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# Fundamental Groups of a Topological Transformation Group

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ABSTRACT. Some properties of a path space and the fundamental group $\sigma(X, x_0, G)$ of a topological transformation group $(X, G, \pi)$ are described. It is shown that $\sigma(X, x_0, H)$ is a normal subgroup of $\sigma(X, x_0, G)$ if $H$ is a normal subgroup of $G$ ; Let $(X, G, \pi)$ be a transformation group with the open action property. If every identification map $p : \Sigma(X, x, G) \longrightarrow \sigma(X, x, G)$ is open for each $x \in X$, then $\lambda$ induces a homeomorphism between the fundamental groups $\sigma(X, x_0, G)$ and $\sigma(X, y_0, G)$ where $\lambda$ is a path from $x_0$ to $y_0$ in $X$ ; The space $\sigma(X, x_0, G)$ is an $H$-space if the identification map $p$ : $\Sigma(X, x_0, G) \longrightarrow \sigma(X, x_0, G)$ is open in a topological transformation group $(X, G, \pi)$.

This paper deals with some properties of the fundamental group $\sigma(X, x_0, G)$ of a transformation group $(X, G, \pi)$ which is a generalization of the fundamental group $\pi_1(X, x_0)$ of a topological space $X$.

Our results are motivated by the paper of F. Rhodes [**3**]. Unless otherwise stated in this paper most of terminologies and notations come from [**1**] and [**2**].

A *topological transformation group*, or briefly, a *transformation group* is a triple $(X, G, \pi)$ where $X$ is a topological space called the phase space, $G$ is a topological group called the phase group, and

$$\pi : G \times X \longrightarrow X, \qquad (g, x) \longmapsto \pi(g, x) = gx$$

is a continuous map such that

(i) $ex = x$ for all $x \in X$, where $e$ is the identity element of $G$,
(ii) $g(hx) = (gh)x$ for all $g$, $h \in G$ and $x \in X$ [**2**].

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Let $(X, G, \pi)$ be a transformation group. Let $I$ be the unit interval $[0,1]$. For a fixed point $x_0$ of $X$, let $\Sigma(X, x_0, G)$ be a set of all elements $(f, g)$ in $X^I \times G$ for which $f : I \longrightarrow X$ is a continuous map such that $f(0) = x_0$ and $f(1) = gx_0$ for some element $g \in G$. This element $(f, g)$ of $\sigma(X, x_0, G)$ is called a *path* $f$ of order $g$ with base point $x_0$. If $X^I$ is topologized by the compact open topology, then $\Sigma(X, x_0, G)$ is regarded as a subspace of the product space $X^I \times G$ of the function space $X^I$ and the topological group $G$.

Now we define a *homotopic relation* $R$ on $\Sigma(X, x_0, G)$ as follows : Let $(f, g)$ and $(f', g)$ be two elements of $\Sigma(X, x_0, G)$. We say that $(f, g)$ and $(f', g)$ are *homotopic*, denoted by $(f, g) \overset{R}{\sim} (f', g)$, if and only if there exists a continuous map $F : I \times I \longrightarrow X$ such that

$$\begin{aligned} F(s,0) &= f(s), & F(s,1) &= f'(s) & &\text{for } 0 \leq s \leq 1, \\ F(0,t) &= x_0, & F(1,t) &= gx_0 & &\text{for } 0 \leq t \leq 1. \end{aligned}$$

Then, since this homotopic relation $R$ is an equivalence relation on $\Sigma(X, x_0, G)$ for every element $g$ of $G$ the set of paths of order $g$ is divided into homotopy classes. The set of these homotopy classes of $\Sigma(X, x_0, G)$ with the identification topology determined by the identification map $p : \sigma(X, x_0, G) \longrightarrow \sigma(X, x_0, G)/R$ will be denoted by $\sigma(X, x_0, G)$, i.e., $\sigma(X, x_0, G) = \sigma(X, x_0, G)/R$ and the homotopy class of a path $f$ of order $g$ will be denoted by $[f\, ;\, g]$. Two homotopy classes of paths of different orders $g_1$ and $g_2$ are distinct, even if $g_1 x_0 = g_2 x_0$.

