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ORIENTED 5–DIMENSIONAL SUBMANIFOLDS IN THE PURELY IMAGINARY OCTONIANS

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1. Introduction

In this note, we shall consider the gometry of the base space $\operatorname{Im}\mathbf{O}$ with the structure group G_2, where $\operatorname{Im}\mathbf{O}$ is the purely imaginary octonians and G_2 is the automorphism group of the octonians. By the algebraic properties of $\operatorname{Im}\mathbf{O}$, we see that any oriented 5-dimensional submanifold (M^5, Ψ) in $\operatorname{Im}\mathbf{O}$ has the induced almost contact metric structure, that is, the structure group of the tangent bundle is reducible to $U(n) \times 1$ (see [1]). In 2, we shall recall the structure equations of the group G_2 established by Bryant ([2]). In 3, we write the induced structure equations of (M^5, Ψ) in $\operatorname{Im}\mathbf{O}$ derived from these equations. In 4, we shall give the conditions for the induced almost contact structure to be normal. In 5, we determined the quasi-Sasakian (M^5, Ψ) and nearly cosymplectic submanifold (M^5, Ψ) in $\operatorname{Im}\mathbf{O}$. These results are improve slightly one of the work of Kenmotsu ([5]). In 6, we give the relations between the Gauss map and the almost contact metric structure. Lastly, we shall observe the condition that induced almost contact structure to be contact. As an application, we shall show that there does not exist contact structure for the special submanifold (M^5, Ψ).

In this paper, we adopt the same notational convention as in [2] and all the manifolds are assumed to be connected and of class C^∞ unless otherwise stated. Throughout this article, we denoted by (M^5, Ψ) and oriented 5-dimensional submanifold in $\operatorname{Im}\mathbf{O}$. The author would like to express his hearty thanks to Professor K. Sekigawa and Professor N. Innami for their encougagements and many valuable suggestions.

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2. Preliminaries

2.1. We denote by $M_{p\times q}(\mathbf{C})$ the set of $p\times q$ complex matrices and $[a]\in M_{3\times 3}(\mathbf{C})$ is given by

$$[a]=\begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}$$

where $a={}^t(a_1,a_2,a_3)\in M_{3\times 1}(\mathbf{C})$. Then, we have

$$[a]b+[b]a=0 \tag{2.1}$$

where $a,b\in M_{3\times 1}(\mathbf{C})$. We denote by $\mathbf{O}$ the octonians and $<\,,\,>$ the canonical inner product of $\mathbf{O}$ ([2],[3]). For any $x\in\mathbf{O}$, we denoted by $\overline{x}$ the conjugate of x. We remark that the octonians may be regarded as the direct sum $\mathbf{H}\oplus\mathbf{H}$ where $\mathbf{H}$ is the quaternious.

2.2. Now, we shall recall the structure equations of $(\mathrm{Im}\,\mathbf{O},G_2)$ which is established by R. Bryant ([2]). We set a basis of $\mathbf{C}\otimes_{\mathbf{R}}\mathrm{Im}\,\mathbf{O}$ by; ε, $E_1=iN$, $E_2=jN$, $E_3=kN$, $\overline{E}_1=i\overline{N}$, $\overline{E}_2=j\overline{N}$, $\overline{E}_3=k\overline{N}$ where $\varepsilon=(0,1)\in\mathbf{H}\oplus\mathbf{H}$, $N=(1-\sqrt{-1}\varepsilon)/2$, $\overline{N}=(1+\sqrt{-1})/2\in\mathbf{C}\otimes_{\mathbf{R}}\mathbf{O}$ and $\{1,i,j,k\}$ is the canonical basis of $\mathbf{H}$. A basis $(u,f,\overline{f})$ of $\mathbf{C}\otimes_{\mathbf{R}}\mathrm{Im}\,\mathbf{O}$ is said to be *admissible*, if there exists $g\in G_2\subset M_{7\times 7}(\mathbf{C})$ so that $(u,f,\overline{f})=(\varepsilon,E,\overline{E})g$. We shall identify the element of G_2 with the corresponding admissible basis. Then, we have

PROPOSITION 2.1 ([2], PROPOSITION 2.3.). *There exists left invariant 1-forms κ,θ on G_2; θ with values in $M_{3\times 1}(\mathbf{C})$ and $\kappa=(\kappa^i_j)$, $1\le i,j\le 3$, with values in 3×3 skew Hermitian matrices which satisfies $tr\kappa=0$,*

$$\begin{aligned} d(u,f,\overline{f})&=(u,f,\overline{f})\begin{pmatrix} 0 & -\sqrt{-1}\,{}^t\overline{\theta} & \sqrt{-1}\,{}^t\theta \\ -2\sqrt{-1}\theta & \kappa & [\overline{\theta}] \\ 2\sqrt{-1}\,\overline{\theta} & [\theta] & \overline{\kappa} \end{pmatrix} \\ &=(u,f,\overline{f})\phi. \end{aligned} \tag{2.2}$$

