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On Uniform Integrability

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ABSTRACT. In this paper, we show that uniform integrability is equivalent to convergence to a μ-integrable function f in L_1 for μ-integrable functions in the sense of the integral defined by Lewis.

1. Introduction

The concept of an asymptotic martingale, abbreviated as amart, was first given by Meyer [**7**] who proved that a continuous parametered scalar-valued amart converges almost everywhere if it is essentially bounded. Austin, Edgar, and Tulcea [**1**] proved that a real valued amart converges almost everywhere if it is L_1 bounded.

These amarts are integrable in the sense of the integral defined by Lewis [**5**]. Lewis developed the integration theory through the study of the p-semi-variation of the vector measure whose value is in a locally convex topological vector space V, where P is a semi-norm on V. In this paper, we show that (f_n) is uniformly integrable is equivalent to (f_n) converges to a μ-integrable function f in L_1 for μ-integrable functions in the sense of the integral defined by Lewis. Most of the notations and terminologies follow those of Smith [**9**].

2. Main Theorems

PROPOSITION 2.1. *Let (f_n) be a sequence of μ-integrable functions. Then the following are equivalent.*

(a) *(f_n) is L_1-bounded and uniformly absolutely continuous with respect to μ.*
(b) *(f_n) is uniformly integrable.*

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PROOF: First we shall show that (a) implies (b). Let $\epsilon > 0$ be given and P an arbitrary but fixed continuous semi-norm on V. By (a), we have

$$\{\sup \int_E |f_n| dv(x^*\mu) : x^* \leq P, n \in N\} = M < \infty$$

and there is a number $\delta > 0$ such that $\|\phi_n\|_p(E) < \epsilon$ whenever $E \in \sum$ and $\|\mu\|_P(E) < \delta$. Let $a = \frac{M}{\delta}$ and let $\delta = \frac{\epsilon}{a}$. Since $\sup \|\mu\|_p\{|f_n| > a\} \leq \frac{M}{a} = \delta$, we have $\|\phi_n\|_p(\{|f_n| > a\}) < \epsilon$ for all f_n. Thus we get

$$\lim_{a \to \infty} \|\phi_n\|_p(|f_n| > a) = 0$$

uniformly n, for every continuous semi-norm P. Hence (f_n) is uniformly integrable.

Next we want to show that (b) implies (a). Let P be an arbitrary but fixed continuous semi-norm on V. Since (f_n) is uniformly integrable, we have

$$\lim_{a \to \infty} \|\phi\|_p(|f_n| > a) = 0$$

uniformly in n. For nonnegative μ-integrable function, $E \in \sum$ and $a > 0$, we obtain that

$$\begin{aligned}\int_E f_n \, dv(x^*\mu) &= \int_{E \cap \{f_n \leq a\}} f_n \, dv(x^*\mu) + \int_{E \cap \{f_n > a\}} f_n \, dv(x^*\mu) \\ &\leq a \cdot v(x^*\mu, (f_n \leq a)) + \int_{\{f_n > a\}} f_n \, dv(x^*\mu).\end{aligned}$$

Hence we get

$$\int |f_n| \, dv(x^*\mu) \leq a \cdot v(x^*\mu, (f_n \leq a)) + 1 \leq a \cdot \|\mu\|_p(S) + 1,$$

for each $n \in N$ and each $x^* \leq P$. It follows from Proposition 2.1 that $\|\mu\|_p(S)$ is finite. Therefore we obtain

$$\sup\{\int |f_n| \, dv(x^*\mu); n \in N, x^* \leq P\} < 1 + a \cdot \|\mu\|_p(S) < \infty.$$

This means that (f_n) is L_1-bounded. Let $\epsilon > 0$ be given. Since (f_n) is uniformly integrable, there exists an integer N_ϵ such that

$$\|\phi_n\|_p(|f_n| > N_\epsilon) < \frac{\epsilon}{2}$$

for each $n \in N$. Choose $\delta = \frac{\epsilon}{2N_\epsilon}$ and let E be in $\sum$ such that $\|\mu\|_p(E) < \delta$. Then we have