Let two paths $f_1$ and $f_1'$ of order $g_1$ be homotopic and two paths $f_2$ and $f_2'$ of order $g_2$ be homotopic, and let $F$ be a homotopy between $f_1$ and $f_1'$, and let $F'$ be a homotopy between $f_2$ and $f_2'$. If we define a map $H : I \times I \longrightarrow X$ by the equation

$$H(s,t) = \begin{cases} F(2s, t) & \text{if } 0 \leq s \leq 1/2, \\ g_1 F'(2s-1, t) & \text{if } 1/2 \leq s \leq 1, \end{cases}$$

then, clearly, $H$ is a well-defined continuous map. Moreover, the map $H$ is the desired homotopy between $f_1 + g_1 f_2$ and $f_1' + g_1 f_2'$. Thus two paths $f_1 + g_1 f_2$ and $f_1' + g_1 f_2'$ are homotopic. Therefore the map

$$\mu_* : \sigma(X, x_0, G) \times \sigma(X, x_0, G) \longrightarrow \sigma(X, x_0, G)$$

defined by the equation

$$\mu_*([f_1;g_1],[f_2;g_2]) = [f_1;g_1] * [f_2;g_2] = [f_1 + g_1 f_2; g_1 g_2]$$

is a well-defined rule of composition in $\sigma(X, x_0, G)$. This composition is associative. If $e$ denotes the identity element of the group $G$ and $x_0'$ denotes the constant map $x_0' : I \longrightarrow x_0$, then $[x_0'; e]$ is an identity element for this rule of composition. Furthermore, if $\rho$ denotes the map from $I$ to $I$ which maps $s$ to $1-s$, then given any homotopy class $[f;g]$ of order $g$, $[g^{-1}f\rho;\, g^{-1}]$ is a homotopy class of order $g^{-1}$, and

$$[f;g] * [g^{-1}f\rho; g^{-1}] = [g^{-1}f\rho; g^{-1}] * [f;g] = [x_0'; e].$$

Thus the set $\sigma(X, x_0, G)$ with the rule of composition defined above forms a group. This group $\sigma(X, x_0, G)$ will be called the *fundamental group of a transformation group* $(X, G, \pi)$ with base point $x_0$ [**3**]. The fundamental group $\sigma(X, x_0, G)$ clearly contains the fundamental group $\pi_1(X, x_0)$ of a topological space $X$ as a subgroup. So the concept of the fundamental group $\sigma(X, x_0, G)$ of a transformation group $(X, G, \pi)$ is a generalization of the fundamental group $\pi_1(X, x_0)$ of a topological space $X$.

Let $(X, G, \pi)$ be a transformation group and $\lambda$ be a path in $X$ from $x_0$ to $y_0$. We define a map

$$\lambda_* : \sigma(X, x_0, G) \longrightarrow \sigma(X, y_0, G)$$

by the equation

$$\lambda_*([f;g]) = [\lambda\rho + f + g\lambda; g].$$

Then the map $\lambda_*$ is a group isomorphism of $\sigma(X, x_0, G)$ to $\sigma(X, y_0, g)$ [**3**].

## 1. Subgroups of $\sigma(X, x_0, G)$

We will obtain a result that the normality of subgroup of the fundamental group $\sigma(X, x_0, G)$ behaves well whenever the given subgroup of the phase group $G$ is normal.

THEOREM 1. *Let $(X, G, \pi)$ be a transformation group and let $H$ be a subgroup of $G$. Then the fundamental group $\sigma(X, x_0, H)$ is a subgroup of $\sigma(X, x_0, G)$.*

PROOF: It is straightforward.

