G$, the above condition is reduced to*

$$\frac{\bar{a}b}{(1-a)(\alpha-a)} \in N_F^{E^{\theta_1}}((E^{\theta_1})^x) \text{ and } \frac{\bar{a}\beta}{(1-\alpha)(a-\alpha)} \in N_F^{E^{\theta_2}}((E^{\theta_2})^x).$$

(ii) *Otherwise, $\Gamma_{u(\bar{a})}(°t) = 0$.*

REMARKS. (A) Odd residual characteristic is needed for calculation of some Jacobians and the number of unipotent conjugacy classes in $G = S_{p_4}(F)$.

(B) Other normalization of measures is plausible as is seen in Harish-Chandra's related papers. For example, he used to include $|D(°t)|^{\frac{1}{2}}$ into the measure on the homogeneous space $°T\backslash G$.

(C) In general, when a small neighborhood of a unipotent element and a regular element t of a maximal torus are given, it is dependent not only on t but also on elements in a set not of measure zero whether an element in that neighborhood belongs to t^G or not as will be shown in the author's paper in preparation for subregular germs in $S_{p_4}(F)$. As far as the author knows, such a phenomenon has not appeared yet.

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