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A NOTE ON EQUIVALENT INTERVAL COVERING SYSTEMS FOR PACKING DIMENSION OF R

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1. Introduction

A useful outer measure for a fractal set in **R** is Hausdorff measure defined by using economical coverings of bounded sets in **R**. The Hausdorff dimension of a set in **R** is usually defined by considering countable coverings of the set by general intervals. Recently, C.D. Cutler [2] showed that an *open* bounded Vitali covering instead of the general intervals produces the same value for Hausdorff dimension. Recently, Taylor and Tricot [7] introduced a new outer measure, packing measure. The packing measure of a set in **R** is defined by using maximal packings from open intervals centered in the set. In the same way as Hausdorff dimension is defined, one uses packing measure to define packing dimension ([7], [8]). In the present paper, squeezing the packing dimension with respect to a bounded Vitali covering into two equivalent packing dimensions, we show that the bounded Vitali covering gives us the same value for the packing dimension with respect to the centered open intervals. That is, we generalize the conditions of the families [7], from which packings are extracted, to give the same value for packing dimension. Whereas C.D. Cutler [2] introduced a method to compute the Hausdorff dimension of the sets defined by certain types of generalized expansions, using the theories of Billingsley [1], we obtain the corresponding method for packing dimension, using the Vitali covering theorem due to Miguel de Guzman [4]. We end up with one application of our method.

2. Preliminaries

A family $\{I_i\}_{i=1}^{\infty}$ of bounded disjoint intervals, is called a *packing* of E if $\overline{I}_i \cap \overline{E} \neq \emptyset$ for each i [7].

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We say $\{I_i\}_{i=1}^{\infty}$ is a *g–packing* of E if it is a bounded disjoint subfamily of g, and each I_i contains some elements in E, where g is a collection of intervals.

Let $\alpha > 0$. We obtain the *α–dimensional g–packing measure* p_g^{α} by a two–stage definition using the pre–measure $\mathbf{P}_g^{\alpha}$ defined for bounded sets $E \subset \mathbf{R}$ as follows :

For simplicity, we write diam $I = d(I)$ henceforth.

$$(1) \qquad \mathbf{P}_g^{\alpha}(E) = \lim_{\delta \downarrow 0} \sup\{\sum_{i=1}^{\infty}[d(I_i)]^{\alpha} : \{I_i\}_{i=1}^{\infty} \text{ is a } g\text{ -packing of } E \text{ and } d(I_i) \leq \delta \text{ for each } i\}.$$

Since $\mathbf{P}_g^{\alpha}$ is not an outer measure in general [7], we employ Method I by Munroe [5] to obtain the outer measure p_g^{α} :

$$(2) \qquad p_g^{\alpha}(E) = \inf\{\sum_{n=1}^{\infty} \mathbf{P}_g^{\alpha}(E_n) : E_n \text{ are bounded, } E \subset \cup_{n=1}^{\infty} E_n\}.$$

In fact,

$$(3) \qquad p_g^{\alpha}(E) = \inf\{\sum_{n=1}^{\infty} \mathbf{P}_g^{\alpha}(E_n) : E_n \text{ are bounded, } E = \cup_{n=1}^{\infty} E_n\}.$$

Similarly we define $\mathbf{P}_O^{\alpha}(E)[\mathbf{P}^{\alpha}(E)]$ for packing of E consisting of open intervals [open intervals with center in E] instead of g–packing of E in (1).

We also define $p_O^{\alpha}(E)[p^{\alpha}(E)]$ using $\mathbf{P}_O^{\alpha}[\mathbf{P}^{\alpha}]$ in (2). In that case, we call $\mathbf{P}_O^{\alpha}[\mathbf{P}^{\alpha}]$ the *α–dimensional open packing pre–measure* [the *α–dimensional packing pre–measure*], and $p_O^{\alpha}[p^{\alpha}]$ the *α–dimensional open packing measure* [the *α–dimensional packing measure*] (see [7]).