THEOREM 2. *Let $(X, G, \pi)$ be a transformation group. If $H$ is a normal subgroup of $G$, then $\sigma(X, x_0, H)$ is also a normal subgroup of $\sigma(X, x_0, G)$.*

PROOF: From Theorem 1, $\sigma(X, x_0, H)$ is a subgroup of $\sigma(X, x_0, G)$. Let $[f; g]$ be an arbitrary element of $\sigma(X, x_0, G)$ and $[f'; g']$ be an arbitrary element of $\sigma(X, x_0, H)$. Then

$$\begin{aligned}
[f; g]^{-1} * [f'; g'] * [f; g] &= [g^{-1} f\rho; g^{-1}] * [f'; g^{-1}] * [f; g] \\
&= [g^{-1} f\rho + g^{-1} f'; g^{-1} g'] * [f; g] \\
&= [g^{-1} f\rho + g^{-1} f' + g^{-1} g' f; g^{-1} g' g].
\end{aligned}$$

Since $H$ is a normal subgroup of $G$, $g^{-1}g'g$ belongs to $H$. so $[f; g]^{-1} * [f'; g'] * [f; g]$ is an element of $\sigma(X, x_0, H)$. Therefore $\sigma(X, x_0, H)$ is a normal subgroup of $\sigma(X, x_0, G)$.

THEOREM 3. *Let $(X, G, \pi)$ be a transformation group. If $A$ is a subgroup of the fundamental group $\sigma(X, x_0, G)$ of $(X, G, \pi)$, then $H = \{g \in G \mid [f; g] \in A\}$ is a subgroup of $G$.*

PROOF: Let $g$ and $g'$ be two elements of $H$. Then there exist $[f; g]$ and $[f'; g']$ in $A$. $[f; g] * [f'; g'] = [f + gf'; gg']$ belongs to $A$ since $A$ is a group. so $gg'$ is an element of $H$. For $[f; g] \in A$, $[f; g]^{-1} = [g^{-1} f\rho; g^{-1}]$ is an element of $A$. Hence $g^{-1}$ is an element of $H$. This means $H$ is a subgroup of $G$.

THEOREM 4. *Let $(X, G, \pi)$ be a transformation group. Let the orbit $Gx_0$ of an element $x_0 \in X$ be path connected. If $N$ is a normal subgroup of the fundamental group $\sigma(X, x_0, G)$, then $H = \{g \in G \mid [f; g] \in N\}$ is a normal subgroup of $G$.*

PROOF: By Theorem 3, $H$ is a subgroup of $G$. Let $g$ be an element of $G$. Then there exists an element $[f; g]$ in $\sigma(X, x_0, G)$, since the orbit $Gx_0$ of $x_0$ is path connected. If $[f'; g']$ is an element of $N$, then $[f; g]^{-1} * [f'; g'] * [f; g] \in N$. So $g^{-1}g'g \in H$. Thus $H$ is a normal subgroup of $G$.

## 2. The Space $\Sigma(X, x_0, G)$

Let $(X, G, \pi)$ be a transformation group. Given a topological space

$$\Sigma(X, x_0, G)$$
$$= \{(f, g) \in X^I \times G \mid f(0) = x_0, f(1) = gx_0 \text{ for some } g \in G\}$$

as a subspace of the product space $X^I \times G$, we will construct two continuous functions $\varphi_\lambda$ of $\Sigma(X, x_0, G)$ to $\Sigma(X, y_0, G)$ and $\psi_\lambda$ of $\Sigma(X, x_0, G)$ in the following theorems.