$$\begin{aligned}
&\int_E |f_n|\, dv(x^*\mu) \\
&= \int_{E\cap(|f_n|>N_\epsilon)} |f_n|\, dv(x^*\mu) + \int_{E\cap(|f_n|\leq N_\epsilon)} |f_n|\, dv(x^*\mu) \\
&\leq \int_{(|f_n|>N_\epsilon)} |f_n|\, dv(x^*\mu) + N_\epsilon \cdot v(x^*\mu, E) \\
&< \frac{\epsilon}{2} + N_\epsilon \cdot \|\mu\|_p(E) < \epsilon
\end{aligned}$$

for each $n \in N$ and for each $x^* \in V^*$, $x^* \leq p$. Therefore

$$\|\phi_n\|_p(E) = \sup\{\int_E |f_n|\, dv(x^*\mu)\,;\; x^* \leq P\} < \epsilon$$

for each $n \in N$. This means that given any $\epsilon > 0$, there is some $\delta > 0$ such that $\|\phi_n\|_p(E) < \epsilon$ for each $n \in N$, whenever $E \in \sigma$ and $\|\mu\|_p(E) < \delta$. Therefore (f_n) is uniformly absolutely continuous with respect to μ. This completes the proof.

THEOREM 2.2. *Let (f_n) be a sequence of μ-integrable functions. Then the following are equivalent.*

(a) *(f_n) converges to a μ-integrable function f in L_1.*
(b) *(f_n) is uniformly integrable and (f_n) converges to f in μ-measure.*

PROOF: Let P be an arbitrary but fixed continuous semi-norm. Since (f_n) converges to f in L_1, we have

$$\lim_n \int |f_n - f|\, dv(x^*\mu) = 0$$

uniformly in $x^* \in V^*$, $x^* \leq P$. Take $\epsilon_0 = 1$. Then there is an integer n_1 such that $\int |f_n - f| \, dv(x^*\mu) < 1$ for each $n \geq n_1$, $n \in N$ and $x^* \leq P$. Hence we get

$$\begin{aligned}\int |f_n| \, dv(x^*\mu) &\leq \int |f_n - f| \, dv(x^*\mu) + \int |f| \, dv(x^*\mu) \\ &\leq 1 + \|\phi\|_p(S)\end{aligned}$$

for each $n \in N$ with $n \geq n_1$, where $\|\phi\|_p(S) = \sup\{\int |f| \, dv(x^*\mu) : x^* \leq p\}$. Therefore we obtain

$$\begin{aligned}&\sup\{\int |f_n| \, dv(x^*\mu) : x^* \leq P, n \in N\} \\ &= \max\{1 + \|\phi\|_p(S), 1 + \|\phi_1\|, \ldots, 1 + \|\phi_{n_1} - 1\|_p(S)\} < \infty.\end{aligned}$$

where $\|\phi\|_p(S) = \sup\{\int |f_k| \, dv(x^*\mu) : x^* \leq P\}$ for each $k = 1, 2, \ldots, n_1 - 1$. This shows that (f_n) is L_1-bounded. Let $\epsilon > 0$ be given. Since (f_n) converges to f in L_1, we have

$$\lim_n \int |f_n - f| \, dv(x^*\mu) = 0$$

uniformly in $x^* \leq P$. There is an integer n_0 such that $\int |f_n - f| \, dv(x^*\mu) < \frac{\epsilon}{2}$, for each $n \in N$ with $n \geq n_0$ and each $x^* \leq P$. Since f is μ-integrable, there exists a number $\delta_0 > 0$ such that $\|\phi\|_p(E) < \frac{\epsilon}{2}$ whenever $E \in \Sigma$ and $\|\mu\|_p(E) < \delta_0$. Now,

$$\begin{aligned}\int_E |f_n| \, dv(x^*\mu) &\leq \int_E |f_n - f| \, dv(x^*\mu) + \int_E |f| \, dv(x^*\mu) \\ &\leq \int_E |f_n - f| \, dv(x^*\mu) + \|\phi\|_p(E)\end{aligned}$$

for each $x^* \leq P$. Therefore for every integer n, $n \geq n_0$ and every E in $\sum$ with $\|\mu\|_p(E) < \delta_0$, we have

$$\int_E |f_n| \, dv(x^*\mu) < \epsilon$$

for each $x^* \leq P$. Since $f_1, \ldots, f_{n_0-1}$ are μ-integrable, there exist numbers $\delta_1, \delta_2, \ldots, \delta_{n_0-1}$ such that for each $m \in \{1, 2, \ldots, n_0 - 1\}$,

