JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 4, June 1991

Applications of Floquet Theory

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ABSTRACT. In this paper we obtain the asymptotic behavior of solutions of the perturbed system $x' = (A(t) + B(t))x$ of $x' = A(t)x$ by using the Floquet theorem.

Consider the system

$$x' = A(t)x \tag{L}$$

where $A(t)$ is a matrix of continuous functions on $\mathbf{R}$ and $A(t+T) = A(t)$ for all $t \in \mathbf{R}$ and some constant $T > 0$. Floquet theory for (L) concerns the representation of a fundamental matrix with the same period T and a solution matrix for the system with constant coefficients, that is, if $Z(t)$ is a fundamental matrix for (L), then there are a nonsingular T-periodic matrix P and a constant matrix R such that $Z(t) = P(t)e^{Rt}$.

Murdock [**5**] studied the Floquet theory for quasiperiodic systems. That is, the problem of reducing (L), where $A(t)$ is a quasiperiodic ($A(t) = \hat{A}(\omega_1 t, \omega_2 t, \ldots, \omega_k t), \hat{A}(\theta_1, \ldots, \theta_k)$ is continuous and periodic with period 2π in each argument) $n \times n$ matrix, to a system with constant coefficient is studied by means of an associated linear partial differential equation.

El-Owaidy and Zaghrout [**3**] investigated the generalized Floquet theory.

Becker, Burton and Krisztin [**1**] presented a Floquet-type theory for a system of Volterra equations

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)\,ds + f(t),$$

Received by the editors on 30 June 1991.
1980 *Mathematics subject classifications*: Primary 34D.

where A and C are $n \times n$ matrices, y and f are vectors, $A(t+T) = A(t)$, and $C(t+T, s+T) = C(t,s)$ for some $T > 0$. They also assumed that A and f are continuous on $\mathbf{R}$, while C is continuous for $-\infty < s \leq t < \infty$.

In this paper we obtain the asymptotic behavior of solutions of the perturbed system

$$\text{(PL)} \qquad x' = (A(t) + B(t))x$$

of (L) by using the Floquet theorem.

First, we need the following two lemmas [**2**].

LEMMA 1. (Variation of constants formula). *If Φ is a fundamental matrix for (L), then the function φ defined by*

$$\varphi(t) = \Phi(t) \int_0^t \Phi^{-1}(s) B(s)\, ds$$

is the solution of (PL) *satisfying* $\varphi(0) = x(0)$.

LEMMA 2. (Gronwall's inequality) *Let k be nonnegative constant and let f and g be continuous functions mapping on an interval $[0,\alpha]$ into $[0,\infty)$ with*

$$f(t) \leq k + \int_0^t f(s)g(s)\, ds \quad \text{for } 0 \leq t \leq \alpha,$$

then

$$f(t) \leq k \exp \int_0^t g(s)\, ds \qquad \text{for } 0 \leq t \leq \alpha.$$

We consider the perturbed system

$$\text{(PL)} \qquad x' = (A(t) + B(t))x$$

of (L), where $B(t)$ is an $n \times n$ matrix of continuous functions on $\mathbf{R}$.

THEOREM 3. *Let $A(t+T) = A(t)$ and all solutions of* (L) *tend to zero as $t \to \infty$. Then there exists an $\alpha > 0$ such that $|B(t)| \le \alpha$ implies all solutions of* (PL) *tend to zero as $t \to \infty$.*

PROOF: The fundamental matrix solution of $y' = A(t)y$ is $P(t)e^{Rt}$ by the Floquet theory. Then the solution of (PL) is given by

$$x(t) = P(t)e^{Rt}x(0) + \int_0^t P(t)e^{R(t-s)}P^{-1}(s)B(s)x(s)\,ds.$$

Thus

$$|x(t)| \le |P(t)|\,|e^{Rt}|\,|x(0)| + \int_0^t |P(t)|\,|e^{R(t-s)}|\,|P^{-1}(s)|\,|B(s)|\,|x(s)|\,ds.$$

By the assumption, we can choose constants $K > 0$ and $\beta > 0$ such that $|e^{Rt}| \le Ke^{-\beta t}$. And $P(t)$ is periodic implies $|P(t)| \le M$ for some constant $M > 0$. Thus we have

$$|x(t)| \le MKe^{-\beta t}|x(0)| + \int_0^t Ke^{-\beta(t-s)}|B(s)|\,|x(s)|\,ds$$

or

$$|x(t)|e^{\beta t} \le MK|x(0)| + \int_0^t K|B(s)|\,|x(s)|e^{\beta s}\,ds.$$

Application of Gronwall's inequality yields

$$|x(t)|e^{\beta t} \le MK|x(0)| \exp \int_0^t K|B(s)|\,ds$$

or

$$|x(t)| \le MK|x(0)| \exp \int_0^t (K|B(s)| - \beta)\,ds.$$

Then $|B(s)| \le \alpha$ implies

$$|x(t)| \le MK|x(0)| \exp \int_0^t (\alpha K - \beta)\,ds \to 0 \text{ as } t \to \infty.$$

THEOREM 4. *Let $A(t+T) = A(t)$ and all solutions of $y' = A(t)y$ tend to zero as $t \to \infty$. If $B(t) = B_1(t) + B_2(t)$ in which $\int_0^\infty |B_1(t)|\,dt < \infty$, then there exists an $\alpha > 0$ such that $|B_2(t)| \le \alpha$ implies that all solutions of $x' = (A(t) + B(t))x$ tend to zero as $t \to \infty$.*

PROOF: The solution of (PL) is given by

$$x(t) = P(t)e^{Rt}x(0) + \int_0^t P(t)e^{R(t-s)}P^{-1}(s)(B_1(s) + B_2(s))x(s)\,ds.$$

Hence

$$|x(t)| \le |P(t)|\,|e^{Rt}|\,|x(0)| + \int_0^t |P(t)|\,|e^{Rt}|\,|P^{-1}(s)|(|B_1(s)| + |B_2(s)|)|x(s)|\,ds.$$

Choose $K > 0$ and $\beta > 0$ with $|e^{Rt}| \le Ke^{-\beta t}$ and the periodicity of $P(t)$ implies $|P(t)| \le M$. Thus

$$|x(t)| \le MKe^{-\beta t}|x(0)| + \int_0^t Ke^{-\beta(t-s)}(|B_1(s)| + |B_2(s)|)|x(s)|\,ds$$

or

$$|x(t)|e^{\beta t} \le MK|x(0)| + \int_0^t K(|B_1(s)| + |B_2(s)|)|x(s)|e^{\beta s}\,ds.$$

By Gronwall's inequality

$$|x(t)|e^{\beta t} \le MK|x(0)| \exp \int_0^t K(|B_1(s)| + |B_2(s)|)\,ds.$$

But $\int_0^\infty |B_1(t)|\,dt < \infty$ implies $\exp \int_0^t K|B_1(s)|\,ds \le L$. Now we get

$$|x(t)| \le MKL|x(0)| \exp \int_0^t (K|B_2(s)| - \beta)\,ds.$$

Again choose $\alpha > 0$ with $\alpha < \beta/K$. Then

$$|B_2(s)| \le \alpha \text{ implies } |x(t)| \le MKL|x(0)| \exp \int_0^t (\alpha K - \beta)\,ds \to 0$$

as $t \to \infty$.

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