Then, ϕ satisfies $d\phi=-\phi\wedge\phi$, or equivalently,

$$d\theta=-\kappa\wedge\theta+[\overline{\theta}]\wedge\overline{\theta}, \tag{2.3}$$

$$d\kappa=-\kappa\wedge\kappa+3\theta\wedge{}^t\overline{\theta}-{}^t\theta\wedge\overline{\theta}I_3. \tag{2.4}$$

Let $\mathcal{F} = \operatorname{Im}\mathbf{O} \times G_2$ and $x : \mathcal{F} \to \operatorname{Im}\mathbf{O}$ denote the projection onto the first factor. We regareded $\mathcal{F}$ as the space of pairs $(y; (u, f, \overline{f}))$ consisting of a base point $y \in \operatorname{Im}\mathbf{O}$ and admissible basis at that point. Then, we have

PROPOSITION 2.2. *There exists the dual basis $(\eta, \omega, \overline{\omega})$ of $(u, f, \overline{f})$ on $\mathcal{F}$ so that*

$$dx = (u, f, \overline{f}) \begin{pmatrix} \eta \\ \omega \\ \overline{\omega} \end{pmatrix} = (u, f, \overline{f})\psi \tag{2.5}$$

Then, ψ satisfies

$$d\psi = -\phi \wedge \psi. \tag{2.6}$$

2.3. Let $\tilde{G}(2, \operatorname{Im}\mathbf{O})$ denote the Grassmannian manifold of oriented 2-planes in $\operatorname{Im}\mathbf{O}$. We define the map $\tilde{\eta} : G_2 \to \tilde{G}(2, \operatorname{Im}\mathbf{O})$ by $\tilde{\eta}(g) = -2\sqrt{-1}\, f_1 \wedge \overline{f}_1$. By (2.2), we have

$$\begin{aligned} d\tilde{\eta} = (-2\sqrt{-1})\{ & u \wedge \overline{f}_1(-2\sqrt{-1}\theta^1) + f_1 \wedge u(2\sqrt{-1}\theta^1) \\ & + \sum_{i=2}^{3}(f_i \wedge \overline{f}_1\kappa^i_1 + f_1 \wedge \overline{f}_i\overline{\kappa}^i_1) - \overline{f}_2 \wedge \overline{f}_1\theta^3 - f_1 \wedge f_2\overline{\theta}^3 \\ & + \overline{f}_3 \wedge \overline{f}_1\theta^2 + f_1 \wedge f_3\overline{\theta}^2\} \end{aligned} \tag{2.7}$$

Hence, we have

$$\ker d\tilde{\eta} = \left\{ \begin{pmatrix} 0 & & 0 & & & 0 & \\ & \kappa^1_1 & 0 & 0 & & & \\ 0 & 0 & \kappa^2_2 & \kappa^2_3 & & 0 & \\ & 0 & \kappa^3_2 & \kappa^3_3 & & & \\ & & & & \overline{\kappa}^1_1 & 0 & 0 \\ 0 & & 0 & & 0 & \overline{\kappa}^2_2 & \overline{\kappa}^2_3 \\ & & & & 0 & \overline{\kappa}^3_2 & \kappa^3_3 \end{pmatrix} \in \mathcal{G}_2 \right\}$$

So, the orthogonal complement $\mathcal{H}$ of ker $d\tilde{\eta}$ is given by

$$\mathcal{H}=\left\{\left(\begin{array}{ccccccc} 0 & & -\sqrt{-1}\,\overline{\theta} & & & \sqrt{-1}\theta & \\ & 0 & \kappa_2^1 & \kappa_3^1 & & & \\ -2\sqrt{-1}\theta & \kappa_1^2 & 0 & & & [\overline{\theta}] & \\ & \kappa_1^3 & & & & & \\ & & & & 0 & \overline{\kappa}_2^1 & \overline{\kappa}_3^1 \\ 2\sqrt{-1}\,\overline{\theta} & & [\theta] & & \overline{\kappa}_1^2 & & \\ & & & & \overline{\kappa}_1^3 & 0 & \end{array}\right)\in\mathcal{G}_2\right\}.$$

Thus, we have $\mathcal{G}_2=\mathcal{H}\oplus\ker\, d\tilde{\eta}$. From the above observations, we may observe that

PROPOSITION 2.3.

