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PERIODIC PROPERTIES FOR MAPPINGS OF CONTINUA*

JONG SOOK BAE, SEONG GIL LEE AND SEUNG KAB YANG

1. Introduction

Let X be a topological space and let $C^0(X)$ denote the set of continuous maps of X into itself. For any $f \in C^0(X)$, let $f^0 : X \to X$ be the identity, and define, inductively, $f^n = f \circ f^{n-1}$ for any positive integer n. A point $x \in X$ is a *periodic point* of f of *period* $n > 0$ if $f^n(x) = x$ but $f^i(x) \neq x$ for all $0 < i < n$. Hence x is a fixed point if $n = 1$. A point $x \in X$ is called a *recurrent point* of f if there exists a sequence $n_i \to \infty$ of positive integers such that $f^{n_i}(x) \to x$. We denote the sets of periodic and recurrent points of f by $P(f)$ and $R(f)$, respectively.

By a *continuum* we mean a compact connected Hausdorff space and by a *tree* we mean a continuum in which every pair of distinct points is separated by a third point.

In this paper we give several properties of a tree as a partially ordered topological space and obtain results concerning periodic and recurrent points for maps of the tree. Also, we study recursive properties for maps of the various spaces, that is, linear continua, $Y_n = \{z \in \mathbf{C} : 0 \leq z^n \leq 1\}$ and a Sharkovsky space Z which is not homeomorphic to any linear continuum.

2. Properties of the tree

A *partially ordered topological space* (denoted by POTS) X consists of a set with a partial order $\leq$ and a topology which has a subbasis for its closed sets consisting of the sets:

$$\{x \in X : x \leq a\}, \quad \{x \in X : x \geq a\}$$

for all $a \in X$ (see [15]). If the partial order is linear, then this topology coincides with the order topology. We say that a tree is a *dendrite* if

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it is metrizable (see [9] or [17]). The triod is an example of a dendrite, and a linearly ordered dendrite is an arc, i.e., a homeomorphic image of the unit interval. A characterization of a dendrite and a tree as a POTS was given by L.E.Ward, Jr., [16], that is, a tree admits an order $\leq$ such that it is order dense and has the *lub* and *glb* property, and also the induced order topology coincides with the original one.

Hereafter we will assume that a tree is a POTS with an order which is order dense and has the *lub* and *glb* property.

Now, we give several properties of a tree, which will be used in the following sections.

LEMMA 2.1. *Let X be a tree and $x, y \in X$ with $x < y$. Then there exists a linear continuum $M_0 \subset X$ with $glb\, M_0 = x$ and $lub\, M_0 = y$.*

Proof. Let $X_0 = \{z \in X : x \leq z \leq y\}$ and $\Gamma = \{M \subset X_0 : x, y \in M$ and M is linearly ordered$\}$. Then since $\{x, y\} \in \Gamma$, $\Gamma \neq \phi$. Let $\{M_\alpha\}$ be any chain in Γ. Then it is easy to show that $\bigcup_\alpha M_\alpha$ is linearly ordered. Therefore by Zorn's lemma, Γ has a maximal element M_0. Then clearly $glb\, M_0 = x$ and $lubM_0 = y$.

Now we claim that M_0 is connected. Suppose that M_0 is not connected. Then there is a separation $A, B \subset M_0$ such that $x \in A$ and $y \in B$. Since A has an upper bound y, $x_0 = lub\, A$ exists and $x_0 \leq y$. It is easy to see that $M_0 \cup \{x_0\}$ is also linearly ordered, so that $x_0 \in M_0$ by the maximality of M_0. Since A is closed, $x_0 \in A$ and hence $x_0 < y$. This means that $\{z \in B : z > x_0\} \neq \phi$. Let $y_0 = glb\{z \in B : z > x_0\}$. Then $y_0 \geq x_0$ and $y_0 \in M_0$ by the same reason as above. Therefore since B is closed $y_0 \in B$. Since $A \cap B = \phi$, we know that $x_0 < y_0$. Since X is order dense, there exists a $z \in X$ with $x_0 < z < y$. But then $z \in X_0$ and $M_0 \cup \{z\}$ is linearly ordered, which contradicts to the maximality of M_0. This completes the proof.

COROLLARY 2.2. *Let X be a tree and $x, y \in X$ with $x < y$. Then the set*

$$X_0 = \{z \in X : x \leq z \leq y\}$$

is connected.