THEOREM 5. *Let $\lambda : I \longrightarrow X$ be a path with $\lambda(0) = x_0$ and $\lambda(1) = y_0 = hx_0$ for an element $h$ of $G$. If*

$$\varphi_\lambda : \Sigma(X, x_0, G) \longrightarrow \Sigma(X, y_0, G)$$

*is a function such that*

$$\varphi_\lambda((f, g)) = (\lambda\rho + f, gh^{-1}) \quad \text{for all} \quad (f, g) \in \Sigma(X, x_0, G),$$

*then $\varphi_\lambda$ is continuous.*

PROOF: Let $(f, g)$ be an arbitrary element of $\Sigma(X, x_0, G)$ and let $((K, W) \times H) \cap \Sigma(X, y_0, G)$ be a subbasic open neighborhood of $\varphi_\lambda((f, g)) = (\lambda\rho + f, gh^{-1})$ in $\Sigma(X, y_0, G)$, where $K$ is compact in $I$, $W$ is open in $X$ and $H$ is an open neighborhood of $gh^{-1}$ in $G$. It is clear that $((K', W) \times Hh) \cap \Sigma(X, x_0, G)$ is an open neighborhood of $(f, g)$, where $K' = \{2k - 1 \mid k \in K\}$. Suppose that $(f_1, g_1) \in ((K', W) \times Hh) \cap \Sigma(X, x_0, G)$. Then $\varphi_\lambda((f_1, g_1)) = (\lambda\rho + f_1, g_1h^{-1})$. We have to show that $\varphi_\lambda((f_1, g_1))$ belongs to $((K, W) \times H) \cap \Sigma(X, y_0, G)$. For each $k \in K$,

$$(\lambda\rho + f_1)(k) = \begin{cases} \lambda\rho(2k) & \text{if } 0 \leq k \leq 1/2, \\ f_1(2k - 1) & \text{if } 1/2 \leq k \leq 1, \end{cases}$$

which is clearly an element of $W$. Since $g_1$ belongs to $Hh$, $g_1h^{-1}$ is an element of $W$. Since $g_1$ belongs to $Hh$, $g_1h^{-1}$ is an element of $H$. Therefore $\varphi_\lambda((f_1, g_1)) = (\lambda\rho + f_1, g_1h^{-1})$ belongs to $((K, W) \times H) \cap \Sigma(X, y_0, G)$. Thus we have that $\varphi_\lambda$ is continuous.

DEFINITION 1: A transformation group $(X, G, \pi)$ has the *open action property* if for every point $x \in X$ and for every open set $O$ in $G$, $\pi(O \times \{x\}) = Ox$ is an open subset in $X$.

LEMMA 6. *Let $(X, G, \pi)$ be a transformation group with the open action property. Let $A$ be a compact subset of $X$ and $g \in G$. If $\pi(\{g\} \times A) = gA$ is contained in an open subset $W$ of $X$, then there exists an open neighborhood $O$ of the identity element $e$ of $G$ such that $OgA \subset W$, where $OgA = \pi(Og \times A)$.*

PROOF: Since $(X, G, \pi)$ has the open action property, the set $O_x gx$ is an open neighborhood of $gx$ for every $x \in A$ and for every open neighborhood $O$ of $e$. We can assume that every $O_x gx$ is contained in $W$ and $O_x \cdot O_x \subset O_x$ without loss of generality, because $\pi$ is continuous and $G$ is a topological group. $gA = \pi(\{g\} \times A)$ is compact in $X$, since $\pi$ is continuous and $\{g\} \times A$ is compact in $G \times X$. The family $\{O_x gx \mid x \in A\}$ is an open covering of the compact subset $gA$ of $X$. So there exists a finite subcovering $\{O_{x_1} gx_1, \ldots, O_{x_n} gx_n\}$ of $\{O_x gx \mid x \in A\}$. Let $O = O_{x_1} \cap \cdots \cap O_{x_n}$. The set $O$ is an open neighborhood of $e$.

$$\begin{aligned} O \cdot gA &\subset O \cdot (O_{x_1} gx_1 \cup \cdots \cup O_{x_n} gx_n) \\ &\subset O \cdot O_{x_1} gx_1 \cup \cdots \cup O \cdot O_{x_n} gx_n \\ &\subset O_{x_1} gx_1 \cup \cdots \cup O_{x_n} gx_n \\ &\subset W. \end{aligned}$$

So $OgA$ is contained in $W$.