$$\tilde{G}(2,\mathrm{Im}\,\mathbf{O})\simeq G_2/U(2).$$

3. The induced almost contact structure and structure equations on (M^5,Ψ)

Let ξ_1,ξ_2 be mutually orthogonal unit normal vector fields on a neighborhood of $p\in M^5$. The vector field u, 1-form η and (1.1) tensor field φ are defined by $u=\xi_1\times\xi_2$, $\eta(X)=\langle x,u\rangle$ and $\varphi X=X\times u$ for any $X\in\mathcal{X}(M^5)$, respectively, where $\times$ is the exterior product of $\mathbf{O}$ which is defined by $x\times y=(\overline{y}x-\overline{x}y)/2$ for any $x,y\in\mathbf{O}$. We noted that $\xi_1\times\xi_2$ is depend on the orientation of M^5 in Im $\mathbf{O}$, so u is a global vector field on M^5. Then, $(\varphi,u,\eta,<\ ,\ >)$ is the almost contact metric structure. In fact, for any $X\in\mathcal{X}(M^5)$, we get

$$\begin{aligned}\varphi^2X&=(X\times u)\times u\\&=\{(X-<X,u>u+<X,u>u)\times u\}\times u\\&=\{\overline{u}(X-<X,u>u)\}\times u.\end{aligned}$$

On one hand, we have

$$<\overline{u}(X-<X,u>u),u>=0.$$

From these, we get

$$\varphi^2X=\overline{u}\{\overline{u}(X-<X,u>u)\}=(\overline{u})^2(X-<X,u>u)=-X+\eta(X)u.$$

We get, also

$$\begin{aligned} < \varphi X, \varphi Y > &=< X \times u, Y \times u > \\ &=< \overline{u}(X- < X, u > u), \overline{u}(Y- < Y, u > u) > \\ &=< X - \eta(X)u,\ Y - \eta(Y)u > \\ &=< X, Y > -\eta(X)\eta(Y). \end{aligned}$$

Next, we shall give the structure equations of (M^5, Ψ). We set $\mathcal{F}_\Psi(M^5) = \{(p; (u, f, \overline{f}))| -2\sqrt{-1} f_1 \wedge \overline{f}_1 = T_p^\perp M^5\}$. From Fact 2.3, there exists $g \in G_2$ such that $-2\sqrt{-1}\, f_1 \wedge \overline{f}_1 = T_p^\perp M^5$. By [2: p.194. (1.31)], we get

$$\begin{aligned} \varphi(f_\alpha) &= f_\alpha \times u = g(E_\alpha) \times g(\varepsilon) = \chi_g(E_\alpha \times \varepsilon) = \sqrt{-1}\, \chi_g(E_\alpha) \\ &= \sqrt{-1}\, \chi_g(E_\alpha \times 1) = \sqrt{-1}\, g(E_\alpha) \times g(1) = \sqrt{-1}\, f_\alpha. \end{aligned}$$

By the definition of φ, we get $\varphi(u) = u \times u = 0$. From these facts, we get

PROPOSITION 3.1. *$\mathcal{P} : \mathcal{F}_\Psi(M^5) \to M^5$ is the $U(2)$ principal right bundle over M^5 with the natural projection $\mathcal{P}$.*

We call $\mathcal{P} : F_\Psi(M^5) \to M^5$ the adapted frame bundle of M^5. By proposition 3.1, we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{F}_\Psi(M^5) & \xrightarrow{\tilde{\Psi}} & \mathcal{F} \\ {\scriptstyle U(2)}\downarrow & & \downarrow{\scriptstyle G_2} \\ M^5 & \xrightarrow{\Psi} & \mathbf{O} \end{array}$$

where $\tilde{\Psi}$ is the natural inclusion. Then, from (2.2) and (2.6), we have the following structure equations on M^5:

$$\omega^1 = \overline{\omega}^1 = 0 \quad \text{on} \quad \mathcal{F}_\Psi(M^5), \tag{3.1}$$

$$dx = u \otimes \eta + \sum_{i=2}^{3} \{f_i \otimes \omega^i + \overline{f}_i \otimes \bar{\omega}^i\}, \tag{3.2}$$

$$du = \sum_{\alpha=1}^{3}\{f_\alpha(-2\sqrt{-1}\,\theta^\alpha) + \overline{f}_\alpha(2\sqrt{-1}\,\overline{\theta}^\alpha)\}, \tag{3.3}$$

(Gauss formula)

$$df_2 = u(-\sqrt{-1}\,\overline{\theta}^2) + \sum_{\alpha=1}^{3} f_\alpha\kappa_2^\alpha - \overline{f}_3\theta^1 + \overline{f}_1\theta^3, \tag{3.4}$$

$$df_3 = u(-\sqrt{-1}\,\overline{\theta}^3) + \sum_{\alpha=1}^{3} f_\alpha\kappa_3^\alpha - \overline{f}_1\theta^2 + \overline{f}_2\theta^1, \tag{3.5}$$

$$df_1 = u(-\sqrt{-1}\,\overline{\theta}^1) + \sum_{\alpha=1}^{3} f_\alpha\kappa_1^\alpha - \overline{f}_2\theta^3 + \overline{f}_3\theta^2, \tag{3.6}$$