Proof. For any $z \in X_0$, by Lemma 2.1. there exists a linear continuum $M \subset X$ such that $glb\, M = x$ and $lub\, M = z$. This means that every $z \in X_0$ can be lie in the same component of x in X_0. Therefore X_0 is connected.

LEMMA 2.3. *Let X be a tree and $x, y, z \in X$ with $x < z < y$. Then x and y cannot belong to the same component of $X - \{z\}$.*

Proof. By Lemma 2.1, we can choose linear continua $L_1, L_2 \subset X$ such that

$$glb\, L_1 = x,\ lub\, L_1 = z = glb\, L_2,\ lub\, L_2 = y.$$

Then actually $L = L_1 \cup L_2$ is a linear continuum. Therefore x and y cannot be separated by a point in $X - L$. Since X is a tree, x and y can be separated by a third point in X, which means that x and y can be separated by only a point in L. Since L is a linear continuum, actually x and y are separated by z.

THEOREM 2.4. *Let X be a tree and $x, y \in X$ with $x < y$. Then the set*

$$X_0 = \{z \in X : x \leq z \leq y\}$$

is a linear continuum.

Proof. By Lemma 2.1, it suffices to show that X_0 is linearly ordered. On the contrary, suppose that there are non-comparable points z_1, z_2 in X_0. Then by Lemma 2.1, x and y can not be separated by z_1 and z_2, which contradicts to Lemma 2.3.

Now, in Theorem 2.4, we denote the set X_0 by $[x, y]$. Then Theorem 2.4 says that $[x, y]$ is a linear continuum.

3. Fixed points for maps of the tree

Let I be a closed interval of the real lines and let $f \in C^0(I)$. If a closed subinterval K of I satisfies $f(K) \supset K$, then there is a fixed point in K ([15], [10], [13] and [14]). But this is not true for a continuous map of the circle S^1. For $f \in C^0(S^1)$, however, if N is a proper closed interval on S^1 such that $f(K) = N$ for some closed interval $K \subset N$, then f has a fixed point in N [6].

Now, for maps of a linear continuum, we have

LEMMA 3.1. *Let L be a linear continuum and I be a closed interval in L. If $f : L \to L$ is a continuous function such that $f(I) \subset I$ or $f(I) \supset I$. Then f has a fixed point in I.*

Proof. First, suppose that $f(I) \subset I$ and $I = [a, b]$. Assume that f is fixed point free. And put

$$A = \{x \in I : f(x) > x\},$$
$$B = \{x \in I : f(x) < x\}.$$

Then $a \in A$ and $b \in B$, so that $A \neq \phi$ and $B \in \phi$. Since f is continuous, it is easy to show that A and B are open sets. This means that A and B form a separtion of a connected set I, which is a contradiction. Therefore f has a fixed point in I.

For the case $f(I) \supset I$, see the lemma 2.3 of [13].

THEOREM 3.2. *Let f be a continuous selfmap of a tree X and J be an open subset of X such that $\bar{J}$ is a linear continuum. If $f(\bar{J}) \supset \bar{J}$, then there exists a closed subinterval Q of $\bar{J}$ such that $f(Q) = \bar{J}$.*

Proof. Let $p = glb\, J$ and $q = lub\, J$. Then by Theorem 2.4, $\bar{J} = [p, q]$. We can choose $a, b \in \bar{J}$ such that $f(a) = p$, $f(b) = q$. Now we may assume that $a < b$. Define $a \leq r \leq b$ by

$$r = lub\{x \in [a, b] : f(x) = p\}.$$

Then by the continuity of f and since $[a, b]$ is a linear continuum, $f(r) = p$. Now we define $r < s \leq b$ by

$$s = glb\{x \in [r, b] : f(x) = q\}.$$

Then $f(s) = q$ by the same argument. We put $Q = [r, s] \subset \bar{J}$ and show that $f(Q) = \bar{J}$.

First, we show that $f(Q) \supset \bar{J}$. On the contrary, suppose that $z \in \bar{J} - f(Q)$. Then p and q can be separated by z, which contradicts to the fact $f(Q)$ is connected. Hence $f(Q) \supset \bar{J}$.