As Hausdorff dimension is defined, we use packing measure to define packing dimension. Before we define packing dimension, we define *rarefaction index* :

$$(4) \qquad \Delta(E) = \sup\{\alpha > 0 : \mathbf{P}^{\alpha}(E) = \infty\} = \inf\{\alpha > 0 : \mathbf{P}^{\alpha}(E) = 0\}$$

for a bounded set $E \subset \mathbf{R}$.

In fact, $\dim(E) \leq \Delta(E) \leq 1$ for a bounded set $E \subset \mathbf{R}$, where $\dim(E)$ is the Hausdorff dimension of E [7].

Now, we define *packing dimension* :

$$(5) \quad \text{Dim}\,(E) = \sup\{\alpha > 0 : p^{\alpha}(E) = \infty\} = \inf\{\alpha > 0 : p^{\alpha}(E) = 0\}$$

for any set $E \subset \mathbf{R}$.

We recall $\dim(E) \leq \text{Dim}\,(E) \leq \Delta(E)$ for any bounded set $E \subset \mathbf{R}$.

Similarly, we can define $\Delta_O[\Delta_g]$ and Dim_O $[\text{Dim}_g]$ for the pre-measure $\mathbf{P}_O[\mathbf{P}_g]$ and the outer measure $p_O[p_g]$.

In fact, $\Delta(B) = \Delta_O(B)$ for any bounded $B \subset \mathbf{R}$ (Cor. 3.9, [7]), hence $\text{Dim}(E) = \text{Dim}_O(E)$ for any $E \subset \mathbf{R}$ (Cor. 5.9, [7]). (6)

To compare Dim_g with Dim, we first prove the followings.

LEMMA 2.1. *Let g be a collection of intervals. Then $\Delta_g(B) \leq \Delta(B)$ for all bounded $B \subset \mathbf{R}$, hence $\text{Dim}_g(E) \leq \text{Dim}(E)$ for all $E \subset \mathbf{R}$.*

Proof. Since $\Delta(B) = \Delta_O(B)$, it is sufficient to show that $\mathbf{P}_g^{\alpha}(B) \leq \mathbf{P}_O^{\alpha}(B)$ for bounded $B \subset \mathbf{R}$.

Suppose that $\{I_i\}_{i=1}^{\infty}$ is a g-packing of B with $d(I_i) \leq \delta$. Then $\{I_i^o\}_{i=1}^{\infty}$ is a packing of B consisting of open intervals with $d(I_i^o) \leq \delta$, since I_i are intervals.

Hence, by the definitions of packing pre-measures, $\mathbf{P}_g^{\alpha}(B) \leq \mathbf{P}_O^{\alpha}(B)$.

We finish this section by stating the following fact.

LEMMA 2.2. *$\text{Dim}_g(E) = \sup_n \text{Dim}_g(E_n)$ where $E = \cup_{n=1}^{\infty} E_n$.*

Proof. It is immediate from the subadditivity of the outer measure p_g^{α}.

3. Alternative packing dimension

Let F be a closed interval of $\mathbf{R}$.

A collection g of intervals is called a *bounded Vitali covering* of F if, for each $x \in F$, there exists a sequence of intervals from g (which we will denote by $\{I_i(x)\}_i$) such that

(i) $x \in I_i$ for each i,

(ii) $d(I_i(x)) \downarrow 0$, (7)

(iii) $\sup_i \frac{d(I_i(x))}{d(I_{i+1}(x))} = b(x) < \infty$.

The next result is one of our main theorems. It establishes sufficient conditions under which packings whose members are restricted to a particular family g of intervals produce the same value for packing dimension.

We note that g is not restricted to an *open* bounded Vitali covering (cf. Theorem 3.1, [2]).

THEOREM 3.1. *Let g be a bounded Vitali covering of a closed interval $F \subset \mathbf{R}$. Then*

$$Dim_g(E) = Dim(E) \text{ for all } E \subset F.$$

Proof. By Lemma 2.1, we only need to show that $\mathrm{Dim}_g(E) \geq \mathrm{Dim}(E)$. For each $x \in E$, we can choose $\{I_i(x)\}_i$ satisfying (7) from g.