(Weingarten formula)

$$d\omega^2 = 2\sqrt{-1}\,\theta^2\wedge\eta - \sum_{i=2}^{3}\kappa_i^2\wedge\omega^i - \overline{\theta}^1\wedge\overline{\omega}^3, \tag{3.7}$$

$$d\omega^3 = 2\sqrt{-1}\,\theta^3\wedge\eta - \sum_{i=2}^{3}\kappa_i^3\wedge\omega^i + \overline{\theta}^1\wedge\overline{\omega}^2, \tag{3.8}$$

$$d\eta = \sqrt{-1}\sum_{i=2}^{3}(\overline{\theta}^i\wedge\omega^i - \theta^i\wedge\overline{\omega}^i), \tag{3.9}$$

$$\Phi = -(\sqrt{-1}/2)\sum_{i=2}^{3}(\omega^i\wedge\overline{\omega}^i), \tag{3.10}$$

where $\Phi(X,Y) =< X,\varphi Y >$ for any $X,Y \in \mathcal{X}(M^5)$. By (3.1), we get $d\omega^1 = 0$. From (2.6) and Cartan's lemma, there exist $a \in \mathbf{C}$, $b,c \in M_{2\times1}(\mathbf{C})$, $A,B,C \in M_{2\times2}(\mathbf{C})$ satisfying

$$^tA = A, \quad ^tC = C,$$

$$\begin{pmatrix}\theta^1\\ \zeta\\ \Theta\end{pmatrix} = \begin{pmatrix} a & b & c\\ -2\sqrt{-1}\,{}^tb & A & B\\ 2\sqrt{-1}\,{}^t\overline{c} & C & {}^t\overline{B}\end{pmatrix}\begin{pmatrix}\eta\\ \mu\\ \overline{\mu}\end{pmatrix}, \tag{3.11}$$

where $\zeta = {}^t(\kappa_2^1,\kappa_3^1)$, $\Theta = {}^t(\theta^3,-\theta^2)$ and $\mu = {}^t(\omega^2,\omega^3)$. By (3.11), the second fundamental from II is given by

$$\begin{aligned} II &= -2Re\{(2\sqrt{-1}\eta \circ \theta^1 - {}^t\mu \circ \zeta - {}^t\mu \circ \overline{\Theta})f_1\} \\ &= -2Re\{[2\sqrt{-1}\eta \circ (a\eta + b\mu + c\overline{\mu}) \\ &- {}^t\mu \circ (-2\sqrt{-1}{}^t b\eta + A\mu + B\overline{\mu}) \\ &- {}^t\overline{\mu} \circ (-2\sqrt{-1}{}^t c\eta + \overline{C}\overline{\mu} + {}^t B\mu)]f_1\} \end{aligned}$$

Hence, we have the following canonical splitting:

$$\begin{aligned} II^{(2,0)} &= -({}^t\mu \circ A\mu)f_1, \\ II^{(1,1)} &= -({}^t\mu \circ B\overline{\mu} + {}^t\overline{\mu} \circ {}^t B\mu)f_1, \\ II^{(0,2)} &= -({}^t\overline{\mu} \circ \overline{C}\overline{\mu})f_1. \end{aligned}$$

The mean curvature $\mathfrak{H}$ is given by

$$\mathfrak{H} = -(4/5)Re\{(\sqrt{-1}\,a - 2trB)f_1\}. \tag{3.13}$$

4. Normal almost contact submanifold (M^5, Ψ)

Let D (resp. $D\pm$) be the contact distribution which is defined by $D(p) = \{X \in T_pM^5 \,|\, \eta(X) = 0\}$ (resp. $D \pm (p) = \{X \in \mathbf{C} \otimes_{\mathbf{R}} D(p) \,|\, \varphi X = \pm\sqrt{-1}X\}$).

THEOREM 4.1. *The induced almost contact metric structure of (M^5, Ψ) is normal, if and only if*

(1) *$D\pm$ are involutive distributions,*
(2) *Each integral curve of u is a line in $Im \vee$,*
(3) *$II^{(1,1)} = 0$, rank $II^{(0,2)} \leq 1$,*
(4) *M^5 is a minimal submanifold.*

Proof. The induced almost contact metric structure is normal if and only if

$$\varphi((\nabla_X\varphi)Y) - ((\nabla_{\varphi X}\varphi)Y) - ((\nabla_X\eta)Y)u = 0 \tag{4.1}$$

for any $X, Y \in \mathcal{X}(M^5)$. Any vector field Y on M^5 is given by

$$Y = uY^0 + \sum_{i=2}^{3}(f_iY^i + \overline{f}_i\overline{Y}^i). \tag{4.2}$$

On one hand, by (3.3) $\sim$ (3.5), we get

$$\nabla_X u = \sum_{i=2}^{3}\{f_i(-2\sqrt{-1}\theta^i(X) + \overline{f}_i(2\sqrt{-1}\,\bar{\theta}^i(X)\},$$

$$\nabla_X f_2 = u(-\sqrt{-1}\,\overline{\theta}^2(X)) + \sum_{i=2}^{3} f_i\kappa_2^i(X) - \overline{f}_3\theta^1(X), \tag{4.3}$$

$$\nabla_X f_3 = u(-\sqrt{-1}\,\overline{\theta}^3(X)) + \sum_{i=2}^{3} f_i\kappa_3^i(X) + \overline{f}_2\theta^1(X),$$