THEOREM 7. *Let $(X, G, \pi)$ be a transformation group with the open action property and let $\lambda : I \longrightarrow X$ be a path with $\lambda(0) = x_0$ and $\lambda(1) = y_0 = hx_0$ for some element $h$ of $G$. If*

$$\psi_\lambda : \Sigma(X, x_0, G) \longrightarrow \Sigma(X, x_0, G)$$

*is defined by*

$$\psi_\lambda((f,g)) = (f + g\lambda, gh) \quad \text{for all} \quad (f,g) \in \Sigma(X, x_0, G),$$

*then $\psi_\lambda$ is a continuous function.*

PROOF: Let $(f,g)$ be an arbitrary element of $\Sigma(X, x_0, G)$ and let $((K, W) \times H) \cap \Sigma(X, x_0, G)$ be a subbasic open neighborhood of $\psi_\lambda((f, g)) = (f + g\lambda, gh)$, where $K$ is compact in $I$, $W$ is open in $X$ and $H$ is an open neighborhood of $gh$ in $G$. Let $\alpha : I \longrightarrow I$ be defined by

$\alpha(t) = 2t - 1$ for all $t \in I$. Since $K$ is compact, $\lambda\alpha(K)$ is a compact subset of $X$. Set $A = \lambda\alpha(K)$. From the above lemma, there exists an open neighborhood $O$ of $e$ in $G$ such that $OgA$ is contained in $W$, since $gA \subset W$.

Suppose that $(f_1, g_1)$ is an element of $((2K \cap I, W) \times O_g) \cap \Sigma(X, x_0, G)$. For every $k \in K$,

$$\psi_\lambda((f_1, g_1))(k) = (f_1 + g_1\lambda)(k) = \begin{cases} f_1(2k) & \text{if } 0 \leq k \leq 1/2, \\ g_1\lambda(2k-1) & \text{if } 1/2 \leq k \leq 1. \end{cases}$$

It is clear that $f_1(2k) \in W$ for $0 \leq k \leq 1/2$ and $g_1\lambda(2k-1) \in W$ for $1/2 \leq k \leq 1$ and $k \in K$ since $g_1 \in Og$. Hence $\psi_\lambda(((2K \cap I, W) \times Og) \cap \Sigma(X, x_0, g))$ is contained in $((K, W) \times H) \cap \Sigma(X, x_0, G)$. Therefore $\psi_\lambda$ is a continuous function of $\Sigma(X, x_0, G)$.

## 3. Fundamental Group $\sigma(X, x_0, G)$

We define a multiplication

$$\mu : \Sigma(X, x_0, g) \times \Sigma(X, x_0, G) \longrightarrow \Sigma(X, x_0, G)$$

by

$$\mu((f_1, g_1), (f_2, g_2)) = (f_1, g_1) \cdot (f_2, g_2) = (f_1 + g_1 f_2, g_1 g_2)$$

where the path $f_1 + g_1 f_2 : I \longrightarrow X$ is defined by

$$(f_1 + g_1 f_2)(s) = \begin{cases} f_1(2s) & \text{if } 0 \leq s \leq 1/2, \\ g_1 f_2(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Then we have the following theorem :

THEOREM 8. *The multiplication map $\mu$ in the space $\Sigma(X, x_0, G)$ is continuous.*

PROOF: Let $((f_1, g_1), (f_2, g_2))$ be an arbitrary element of $\Sigma(X, x_0, G) \times \Sigma(X, x_0, g)$ and let $((K, W) \times H) \cap \Sigma(X, x_0, G)$ be a subbasic open neighborhood of $\mu((f_1, g_1), (f_2, g_2)) = (f_1 + g_1 f_2, g_1 g_2)$, where