Then $E = \cup_{n=2}^{\infty} \cup_{m=1}^{\infty} E_{n,m}$, where

$$E_{n,m} = \{x \in E : \sup_j \frac{d(I_j(x))}{d(I_{j+1}(x))} \leq n, \quad d(I_1(x)) \geq \frac{1}{3m}\}.$$

Recalling Lemma 2.2, we only need to show that

$$\mathrm{Dim}_g(E_{n,m}) \geq \mathrm{Dim}(E_{n,m}) \text{ for all } n, m.$$

Fix n and m. By (3), it is sufficient to show that $\mathbf{P}^\alpha(A) \leq C_\alpha \mathbf{P}_g^\alpha(A)$ for all bounded $A \subset E_{n,m}$, where C_α is a positive constant.

Let $0 < \delta < \frac{1}{m}$. Consider a packing $\{V_k\}_{k=1}^{\infty}$ of open intervals centered in A and $d(V_k) \leq \delta$ for each k.

For the center x_k of V_k, there exists $I_{j(k)}(x_k)$ such that $3d(I_{j(k)-1}(x_k)) \geq d(V_k) > 3d(I_{j(k)}(x_k))$, where $j(k) = \min\{i : d(V_k) > 3d(I_i(x_k))\}$. Therefore $I_{j(k)}(x_k) \subset V_k$.

Thus, $\{I_{j(k)}(x_k)\}_{k=1}^{\infty}$ is a g–packing of A with $d(I_{j(k)}(x_k)) \geq \frac{d(V_k)}{3n}$ for each k.

Hence, $\sup\{\sum_{k=1}^{\infty}[d(V_k)]^\alpha : \{V_k\}$ is a packing of open intervals centered in A with $d(V_k) \leq \delta\} \leq \sup\{\sum_{k=1}^{\infty}[3nd(I_{j(k)}(x_k)]^\alpha : \{V_k\}$ is a packing of open intervals centered in A with $d(V_k) \leq \delta\} \leq (3n)^\alpha \sup\{\sum_{i=1}^{\infty}[d(I_{h_i}(y_i)]^\alpha$: $\{I_{h_i}(y_i)\}_{i=1}^{\infty}$ is a g–packing of A with $d(I_{h_i}(y_i)) \leq \frac{\delta}{3}\}$.

Thus, $\mathbf{P}^\alpha(A) \leq (3n)^\alpha \mathbf{P}_g^\alpha(A)$.

REMARK 3.2. If $g = \Gamma^*$, the collection of dyadic intervals or $g = \Gamma^{**}$, the collection of subdyadic intervals, then g is a bounded Vitali covering.

Hence $\mathrm{Dim}_g(E) = \mathrm{Dim}(E)$ by our theorem, as it was shown in the corollary 5.9 of [7].

Moreover our theorem shows that $\mathrm{Dim}_g(E) = \mathrm{Dim}(E)$, where g is the collection of r–adic intervals.