By (4.1) $\sim$ (4.3), we get

$$\begin{aligned}(\nabla_X\varphi)Y &= \nabla_X(\varphi Y) - \varphi(\nabla_X Y) \\ &= \nabla_X\{\sqrt{-1}\sum_{i=2}^{3}(f_iY^i + \overline{f}_i\bar{Y}^i)\} \\ &\quad - \varphi\{\nabla_X(uY^0 + \sum_{i=2}^{3}(f_iY^i + \overline{f}_i\bar{Y}^i))\} \\ &= u\sum_{i=2}^{3}(\theta^i(X)\bar{Y}^i + \bar{\theta}^i(X)Y^i) \\ &\quad - 2\{\sum_{i=2}^{3}(f_i\theta^i(X) + \overline{f}_i\bar{\theta}^i(X)\}Y^0 \\ &\quad - 2\sqrt{-1}(f_2\overline{\theta}^1(X)\overline{Y}^3 - \overline{f}_2\theta^1(X)Y^3) \\ &\quad + 2\sqrt{-1}(f_3\overline{\theta}^1(X)\overline{Y}^2 - \overline{f}_3\theta^1(X)Y^2).\end{aligned} \tag{4.4}$$

From (4.2) and (4.3), we get

$$(\nabla_X\eta)Y = \sqrt{-1}\sum_{i=2}^{3}\{\bar{\theta}^i(X)Y^i - \theta^i(X)\overline{Y}^i\}. \tag{4.5}$$

By (4.1), (4.4) and (4.5), we get

$$
\begin{aligned}
(4.6) \qquad 0 = & -2\sqrt{-1}\{\sum_{i=2}^{3}(f_i\theta^i(X) + \overline{f}_i\bar{\theta}^i(X)\}Y^0 \\
& + 2(f_2\overline{\theta}^1(X)\overline{Y}^3 + \overline{f}_2\theta^1(X)Y^3) \\
& - 2(f_3\overline{\theta}^1(X)\overline{Y}^2 + \overline{f}_3\theta^1(X)Y^2) \\
& - \{u\sum_{i=2}^{3}(\theta^i(\varphi X)\bar{Y}^i + \bar{\theta}^i(\varphi X)\bar{Y}^i) \\
& - 2\sum_{i=2}^{3}(f_i\theta^i(\varphi X) + \overline{f}_i\bar{\theta}^i(\varphi X))Y^0 \\
& - 2\sqrt{-1}(f_2\overline{\theta}^1(\varphi X)\overline{Y}^3 - \overline{f}_2\theta^1(\varphi X)Y^3) \\
& + 2\sqrt{-1}(f_3\overline{\theta}^1(\varphi X)\overline{Y}^2 - \overline{f}_3\theta^1(\varphi X)Y^2)\} \\
& - u\sqrt{-1}\sum_{i=2}^{3}\{\bar{\theta}^i(X)Y^i - \theta^i(X)\bar{Y}^i\}.
\end{aligned}
$$

By (4.6), we get

$$
(4.7) \qquad \theta^1(\varphi X) = -\sqrt{-1}\theta^1(X), \quad \theta^i(\varphi X) = \sqrt{-1}\theta^i(X)
$$

for any $i = 2, 3$. By (4.7), we get

$$
(4.8) \qquad \theta^\alpha(u) = \theta^1(f_j) = \theta^i(\overline{f}_j) = 0,
$$

for any $\alpha = 1, 2, 3$ and $i, j = 2, 3$. By (2.15) and (4.8), we get

$$
(4.9) \qquad \begin{pmatrix} \theta^1 \\ \zeta \\ \Theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & C & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \mu \\ \overline{\mu} \end{pmatrix}.
$$

By (4.9), we have (4). Since, $\theta^1 = 0$, by (2.3), we get

$$
(4.10) \qquad -\kappa_2^1 \wedge \theta^2 - \kappa_3^1 \wedge \theta^3 - 2\overline{\theta}^2 \wedge \overline{\theta}^3 = 0
$$

By (4.9) and (4.10), we have (3). From (3.7), (3.8) and (4.9), we get

$$d\omega^2 = d\omega^3 = 0 \mod (\omega^2, \omega^3).$$

Thus, we have (1). By (3.3) and (4.9), we get

$$du = 0 \mod (\omega^i, \overline{\omega}^i).$$

Thus, we obtain (2).