Next, if we prove that the set

$$A = \{x \in [r, s] : f(x) \notin \bar{J}\}$$

is empty, then $f(Q) \subset \bar{J}$. Now assume that $A \neq \phi$. Since $f(Q) \supset \bar{J}$, we can choose $r < c < s$ such that $f(c) \in J$. Then since $A \neq \phi$, we know that either

$$A_1 = \{x \in [r, c] : f(x) \notin \bar{J}\}$$

or

$$A_2 = \{x \in [c, s] : f(x) \notin \bar{J}\}$$

is empty. Now assume that $A_1 \neq \phi$ and let $d = lub\, A_1$. Then $d \leq c$ and $f([d, c]) \subset \bar{J}$ and $f(d) \in \overline{X - J}$ by the continuity of f. Since J is an open subset of X, we know that $\bar{J} \cap \overline{X - J} \subset \{p, q\}$, so that $f(d) = p$ or q. But since $d < s$ means $f(d) = p$, by the choice of r, in this case we can see that $d = r$. But $lub\, A_1 = r$ means $A_1 = \phi$ because of $r \notin A$, which is a contradiction. Also we can show that $A_2 = \phi$ by the same way. Hence $A = \phi$. Thus $f(Q) \subset \bar{A}$, which completes the proof.

THEOREM 3.3. *Let f be a continuous selfmap of a tree X and $x, y \in X$ with $x < y$.*

(1) *If $f([x, y]) \subset [x, y]$, then f has a fixed point in $[x, y]$.*

(2) *If $(x, y) = [x, y] - \{x, y\}$ is open in X and $f([x, y]) \supset [x, y]$, then f has a fixed point in $[x, y]$.*

Proof. (1) Since $[x, y]$ is a linear continuum by Theorem 2.4, f has a fixed point in $[x, y]$ by Lemma 3.1.

(2) By Theorem 3.2, we can have a closed subinterval Q of $[x, y]$ such that $f(Q) = [x, y]$. Then we claim that f has a fixed point in Q. Suppose that f has no fixed point in Q. Now let

$$A = \{z \in Q : f(z) < z\}$$
$$B = \{z \in Q : f(z) > z\}.$$

Then since f is continuous, A and B are open in Q. Also $Q \subset [x, y]$ and $f(Q) = [x, y]$ mean that $A \neq \phi$ and $B \neq \phi$. Therefore A and B form a separation of Q, which gives a contradiction because Q is also a linear continuum.

REMARK. For the above results on the real line and the linear continuum, see lemmas 0 and 1 of [10] and lemmas 2.2 and 2.3 of [13], respectively.

4. Recurrent Properties for Maps of the tree

In this section we describe the recurrent properties for maps of the tree.

We begin with the lemma of [3].

LEMMA 4.1 ([3], LEMMA 1). *Let f be a continuous selfmap of a compact Hausdorff space. Then $R(f) = R(f^n)$ for each positive integer n.*

PROPOSITION 4.2. *Let f be a continuous map of a tree X and let J be a subset of X with*

$$\bar{J} = [a, b] = \{x \in X : a \leq x \leq b\}$$

for some $a, b \in X$ with $a < b$. If $J \cap P(f) = \phi$ and $f(J) \subset J$, then $J \cap R(f) = \phi$.

Proof. Since $f(x) \neq x$ for all $x \in J$, two sets

$$A = \{x \in J : f(x) > x\}, \quad B = \{x \in J : f(x) < x\}$$

form a separation of J. Since $[a, b]$ is closed, $\bar{J} = [a, b]$. And by Theorem 2.4, $\bar{J}$ is a linear continuum, so that J is connected. Therefore $A = J$ or $B = J$. Now assume that $A = J$. Then for each fixed $x \in J$, we have

$$x < f(x) < f^2(x) < \cdots.$$

Since $U = X - \{y \in X : y \geq f(x)\}$ is an open set containing x and $f^n(x) \notin U$ for all $n \geq 2$, $x \notin R(f)$. This completes the proof.