4. Computing packing dimension

A *generalized expansion* of a number in $[0,1]$ will be defined as follows [2]. For each $n = 1, 2, \cdots$ let $k_n \geq 2$ be an integer and choose values $0 < \alpha_{n,1} < \cdots < \alpha_{n,k_n-1} < 1$, setting $\alpha_{n,0} = 0$ and $\alpha_{n,k_n} = 1$. The initial proportions $\alpha_{1,1}, \cdots, \alpha_{1,k_1-1}$ determine a division of $[0,1]$ into the disjoint intervals $[\alpha_{1,i}, \alpha_{1,i+1}), i = 0, 1, \cdots, k_1 - 2$, and $[\alpha_{1,k_1-1}, 1]$. We will indicate that a point x in $[0,1]$ falls into the i^{th} interval ($i = 0, 1, \cdots, k_1 - 1$) by the notation $I_1(x) = i$. $I_1(x)$ will be the first term in the expansion of x (with respect to the choices $\alpha_{n,i}$). At the second stage each interval $\{x : I_1(x) = i\}$ is divided into k_2 disjoint subintervals determined by the given proportions $a_{2,1}, \cdots, a_{2,k_2-1}$. This splits $[0,1]$ into $k_1 k_2$ disjoint intervals which are most conveniently expressed in the form $\{x : I_1(x) = i, I_2(x) = j\}$ for some choice of $i = 0, 1, \cdots, k_1 - 1$ and $j = 0, 1, \cdots, k_2 - 1$. Letting $d_{n,i} = \alpha_{n,i+1} - \alpha_{n,i}$ for each n and i, we can alternately write $\{x : I_1(x) = i, I_2(x) = j\}$ $= \{x : \alpha_{1,i} + \alpha_{2,j} d_{1,i} \leq x < \alpha_{1,i} + \alpha_{2,j+1} d_{1,i}\}$ (but including the right hand endpoint if $i = k_1 - 1$ and $j = k_2 - 1$). $I_2(x)$ will be the second term in the expansion of x. Each interval $\{x : I_1(x) = i, I_2(x) = j\}$ is then divided according to the proportions $\alpha_{3,1}, \cdots, \alpha_{3,k_3-1}$. Continuing this subdivision process, the n^{th} stage produces a splitting of $[0,1]$ into $k_1 k_2 \cdots k_n$ disjoint intervals $\{x : I_1(x) = i_1, I_2(x) = i_2, \cdots, I_n(x) = i_n\} = \{x : \alpha_{1,i_1} + \alpha_{2,i_2} d_{1,i_1} + \alpha_{3,i_3} d_{2,i_2} d_{1,i_1} + \cdots + \alpha_{n,i_n} d_{n-1,i_{n-1}} \cdots d_{1,i_1} \leq x < \alpha_{1,i_1} + \alpha_{2,i_2} d_{1,i_1} + \cdots + \alpha_{n,i_n+1} d_{n-1,i_{n-1}} \cdots d_{1,i_1}\}$.

The sequence $I_1(x), I_2(x) \cdots$ is the *generalized expansion* of x, taking values in the countable set $S = \{0, 1, \cdots, k_n - 1 : n = 1, 2, \cdots\}$. If $r \geq 2$ is a positive integer and $k_n = r$, $\alpha_{n,i} = i/r$ for each n, then the result is the usual r–adic expansion of x. (If x has more than one r–adic expansion, this method produces the terminating one.)

In this section, we wil use g–packings, where g is the collection of n–cylinders

$$c(i_1, \cdots, i_n) = \{x : I_1(x) = i_1, \cdots, I_n(x) = i_n\}$$

generated by the generalized expansion satisfying $\sup_n \frac{d(c_n(x))}{d(c_{n+1}(x))} = b(x) < \infty$, where $c_n(x)$ is the n–cylinders containing the point x (8), hence $d(c_n(x)) \downarrow 0$ and S is a finite set [2].

We now show that p_g^α is a regular outer measure.

LEMMA 4.1. *If g is the collection of n-cylinders generated by the generalized expansion satisfying (8), then, for all bounded $E \subset \mathbf{R}$, there is a Borel set $B \supset E$ such that $\mathbf{P}_g^\alpha(B) = \mathbf{P}_g^\alpha(E)$.*

Proof. Let $B = \cap_{n=1}^\infty \cup_{x \in E} c_n(x)$, where $c_n(x)$ is n–cylinder containing x.

Then $B \supset E$, hence $\mathbf{P}_g^\alpha(B) \geq \mathbf{P}_g^\alpha(E)$.

Further, if $\{J_k\}$ is a g–packing of B with $d(J_k) \leq \delta$ for any k, then $\{J_k\}$ is a g–packing of E.

Hence $\mathbf{P}_g^\alpha(B) \leq \mathbf{P}_g^\alpha(E)$.

LEMMA 4.2. *Let g be as in Lemma 4.1. Then p_g^α is a regular outer measure.*

Proof. Since $\mathbf{P}_g^\alpha$ is a metric pre–measure (see Lemma 3.1 ii), [7]), p_g^α is a metric outer measure. Hence every Borel set is p_g^α–measurable. Further p_g^α is Borel regular by using Lemma 4.1.