5. Quasi-Sasakian (M^5, Ψ) and nearly cosymplectic (M^5, Ψ)

In this section, we shall prove the following:

THEOREM 5.1. *The induced almost contact submanifold (M^5, Ψ) is quasi-Sasakian manifold (i.e., normal and $d\Phi = 0$) if and only if*

(1) *u is a constant vector field on M^5,*

(2) *$II^{2,0} = II^{1,1} = 0$ on M^5,*

(3) *$\Psi(M^5)$ is locally isometric to $N^4 \times \mathbf{R}$ where N^4 is a complex hypersurface in $\mathbf{C}^3 = ((span_{\mathbf{R}}\{u\})^{\perp}, J_u)$.*

Proof. By (4.9), we get

$$d\Phi = -\sum_{i=2}^{3}[\theta^i \wedge \bar{\omega}^i + \overline{\theta}^i \wedge \omega^i] \wedge \eta = 0. \tag{5.1}$$

By (4.9) and (5.1), we get $C = 0$. Hence, we have $II^{2,0} = 0$. From this and (4.9), we get

$$\theta^{\alpha} = 0, \tag{5.2}$$

for $\alpha = 1, 2, 3$ on M^5. By (3.3), (3.9), (4.4) and (5.2), we get

$$du = d\eta = (\nabla_X \varphi)Y = 0$$

for any $X, Y \in \mathcal{X}(M^5)$. Thus, we get the desired conclusion.

THEOREM 5.2. *If the induced contact submanifold* (M^5, Ψ) *is nearly cosymplectic manifold (i.e.,* $(\nabla_X \varphi)X = 0$ *for any* $X \in \mathcal{X}(M^5)$*), then* Ψ *is totally umbilic.*

Proof. For any $X \in \mathcal{X}(M^5)$, we get

$$(5.3) \qquad \begin{aligned}(\nabla_X\varphi)X =& u\sum_{i=2}^{3}(\theta^i(X)\bar{X}^i + \overline{\theta}^i(X)X^i)\\ &- 2\{\sum_{i=2}^{3}(f_i\theta^i(X) + \overline{f}_i\bar{\theta}^i(X)\}X^0\\ &- 2\sqrt{-1}(f_2\overline{\theta}^1(X)\overline{X}^3 - \overline{f}_2\theta^1(X)X^3)\\ &+ 2\sqrt{-1}(f_3\overline{\theta}^1(X)\overline{X}^2 - \overline{f}_3\theta^1(X)X^2),\end{aligned}$$

where $X = uX^0 + \sum_{i=2}^3\{f_iX^i + \overline{f}_i\bar{X}^i\}$. By (5.3), we get

$$(5.4) \qquad \begin{aligned}\theta^i(u) = \theta^i(\overline{f}_i) = \theta^i(f_j) = \theta^1(f_i) = 0,\\ \theta^2(\overline{f}_3) = -\theta^3(\overline{f}_2) = -\sqrt{-1}\theta^1(u),\end{aligned}$$

By (5.4), we get

$$(5.5) \qquad \begin{pmatrix}\theta^1\\ \zeta\\ \Theta\end{pmatrix} = \begin{pmatrix}a & 0 & 0\\ 0 & A & -\sqrt{-1}aI_2\\ 0 & 0 & \sqrt{-1}\,\overline{a}I_2\end{pmatrix}\begin{pmatrix}\eta\\ \mu\\ \overline{\mu}\end{pmatrix}.$$

By (5.5), the mean curvature $\mathfrak{H}$ is given by

$$(5.6) \qquad \mathfrak{H} = -4Re\{(\sqrt{-1}a)f_1\}.$$

By (2.3), (3.9) and (5.5), we get

$$(5.7) \qquad \begin{aligned}d\theta^1 &= -\sum_{\alpha=1}^{3}\kappa^1_\alpha \wedge \theta^\alpha - 2\theta^2 \wedge \theta^3\\ &= d(a\eta) = da \wedge \eta + \sqrt{-1}a({}^t\overline{\theta} \wedge \mu - {}^t\theta \wedge \overline{\mu}).\end{aligned}$$

By (5.5) and (5.7), we get

$$(5.8)\qquad \begin{aligned} &da \wedge \eta + \kappa_1^1 \wedge a\eta = 0, \\ &\sqrt{-1}a({}^t\overline{\theta} \wedge \mu - {}^t\theta \wedge \overline{\mu}) = {}^t\theta \wedge \zeta - 2\theta^2 \wedge \theta^3. \end{aligned}$$

If we put $F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then we get

$$(5.9)\qquad \theta = \sqrt{-1}aF\overline{\mu}.$$

By $(5.8)_2$ and (5.9), we get

$$(5.10)\quad \sqrt{-1}a\{\sqrt{-1}\,\overline{a}{}^t\mu F \wedge \mu - \sqrt{-1}a{}^t\overline{\mu}F \wedge \overline{\mu}\} \\ = (-\sqrt{-1}a{}^t\overline{\mu}F) \wedge (A\mu - \sqrt{-1}a\overline{\mu}) - 2(-\sqrt{-1}\,\overline{a}\omega^3) \wedge (\sqrt{-1}\,\overline{a}\omega^2).$$

We take the (1,1)-part of (5.10), (Since the form η is a contact form, $a \neq 0$ (see 7, Proposition 7.1)), we get $A = 0$. Hence, by (5.5) and (5.6), the second fundamental form is given by

$$II = -4Re\{\sqrt{-1}a(\eta \circ \eta + {}^t\mu \circ \overline{\mu})f_1\} = g \otimes \mathfrak{H}.$$