THEOREM 4.3. *Let $f : X \to X$ be continuous, where X is a tree and let J be an open subset of X with $J \cap P(f) = \phi$. Suppose that $\bar{J} = [a, b]$ for some $a, b \in X$ with $a < b$. If $f(J)$ contains a fixed point of f, then $J \cap R(f) = \phi$.*

Proof. Suppose that $f^n(J) \cap J = \phi$ for all $n \geq 1$. Then $J \cap R(f) = \phi$ since J is open. Suppose that $f^n(J) \cap J \neq \phi$ for some integer $n \geq 1$. Let $p \in J$ such that $f^2(p) = f^3(p) = \cdots$, and $q \in J$ such that $f^n(q) \in J$. Without loss of generality we may assume that $p < q$. Since $f^n(p) \notin J$,

$$A = \{x \in (a, q] : f^n(x) \notin J\} \neq \phi,$$

where $(a,q] = \{x \in X : a < x \leq q\}$. Let $lub\, A = r \in (a,b)$. Then, since f is continuous, $f^n(r) \notin J$. But since $f^n((r,q)) \subset J$, $f^n(r) \in \bar{J} = [a,b]$. This means that $f^n(r) = a$ or b. Now we may assume that $f^n(r) = a$. Then we claim that

($*$) for each $x \in J$ with $f^k(x) \in J$ for some integer $k \geq 1$, then we have $f^k(x) < x$.

To prove ($*$), we need the following results :

(1) For each $k \geq 1$, $f^k(J)$ does not contain both a and b. In particular $b \notin f^n(J)$.

(2) If $f^k(J) \cap J \neq \phi$ for some integer $k \geq 1$, then $a \in f^k(J)$.

Proof of (1). Suppose that $a, b \in f^k(J)$ for some $k \geq 1$. Then actually $[a,b] \subset f^k(J)$. To see this, suppose that there is a point $c \in [a,b]$ such that $f^k(J) \not\ni c$. Then since $f^k(J)$ is connected, by Lemma 2.3 a and b can not be separated by c. Therefore by Theorem 3.2 and Theorem 3.3, f^k has a fixed point in J, which is a contradiction.

Proof of (2). The above argument shows that there is a point $s \in J$ such that $f^k(s) = a$ or b. Suppose that $f^k(s) = b$. Then by (1) we know that $f(p) = f^n(p) \neq b$ and also $f(p) = f^k(p) \neq a$. Since $f(p) \in f^k(J)$, b and $f(p)$ can not be separated by a and also since $f^n(r) = a$, a and $f(p)$ can not be separated by b. But since $f(p) \notin J$, this means that a and b can not be separated by any element of $a < x < b$ by (1), which contradicts to Lemma 2.3.

Now we will prove ($*$). Let $x \in J$ with $f^k(x) \in J$ for some integer $k > 1$. Then by (2) we know that there is a point $c \in J$ such that $f^k(c) = a$. Now assume that $x < f^k(x)$. Then $f^k(c) < c$ and $x < f^k(x)$ means that $f^k(J_1) \supset J_1$, where $J_1 = [c,x]$ or $[x,c]$. Therefore by Theorem 3.2 and Theorem 3.3, $J_1 \cap P(f) \neq \phi$, a contradiction.

Now suppose that $x \in J \cap R(f)$. Then, by definition, there exists a subsequence $\{f^{n_k}(x)\}$ of $\{f^n(x)\}$ such that $f^{n_k}(x) \to x$ and $f^{n_k}(x) \in J$ for all $k \geq 1$. Then by ($*$)

$$a < \cdots < f^{n_{k+1}}(x) < f^{n_k}(x) < \cdots < f^{n_1}(x) < x.$$

But then

$$f^{n_k}(x) \notin X - \{y \in X : f^{n_1}(x) \leq y\}$$

for all $k \geq 1$. Since $\{y \in X : f^{n_1}(x) \leq y\}$ is closed, it leads to a contradiction. Therefore $J \cap R(f) = \phi$.

5. Applications to various spaces

In 1980, E.M. Coven and G.A. Hedlund showed the following result:

THEOREM 5.1 ([8], THEOREM 1). *For any $f \in C^0(I)$, where I is a closed interal of the real line,*

$$\overline{P(f)} = \overline{R(f)}.$$

Special cases of this theorem have been proved in [4] and [7].

For homeomorphisms of the circle S^1, it is easily shown that the above result is not true since there are no periodic points for any irrational rotations on the circle, but every point is recurrent. By adding the necessary condition that the set of periodic points is nonempty, in 1982, I. Mulvey [12] proved in his doctoral dissertation that for any $f \in C^0(S^1)$ with $P(f) \neq \phi$, $\overline{P(f)} = \overline{R(f)}$. Also, in 1986, G.F. Liao and J.C. Xiong [11] proved the same result for any $f \in C^0(S^1)$ with $P(f) \neq \phi$ in a different method. Recently, J.S. Bae and S.K. Yang [3] found a very simple and refined independent proof of the above theorem of Mulvey.