Hence p_g^α is a regular outer measure.

Now, we introduce a nice method of computing packing dimension of certain sets.

THEOREM 4.3. *Let $I_1(x), I_2(x), \cdots$ represent the generalized expansion of $x \in [0,1]$ with respect to a choice of proportions $\alpha_{n,i}, i = 1, \cdots, k_{n-1}$, $n = 1, 2, \cdots$ and suppose the resulting interval collection g of n–cylinders satisfies (8). Let γ be defined over the n–cylinders by the relations (9):*

$$\gamma(c(i_1, i_2, \cdots, i_n)) = p_n(i_1, i_2, \cdots, i_n),$$

where $0 \le p_n(i_1, \cdots, i_n) \le 1$, $p_n(i_1, \cdots, i_n) = 0$ *if one or more* $i_j \ge k_j$, $\sum_{i=0}^{n} p_n(i_1, \cdots, i_{n-1}, i) = p_{n-1}\ (i_1, \cdots, i_{n-1})$ *(consistency condition)*, $\sum_{i=0}^{k_1-1} p_1(i) = 1$, *and* $\lim_{n\to\infty} p_n(i_1, \cdots, i_n) = 0$.

Then γ *can be extended to a regular outer measure on* $[0,1]$, *and*

(a) *if* $E \subset \{x \in [0,1]$:

$$\lim_{n\to\infty} \sup \frac{\log p_n(I_1(x), I_2(x), \cdots, I_n(x))}{\log d_{1,I_1(x)} d_{2,I_2(x)} \cdots d_{n,I_n(x)}} \le \Theta\},$$

then $\mathrm{Dim}(E) \le \Theta$.

(b) *if* $E \subset \{x \in [0,1]$:

$$\lim_{n\to\infty} \sup \frac{\log p_n(I_1(x), I_2(x), \cdots, I_n(x))}{\log d_{1,I_1(x)} d_{2,I_2(x)} \cdots d_{n,I_n(x)}} \ge \Theta\}$$

and $\gamma(E) > 0$, *then* $Dim(E) \ge \Theta$.

Proof. Using Method I or II by Munroe [5], we easily obtain a regular outer measure on $[0,1]$.

We note $p_n(I_1(x), I_2(x), \cdots, I_n(x)) = \gamma(c_n(x))$ and $d_{1,I_1(x)} d_{2,I_2(x)} \cdots d_{n,I_n(x)} = d(c_n(x))$.

(a) : We only need to prove $p_g^{(\Theta+\varepsilon)}(E) < \infty$ for any $\varepsilon > 0$ because of theorem 3.1. Fix $\varepsilon > 0$, and consider $E_\rho = \{x \in E : d(c_n(x)) < \rho$ implies $\gamma(c_n(x)) \ge [d(c_n(x))]^{\Theta+\varepsilon}$ for all $n\}$. Then $E_\rho \uparrow E$ as $\rho \downarrow 0$.

Now,

$$\begin{aligned}
\mathbf{P}_g^{(\Theta+\varepsilon)}(E_\rho) &= \lim_{\delta\downarrow 0} \sup\{\sum_{k=1}^{\infty} [d(J_k)]^{\Theta+\varepsilon} : \{J_k\} \text{ is a} \\
&\quad g\text{-packing of } E_\rho \text{ and } d(J_k) < \delta\} \\
&\le \sup\{\sum_{k=1}^{\infty} [d(J_k)]^{\Theta+\varepsilon} : \{J_k\} \text{ is a } g\text{-packing of } E_\rho \\
&\quad \text{and } d(J_k) < \rho\} \\
&\le \sup\{\sum_{k=1}^{\infty} \gamma(J_k) : \{J_k\} \text{ is a } g\text{-packing of } E_\rho \text{ and} \\
&\quad d(J_k) < \rho\} \\
&\le \gamma([0,1]) = 1.
\end{aligned}$$

By Lemma 4.2, $\mathbf{P}_g^{(\Theta+\varepsilon)}(E) \leq 1$, so $p_g^{(\Theta+\varepsilon)}(E) \leq 1$.