By the structure equations, we have

PROPOSITION 5.3. *The induced almost contact metric structure is not associated one (i.e., $2\Phi \neq d\eta$).*

Proof. If $2\Phi = d\eta$, then we have

$$(5.11)\qquad -\sqrt{-1}{}^t\mu \wedge \overline{\mu} = \sqrt{-1}\sum_{i=2}^{3}(\bar{\theta}^i \wedge \omega^i - \theta^i \wedge \bar{\omega}^i)$$

By (3.11) and (5.11), we get

$$(5.12)\qquad c = 0,\ {}^t\mu B \wedge F\mu = 0,\ {}^t\mu \wedge \overline{\mu} = {}^t\overline{\mu}\,\overline{C} \wedge F\mu - {}^t\mu C \wedge F\overline{\mu},$$

where $F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If we put $C = \begin{pmatrix} k & 1 \\ 1 & m \end{pmatrix}$, then, by $(5.12)_3$, we have $1 + \overline{1} = 1 = -1$. This is a contradiction.

COROLLARY 5.4. *The induced almost contact submanifold in* $\operatorname{Im}\mathbb{V}$ *is not Sasakian manifold.*

6. The relation of the Gauss map

THEOREM 6.1. *Let* $g : M^5 \to \tilde{G}(2, \operatorname{Im}\mathbb{V}) = G_2/U(2)$ *be the Gauss map and* $\tilde{J}$ *the canonical complex structure on* $\tilde{G}(2, \operatorname{Im}\mathbb{V})$.

(1) *If the Gauss map* g *is* φ*-holomorphic (i.e.,* $dg \circ \varphi = \tilde{J} \circ dg$*), then* M^5 *is a quasi-Sasakian manifold.*

(2) *If the Gauss map* g *is anti-*φ*-holomorphic (i.e.,* $dg \circ \varphi = -\tilde{J} \circ dg$*), then* M^5 *is a normal almost contact manifold and* $II^{2,0} = II^{1,1} = 0$.

Proof. The (1,0) part of the canonical almost complex structure of $\tilde{G}(2, \operatorname{Im}\mathbb{V})$ is given by $\operatorname{span}_{\mathbf{R}}\{\overline{\kappa}_1^2, \overline{\kappa}_1^3, \theta^1, \overline{\theta}^2, \overline{\theta}^3\}$. By (2.7), we get

$$
\begin{aligned}
dg = (-2\sqrt{-1})\{u \wedge \overline{f}_1(-2\sqrt{-1}\theta^1) + \sum_{i=2}^{3} f_i \wedge \overline{f}_1 \kappa_1^i \\
- \overline{f}_2 \wedge \overline{f}_1 \theta^3 + \overline{f}_3 \wedge \overline{f}_1 \theta^2\} \\
+ (2\sqrt{-1})\{f_1 \wedge u(2\sqrt{-1}\theta^1) + \sum_{i=2}^{3} f_1 \wedge \overline{f}_i \overline{\kappa}_1^i \\
- f_1 \wedge f_2 \overline{\theta}^3 + f_1 \wedge f_3 \overline{\theta}^2\}.
\end{aligned} \tag{6.1}
$$

Case (1). g is φ-holomorphic if and only if

$$dg(u) = 0 \quad \text{and} \quad (dg(f_i))^{(0,1)} = 0. \tag{6.2}$$

By (6.2), we get

$$\theta^1(u) = \kappa_1^i(u) = \theta^i(u) = \overline{\theta}^1(f_j) = \theta^i(f_j) = \kappa_1^i(f_j) = 0, \tag{6.3}$$

for any $i, j = 2, 3$. By (3.11) and (6.3), we get $II^{(0,2)} = II^{(1,1)} = 0$. By Theorem 5.1, we get desired result. Similarly, in case (2), we get the conclusion.

7. On the contact manifolds

By the direct calculation, we get the following:

Proposition 7.1. $\eta \wedge (d\eta)^2 = -2\{2\,Re(\det C) + \|C\|^2 - \|tr\,B\|^2\}$.