$K$ is compact in $I$, $W$ is open in $X$ and $H$ is an open neighborhood of $g_1 g_2$ in $G$. Let

$$K_1 = \{2k \mid k \in [0, 1/2] \cap K\}, \quad k_2 = \{2k - 1 \mid k \in [1/2, 1] \cap K\}$$

and $H_1$, $H_2$ be open neighborhoods of $g_1$, $g_2$, respectively such that $H_1 H_2 \subset H$. Then clearly

$$((K_1, W) \times H_1) \cap \Sigma(X, x_0, G) \quad \text{and} \quad ((K_2, g_1^{-1}W) \times H_2) \cap \Sigma(X, x_0, G)$$

are open neighborhoods of $(f_1, g_1)$ and $(f_2, g_2)$, respectively.

Suppose that $(f_1, g_1) \in ((K_1, W) \times H) \cap \Sigma(X, x_0, G)$ and $(f_2, g_2) \in ((K_2, g_1^{-1}W) \times H) \cap \Sigma(X, x_0, G)$. To show that $\mu$ is continuous, it suffices to show that $(f_1 + g_1 f_2, g_1 g_2)$ belongs to $((K, W) \times H) \cap \Sigma(X, x_0, G)$. It is clear that $f_1(2k) \in W$ for $0 \le k \le 1/2$, $k \in K$ and $g_1 f_2(2k - 1) \in W$ for $1/2 \le k \le 1$, $k \in K$. Therefore, for $k \in K$,

$$(f_1 + g_1 f_2)(k) = \begin{cases} f_1(2k) & \text{if } 0 \le k \le 1/2, \\ g_1 f_2(2k - 1) & \text{if } 1/2 \le k \le 1, \end{cases}$$

belongs to $W$. Since $g_1 \in H_1$, $g_2 \in H_2$ and $H_1 \cdot H_2 \subset H$, $g_1 g_2$ is an element of $H$. Thus $\mu((f_1, g_1), (f_2, g_2)) = (f_1 + g_1 f_2, g_1 g_2)$ is an element of $((K, W) \times H) \cap \Sigma(X, x_0, G)$. This means that $\mu$ is continuous.

DEFINITION 2: An *H-space* consists of a topological space $X$ together with a continuous multiplication $\mu : X \times X \longrightarrow X$ for which the constant map $c : X \longrightarrow X$ is a homotopy identity, that is, the diagram

$$\begin{array}{ccccc} X & \xrightarrow{(c,1)} & X \times X & \xleftarrow{(1,c)} & X \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & X & & \end{array}$$

commutes up to homotopy : $\mu \circ (c, 1) \simeq 1 \simeq \mu \circ (1, c)$.

COROLLARY 9. *If the identification map $p : \Sigma(X, x_0, G) \longrightarrow \sigma(X, x_0, G)$ is open, then the space $\sigma(X, x_0, G)$ is an H-space.*

PROOF: We have the commutative diagram

$$\begin{array}{ccc} \Sigma(X,x_0,G)\times\Sigma(X,x_0,G) & \xrightarrow{\ \mu\ } & \Sigma(X,x_0,G) \\ {\scriptstyle p\times p}\downarrow & & \downarrow{\scriptstyle p} \\ \sigma(X,x_0,G)\times\sigma(X,x_0,G) & \xrightarrow[\mu_*]{} & \sigma(X,x_0,G) \end{array}$$

where $\mu_*([f_1;g_1],[f_2;g_2]) = [f_1;g_1]*[f_2;g_2] = [f_1+f_1f_2, g_1g_2]$ for all $([f_1;g_1],[f_2;g_2]) \in \sigma(X,x_0,G)\times\sigma(X,x_0,G)$. It is clear that $p\times p$ is an open map. By Theorem 8, $\mu$ is continuous. Thus the multiplication map $\mu_*$ in $\sigma(X,x_0,G)$ is continuous from the above diagram.