$\|\phi_m\|_p < \epsilon$ whenever $E \in \sum$ and $\|\mu\|_p < \delta_m$. Let $\delta = \min\{\delta_0, \delta_1, \ldots, \delta_{n_0-1}\}$. Then we have $\int_E |f_n|\,dv(x^*\mu) < \epsilon$ for each $x^* \leq P$ and each $n \in N$, whenever $E \in \sum$ and $\|\mu\|_p(E) < \delta$. This means that (f_n) is uniformly absolutely continuous with respect to μ. Hence, by Proposition 2.1, (f_n) is uniformly integrable. Next we show that (f_n) converges to f in μ-measure. Suppose not. Then there exist $\epsilon > 0$ and $\delta > 0$ such that $\|\mu\|_p(\{s \in S : |f_n(s) - f(s)| > \epsilon\}) > \delta$ for infinitely many n. Hence $\|f_n - f\| > \epsilon\delta$ for infinitely many n. This contradiction means that (f_n) converges to f in μ-measure. By now, we showed that (a) implies (b).

Next we show that (b) implies (a). Since (f_n) converges to f in μ-measure, some subsequence (f_{n_k}) converges a.e. to f. Hence also $|f_{n_k}|$ converges a.e. to $|f|$. Now $\int |f_{n_k}|\,dv(x^*\mu)$ is bounded since (f_n) is uniformly integrable. Hence, by Fatou's lemma, we have

$$\int |f|\,dv(x^*v) \leq \liminf_{k\to\infty} \int |f_{n_k}|\,dv(x^*v) < \infty$$

and so $f \in L_1$. For $\epsilon > 0$, we let

$$A_n = \left\{ s \in S : |f_n(s) - f(s)| > \frac{\epsilon}{3\|\mu\|_p(S)} \right\}.$$

Then we have $\|\mu\|_p(A_n) \to 0$ as $n \to \infty$. Now we use Proposition 2.1 to deduce that $\|\phi_n\|_p(A_n) < \frac{\epsilon}{3}$ for n large enough. Then we get $\|\phi\|_p(A_n) < \frac{\epsilon}{3}$, since $f \in L_1$. Consequently we have

$$\begin{aligned}
\int |f_n - f|\,dv(x^*\mu) &= \int_{S-A_n} |f_n - f|\,dv(x^*v) + \int_{A_n} |f_n - f|\,dv(x^*\mu) \\
&\leq \frac{\epsilon}{3\|\mu\|_p(S)} \|\mu\|_p(S - A_n) \\
&\quad + \int_{A_n} |f_n|\,dv(x^*\mu) + \int_{A_n} |f|\,dv(x^*\mu) \\
&< \frac{\epsilon}{3} + \|\phi_n\|_p(A_n) + \|\phi\|_p(A_n) < \epsilon,
\end{aligned}$$

for each $n \in N$, $n \geq n_0$ and each $x^* \leq P$, $x^* \in V^*$. This completes the proof.

THEOREM 2.3. *Let (f_n) be a sequence of μ-integrable functions. If there is a positive increasing function ϕ defined on $[0,\infty)$ such that*

$$\lim_{t\to\infty} \phi(t)/t = +\infty$$

and $\sup \int_S (\phi \cdot f_n)\, dv(x^*\mu) < \infty$, *then (f_n) is uniformly integrable.*

PROOF: Let $M = \sup \int_S (\phi \cdot |f_n|)\, dv(x^*\mu)$, and suppose $\epsilon > 0$ is given. Put $a = \frac{M}{\epsilon}$ and choose t_0 such that $\phi(t)/t \geq a$ for $t > t_0$. Hence on the set $\{|f_n| \geq t_0\}$ we have $|f_n| \leq (\phi \cdot |f_n|)/a$ for f_n for all n. So we get

$$\int_{\{|f_n|>t_0\}} |f_n|\, dv(x^*\mu) \leq \frac{1}{a} \int_{\{|f_n|>t_0\}} (\phi \cdot |f_n|)\, dv(x^*\mu) \leq \frac{M}{a} = \epsilon$$

for all f_n. This means that (f_n) is uniformly integrable.

COROLLARY. *If (f_n) is L^p-bounded for some $p > 1$, then (f_n) is uniformly integrable.*

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