In this section, we will show that Theorem 5.1 still holds for maps of various spaces.

THEOREM 5.2. *Let $f : L \to L$ be continuous where L is a linear continuum. Then $\overline{P(f)} = \overline{R(f)}$.*

Proof. By Theorem 3.3, $P(f) \neq \phi$. Now let J be a component of $L - \overline{P(f)}$. We show that $J \cap R(f) = \phi$. If $f^n(J) \cap J = \phi$ for all integer $n \geq 1$, $J \cap R(f) = \phi$, because J is open. If $f^n(J) \subset J$ for some integer $n \geq 1$, then by Proposition 4.2, $R(f^n) \cap J = \phi$, and hence $R(f) \cap J = \phi$ since $R(f) = R(f^n)$ by Lemma 4.1. Suppose that $f^n(J) \cap J \neq \phi$ and $f^n(J) \cap (L - J) \neq \phi$ for some integer $n \geq 1$. Since $f^n(J)$ is an interval, it contains an end point a of J, $a \notin J$. If $a \in P(f)$, then we can choose a point $x \in J$ and an integer $m \geq n$ such that

$$f^m(x) = f^{2m}(x) = \cdots = a.$$

Therefore by Theorem 4.3, $J \cap R(f) = \phi$. Suppose that $a \notin P(f)$, and let $x \in J$ such that $f^n(x) = a$. Now, we may assume that $a < x$. Then since $[a, x] \cap P(f) = \phi$, $f^n(a) < a$. Therefore $(f^n(a), a] \subset f^n(J)$ since

$f^n(J)$ is connected. Since $a \in \overline{P(f)}$ and $[a,x) \cap P(f) = \phi$, $(f^n(a), a]$ contains a periodic point. Hence $f^n(J)$ contains a periodic point, say q. Then also we can find a point $x \in J$ and an integer $m \geq n$ such that $f^m(x) = f^{2m}(x) = \cdots$. Therefore by Theorem 4.3, $R(f^n) \cap J = \phi$ and hence $R(f) \cap J = \phi$ by Lemma 4.1.

On the other hand, in 1989, L. Alseda, J. Llibre and M. Misiurewicz [1] have characterized the set of periods of periodic orbits for continuous maps of $Y = \{z \in \mathbf{C} : z^3 \in [0,1]\}$ into itself having zero as a fixed point.

Now we will prove that Theorem 5.1 holds for maps of $Y_n = \{z \in \mathbf{C} : 0 \leq z^n \leq 1\}$ with $f(0) = 0$.

THEOREM 5.3. *Let $f : Y_n \to Y_n$ be continuous with $f(0) = 0$ and $n \geq 1$. Then $\overline{P(f)} = \overline{R(f)}$.*

Proof. Let us define an order $\leq$ on Y_n by the following:
For $r_1 e^{i\theta_1}$, $r_2 e^{i\theta_2} \in Y_n$, $0 \leq \theta_1$, $\theta_2 \leq 2\pi$,

$$r_1 e^{i\theta_1} \leq r_2 e^{i\theta_2} \iff \begin{cases} r_1 \leq r_2 \text{ and } \theta_1 = \theta_2 = 0; \\ r_1 \geq r_2 \text{ and } \theta_1 = \theta_2 \neq 0; \\ \theta_1 \neq 0,\ \theta_2 = 0. \end{cases}$$

Then Y_n is a partially ordered topological space and this order topology is just the one inherited from $\mathbf{C}$. Since Y_n is compact connected and order dense, Y_n is in fact a tree.

Let J be a component of $Y_n - \overline{P(f)}$. Then it is easy to show that J is open and $\bar{J}$ is a linear continuum. Actually without loss of generality, we may assume that