(b): We may assume that $\Theta > 0$, and consider $\varepsilon > 0$ such that $\Theta - \varepsilon > 0$. It is sufficient to show that $p_g^{(\Theta-\varepsilon)}(E) > 0$. Now, consider $A \subset E$ with $\gamma(A) > 0$. Then $\gamma(c_n(x)) \leq [d(c_n(x))]^{\Theta-\varepsilon}$ for infinitely many n for any $x \in A$. Hence, $\vartheta_\rho = \{c_n(x) : x \in A, d(c_n(x)) \leq \rho, \gamma(c_n(x) \leq [d(c_n(x))]^{\Theta-\varepsilon}\}$ is a Vitali covering of A for any $\rho > 0$.

There is a countable disjoint sequence $\{c_{n_i}(x_i)\}_{i=1}^{\infty}$ from ϑ_ρ such that $\gamma(A \setminus \cup_{i=1}^{\infty} c_{n_i}(x_i)) = 0$ (see Theorem 4.1 [4], and note that $\gamma(x) = 0$ for every $x \in [0,1]$ since $\lim_{n\to\infty} p_n(i_1, \cdots, i_n) = 0$). Thus,

$$
\begin{aligned}
0 < \gamma(A) &= \sum_{i=1}^{\infty} \gamma(A \cap c_{n_i}(x_i)) + \gamma(A \setminus \cup_{i=1}^{\infty} c_{n_i}(x_i)) \\
&\leq \sum_{i=1}^{\infty} \gamma(c_{n_i}(x_i)) \\
&\leq \sum_{i=1}^{\infty} [d(c_{n_i}(x_i))]^{\Theta-\varepsilon}
\end{aligned}
$$

Hence, we obtain $0 < \gamma(A) \leq \mathbf{P}_g^{(\Theta-\varepsilon)}(A)$ for any $A \subset E$ with $\gamma(A) > 0$. By the lemma 5.1 vii) in [7], $p_g^{(\Theta-\varepsilon)}(E) = \inf_{E_n \uparrow E}\{\lim_n \mathbf{P}_g^{(\Theta-\varepsilon)}(E_n) : E_n$ are bounded $\}$.

If $E_n \uparrow E$, where E_n are bounded sets, $0 < \gamma(E) = \lim \gamma(E_n)$ since γ is a regular measure.

Therefore there is an integer n, such that $\gamma(E_n) > \alpha$ for a positive constant $\alpha < \gamma(E)$.

Then $0 < \alpha < \gamma(E_n) \leq \mathbf{P}_g^{(\Theta-\varepsilon)}(E_n) \leq \lim_n \mathbf{P}_g^{(\Theta-\varepsilon)}(E_n)$.

Hence $p_g^{(\Theta-\varepsilon)}(E) > \alpha$.

COROLLARY 4.4. *Let* $E \subset \{x \in [0,1] :$

$$
\lim_{n\to\infty} \sup \frac{\log p_n(I_1(x), I_2(x), \cdots, I_n(x))}{\log d_{1,I_1(x)} d_{2,I_2(x)} \cdots d_{n,I_n(x)}} = \Theta\}.
$$

If $\gamma(E) > 0$, *then Dim* $(E) = \Theta$.

Proof. It is immediate from Theorem 4.3.

REMARK 4.5. Noting the theorem 2.1 and 2.2 of [1], we easily see that, under the assumption of theorem 3.2 in [2], $\dim(E) = \Theta$, where $E \subset \{x \in [0,1]$:

$$\lim_{n\to\infty} \inf \frac{\log p_n(I_1(x), I_2(x), \cdots, I_n(x))}{\log d_{1,I_1(x)} d_{2,I_2(x)} \cdots d_{n,I_n(x)}} = \Theta\}$$

and $\gamma(E) > 0$

(It is a generalization of the theorem 3.2 of [2]).