Corollary 7.2. *Let (M^5, Ψ) be the normal almost contact manifold with $\eta \wedge (d\eta)^2 = 0$. Then, it is a quasi-Sasakian manifold.*

Next, we shall consider the relation between the induced contact structure and the product immersion. Let $f\varepsilon \times g : M^3 \times N^2 \to \mathbf{H}\varepsilon \oplus \operatorname{Im}\mathbf{H} \cong \operatorname{Im}\mathbb{V}$ be the product immersion where $f : M^3 \to \mathbf{H}$ and $g : N^2 \to \operatorname{Im}\mathbf{H}$ are oriented hypersurfaces, and $\varepsilon = (0,1) \in \operatorname{Im}\mathbb{V}$. We denote by ξ_1 (resp. ξ_2) the unit normal vector field of M^3 in $\mathbf{H}$ (resp. N^2 in $\operatorname{Im}\mathbf{H}$). Then, we have $\xi_2(q) \times \xi_1(p)\varepsilon \in T_pM^3$ for any $(p,q) \in M^3 \times N^2$. In fact, we get

$$\xi_2(q) \times \xi_1(p)\varepsilon = (\xi_1(p)\xi_2(q))\varepsilon. \tag{7.1}$$

On one hand, $\{(\xi_1(p)i)\varepsilon,\ (\xi_1(p)j)\varepsilon,\ (\xi_1(p)k)\varepsilon\}$ is an orthonormal basis of T_pM^3 where $\{1, i, j, k\}$ is the canonical basis of $\mathbf{H}$. Hence, by (7.1), we get

$$\begin{aligned}\xi_2(q) \times \xi_1(p)\varepsilon = \langle \xi_2(q), i\rangle(\xi_1(p)i)\varepsilon \\ + \langle \xi_2(q), j\rangle(\xi_1(p)j)\varepsilon + \langle \xi_2(q), k\rangle(\xi_1(p)k)\varepsilon.\end{aligned} \tag{7.2}$$

We put $\mu_1 = \langle \xi_2(q), i\rangle$, $\mu_2 = \langle \xi_2(q), j\rangle$, $\mu_3 = \langle \xi_2(q), k\rangle$. Let ω^1, ω^2, ω^3 be the dual 1-forms on M^3 of the basis $\xi_1 i$, $\xi_1 j$, $\xi_1 k$, respectively. Then, by (7.2), the forms η, $d\eta$ is represented by

$$\begin{aligned}\eta &= \sum_{\alpha=1}^{3} \mu_\alpha \omega^\alpha, \\ d\eta &= \sum_{\alpha=1}^{3}\{\sum_{i=1}^{2} d\mu_\alpha(e_i)\nu^i \wedge \omega^\alpha\} + \sum_{\alpha=1}^{3} \mu_\alpha d\omega^\alpha,\end{aligned} \tag{7.3}$$

where $\{e_1, e_2\}$ is the orthonormal frame of T_qN^2 and $\{\nu_1, \nu_2\}$ is the dual 1-forms of $\{e_1, e_2\}$ on N^2. Since, $d\omega^\alpha = \sum_{\beta,\gamma=1}^{3} \Lambda^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma$, we get

(7.4) $\eta \wedge (d\eta)^2$

$$= -2 \begin{vmatrix} \mu_1 & \mu_2 & \mu_3 \\ d\mu_1(e_1) & d\mu_2(e_1) & d\mu_3(e_1) \\ d\mu_1(e_2) & d\mu_2(e_2) & d\mu_3(e_2) \end{vmatrix} \nu^1 \wedge \nu^2 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3.$$

If we take $\{e_1, e_2\}$ the principal vector of the shape operator A_{ξ_2}, then (7.4) implies

$$\eta \wedge (d\eta)^2 = -2K\sigma_1 \wedge \sigma_2,$$

where K is the Gauss curvature of N^2 and σ_1 (resp. σ_2) is the volume element of N^2 (resp. M^3). Hence, we get the following;

THEOREM 7.3. *Let $f\varepsilon \times g : M^3 \times N^2 \to \mathbf{H}\varepsilon \oplus \mathrm{Im}\,\mathbf{H} \cong \mathrm{Im}\,\mathbb{V}$ be the product immersion where $f : M^3 \to \mathbf{H}$ and $g : N^2 \to \mathrm{Im}\,\mathbf{H}$ are oriented hypersurfaces, and $\varepsilon = (0,1) \in \mathrm{Im}\,\mathbb{V}$. Then, we have*

$$\eta \wedge (d\eta)^2 = -2K\sigma,$$

where K is the Gauss curvature of N^2 and σ is the volume element of $M^3 \times N^2$.

From this, we see that there exists many contact submanifolds $(M^3 \times N^2, f\varepsilon \times g)$ in $\mathrm{Im}\,\mathbb{V}$. However, we have

COROLLARY 7.4. *If N^2 is diffeomorphic to the torus and M^3 is compact, the induced almost contact structure is not a contact.*

Proof. By Theorem 7.3 and Gauss-Bonnet Theorem, we get

$$\int_{M^3 \times N^2} \eta \wedge (d\eta)^2 = -2\int_{M^3 \times N} K\sigma = -4\pi X(N^2)\mathrm{vol}(M^3),$$

where $X(N^2)$ is the Euler number of N^2. Since N^2 is diffeomorphic to torus, we have

$$\int_{M^3 \times N^2} \eta \wedge (d\eta)^2 = 0.$$

Hence, there exists a point $m \in M^3 \times N^2$ such that $\eta \wedge (d\eta)^2(m) = 0$.

REMARK. Corollary 7.5 is a partial negative answer to Blair's problem in ([1; page 71]).

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