$$J \subset [0,1] = \{x \in \mathbf{R} : 0 \leq x \leq 1\}$$

and the order is the usual one. As in the proof of Theorem 5.2, we may assume that $f^n(J) \cap J \neq \phi$ and $f^n(J) \cap (Y_n - J) \neq \phi$ for some integer $n \geq 1$. Let $x \in J$ with $f^n(x) \in J$. Since J is linearly ordered, x and $f^n(x)$ is comparable. First suppose that $x < f^n(x)$. Then we know that $f^n(y) > y$ for all $y \in J$. In fact if there is a point $y \in J$ such that $f^n(y) < y$, then f^n must have a fixed point between x and y. Now let $\bar{J} = [a,b]$ for some $0 \leq a < b \leq 1$. If $b \in J$, then we must have $b = 1$, so that $f^n(b) > b$ leads to a contradiction. Therefore $b \notin J$. But then $f^n(J) \cap (Y_n - J) \neq \phi$ means that $f^n(J) \subset [0,1]$ and hence there is a point $z \in J$ such that $f^n(z) = b$. But if $f^n(y) \leq b$ for all $y \in J$, then $f^n(b) = b$ by the continuity of f^n. On the other hand if there is a point $y \in J$ such that $f^n(y) > b$, then $f^n(J) \supset [f^n(x), f^n(y)]$ and $b \in \overline{P(f)}$ mean that $f^n(J) \cap P(f) \neq \phi$.

Next suppose that $f^n(x) < x$. Then also we know that $f^n(y) < y$ for all $y \in J$. Since J is open, $a \notin J$ and we know that $f^n(a) \leq a$. Hence, by the same way as above, we can easily see that $f^n(J) \cap P(f) \neq \phi$ since $0 \in P(f)$. Therefore by Theorem 4.3, $J \cap R(f) = \phi$, which completes the proof.

COROLLARY 5.4. *Let $f : Y_n \to Y_n$ be continuous such that $0 \in P(f)$. Then $\overline{P(f)} = \overline{R(f)}$.*

Proof. Since $0 \in P(f)$, 0 is a fixed point of f^n for some integer $n \geq 1$. Then by Theorem 5.3, $\overline{P(f^n)} = \overline{R(f^n)}$. But since $P(f^n) = P(f)$ and $R(f^n) = R(f)$ by Lemma 4.1, we conclude that $\overline{P(f)} = \overline{R(f)}$.

A topological space X is called a *Sharkovsky space* provided that if a continuous map $f : X \to X$ has a periodic point of period k, then f has periodic points of all periods which follow k in Sharkovsky's sequence:

$$3, 5, 7, \cdots, 2\cdot 3, 2\cdot 5, 2\cdot 7, \cdots, 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, \cdots, \cdots, 2^3, 2^2, 2, 1.$$

In 1989, Bae and Sung [2] found a counterexample for Question 3.8 of [13] as follows.

EXAMPLE 5.5. Let Ω be the first uncountable ordinal and let X be a linearly ordered set from $[0, \Omega]$ by implementing a copy of the unit interval $(0,1)$ between each ordinal α and $\alpha + 1$. Let Y be a closed interval $[-1, 1]$ of the real line and let Z be a union of X and Y attached Ω to 0, that is,

$$Z = X \cup Y/0 \sim \Omega.$$

Since X and Y are themselves linear continua, Z with the quotient topology can be a tree by giving a suitable partial order. Then Bae and Sung [2] proved that Z is a Sharkovsky space, which is not homeomorphic to any linear continuum.

Although it seems that Z is similar to Y_3, without the condition $0 \in P(f)$, it can be shown that a continuous map $f : Z \to Z$ has the property that $\overline{P(f)} = \overline{R(f)}$. Note that actually Z is not a dendrite because Z is not path connected, but Y_n is a typical example of the dendrite.

THEOREM 5.6. *Let Z be defined as above. If $f : Z \to Z$ is continuous, then $\overline{P(f)} = \overline{R(f)}$.*

Proof. Since $0 \sim \Omega$ in Z, we denote the class $\{0\} = \{\Omega\}$ by Ω simply. If $f(\Omega) = \Omega$, then the same argument as in the proof of Theorem 5.3 well behave. Let $f(\Omega) \in X - \{\Omega\}$. Then Bae and Sung [2] showed that $f(Z) \subset X$. In fact, since X is not path connected but $X - \{\Omega\}$ and Y are path connected, $f(Z) \cap Y$ must be empty. In this case clearly $(Y - \{\Omega\}) \cap R(f) = \phi$, so that $R(f) \subset X$. But since X is a linear continuum and $f(X) \subset X$, by Theorem 5.2 $\overline{P(f)} = \overline{R(f)}$. Finally if $f(\Omega) \in Y - \{\Omega\}$, then also we have $f(Z) \subset Y$. Therefore by the same argument as above $\overline{P(f)} = \overline{R(f)}$, which completes the proof.

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Department of Mathematics
Myong Ji University
Seoul 120–728, Korea