We say C is a *generalized Cantor set* [2] if it can be expressed in the form $C = \{x : (I_1(x), I_2(x), \cdots) \in S^*\}$, where S^* is some subset of the countable product $X_{n=1}^{\infty}\{0, 1, \cdots, k_n - 1\}$. The simplest case occurs when $S^* = X_{n=1}^{\infty} S_n$ where $S_n \subset \{0, 1, \cdots, k_n - 1\}$ is the set of "allowable" digits at the n^{th} stage. The resulting Cantor set is called "independent" and can be written as $C = \{x : I_n(x) \in S_n$ for all $n\}$. The usual Cantor set (minus a countable collection of "endpoints" corresponding to some numbers with more than one triadic expansion) is an example, resulting when $k_n = 3, a_{n,i} = \frac{i}{3}$, and $S_n = \{0, 2\}$ for all n. We have the following corollary.

COROLLARY 4.6. *Let C be a generalized independent Cantor set built from generalized expansion whose n-cylinders satisfy (8). Let s_n denote the size of the set of allowable digits S_n at the n^{th} stage. Suppose there exists d_n such that $d_{n,i} = d_n$ for each $i \in S_n$. If $\limsup_{n\to\infty} \frac{\log(s_1 s_2 \cdots s_n)^{-1}}{\log d_1 d_2 \cdots d_n} = \Theta$, then $Dim(C) = \Theta$.*

Proof. Define

$$\gamma(c(i_1, \cdots, i_n)) = \begin{cases} (s_1 s_2 \cdots s_n)^{-1} & \text{if } i_j \in S_j \text{ for all } j \\ 0 & \text{otherewise} \end{cases}$$

Apply Corollary 4.4 with C in the role of E.

Finally we give an example which our result can be applied to.

EXAMPLE 4.7. Let $S = \{0, 1, 2\}$. We define n-cylinder $c(i_1, i_2, \cdots, i_n)$ generated by the generalized expansion as follows ;

$$d(c(i_1, i_2, \cdots, i_{n-1}, 0)) : d(c(i_1, \cdots, i_{n-1}, 1))$$

$$: d(c(i_1, \cdots, i_{n-1}, 2)) = \frac{5}{15} : \frac{7}{15} : \frac{3}{15}$$

Then $[0,1] = \{x : x = c(i_1, i_2, \cdots) \text{ where } i_j \in S, j = 1, 2, \cdots\}$.

For any n, we define

$$\gamma(c(i_1, i_2, \cdots, i_n)) = \begin{cases} \frac{1}{2^n} & \text{if } i_j = 0 \text{ or } 2, \text{ where } j = 1, 2, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

Then $g = \{c_n(x) : x \in [0,1], n = 1, 2, \cdots\}$ satisfies (8) and γ satisfies (9). Let $E = \{x \in [0,1] : x = c(i_1, i_2, \cdots), i_j \in \{0, 2\}, j = 1, 2, \cdots\}$.

We will call E a *skew symmetric Cantor set.*

Consider $A = \{c(i_1, i_2, \cdots) \in E : \lim_{n\to\infty} \sum_{j=1}^{n} i_n/n = 1\}$. Then $\gamma(A) = 1$ by the strong law of large numbers.

Thus we call A an *a.s. skew symmetric Cantor set.*

Further, $A \subset \{x \in [0,1] : \lim_{n\to\infty} \frac{\log \gamma(c_n(x))}{\log d(c_n(x))} = \frac{\log 4}{\log 15}\}$.

For,

$$\begin{aligned} \frac{\log \gamma(c_n(x))}{\log d(c_n(x))} &= \frac{\log 2^{-n}}{\log (1/3)^k (1/5)^{n-k}} \\ &= \frac{n \log 2}{k \log 3 + (n-k) \log 5} \text{ for } 0 \le k \le n, \text{ where } x \in A. \end{aligned}$$

If $x \in A$, then $\frac{k}{n} \to \frac{1}{2}$, so $\frac{\log \gamma(c_n(x))}{\log d(c_n(x))} \to \frac{\log 4}{\log 15}$.

By Corollary 4.4, $\mathrm{Dim}(A) = \frac{\log 4}{\log 15}$.

We notice that $\dim(A) = \frac{\log 4}{\log 15}$ (Theorem 3.2, [2]).

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