Now we have to show that the constant map $x_0' : I \longrightarrow x_0$ is a homotopy identity, in other words, the diagram

$$\begin{array}{ccccc} \sigma(X,x_0,G) & \xrightarrow{(x_0',1)} & \sigma(X,x_0,G)\times\sigma(X,x_0,G) & \xleftarrow{(1,x_0')} & \sigma(X,x_0,G) \\ & \searrow_{1} & \downarrow{\scriptstyle \mu_*} & {}_{1}\swarrow & \\ & & \sigma(X,x_0,G) & & \end{array}$$

commutes up to homotopy, i.e., $\mu_*\circ(x_0',1)\simeq 1\simeq\mu_*\circ(1,x_0')$. But since $[x_0';e]$ is the identity element of the fundamental group $\sigma(X,x_0,G)$, for every $[f;g]\in\sigma(X,x_0,G)$,

$$\begin{aligned} (\mu_*\circ(x_0',1))[f;g] &= [x_0';e]*[f;g] \\ &= [f;g] \\ &= [f;g]*[x_0';e] \\ &= (\mu_*\circ(1,x_0'))[f;g]. \end{aligned}$$

So the diagram commutes. Hence $\mu_*\circ(x_0',1)\simeq 1\simeq\mu_*\circ(1,x_0')$.

The fundamental groups $\pi_1(X,x_0)$ and $\pi_1(X,y_0)$ of a topological space $X$ are known to be isomorphic if $y_0$ is in the path connected component of $x_0$. For the fundamental groups $\sigma(X,x_0,G)$ and $\sigma(X,y_0,G)$ of a transformation group $(X,G,\pi)$, if $y_0$ is in the path connected component of $x_0$, then $\sigma(X,x_0,G)$ and $\sigma(X,y_0,G)$ are isomorphic [**3**]. This result leads us to consider a homeomorphism between the topological spaces $\sigma(X,x_0,G)$ and $\sigma(X,y_0,G)$ with respect to the identification topologies.

THEOREM 10. *Let $(X, G, \pi)$ be a transformation group with the open action property. Let $\lambda : I \longrightarrow X$ be a path with $\lambda(0) = x_0$ and $\lambda(1) = y_0 = hx_0$ for some element $h$ in $G$. If every identification map $p : \Sigma(X, x, G) \longrightarrow \sigma(X, x, G)$ is open for each $x \in X$, then the path $\lambda$ induces the homeomorphism $\lambda_*$ between fundamental groups $\sigma(X, x_0, G)$ and $\sigma(X, y_0, G)$.*

PROOF: The path $\lambda : I \longrightarrow X$ induces an isomorphism $\lambda_* : \sigma(X, x_0, G) \longrightarrow \sigma(X, y_0, G)$ of fundamental groups [**3**]. We will show that this $\lambda_*$ is a homeomorphism of $\sigma(X, x_0, G)$ to $\sigma(X, y_0, G)$ with respect to the identification topologies.

It is clear that the diagram

$$
\begin{array}{ccc}
\Sigma(X, x_0, G) & \xrightarrow{\psi_\lambda \circ \varphi_\lambda} & \Sigma(X, y_0, G) \\
(f, g) & & ((\lambda\rho + f) + g\lambda, gh^{-1}) \\
 & \searrow & \Big\downarrow \psi_{\lambda\rho} \circ \varphi_{\lambda\rho} \\
\psi_{\lambda\rho} \circ \varphi_{\lambda\rho} \cdot \psi_\lambda \cdot \varphi_\lambda & & \Sigma(X, x_0, G) \\
 & & ((\lambda + (\lambda\rho + f) + g\lambda) + g\lambda\rho, g)
\end{array}
$$

commutes. The path $((\lambda + (\lambda\rho + f) + g\lambda) + g\lambda\rho, g)$ is homotopic to $(f, g)$ [**3**]. Since the identification map $p : \Sigma(X, x, G) \longrightarrow \sigma(X, x, G)$, $x \in X$, is open, the above diagram induces the following diagram of continuous functions :

$$
\begin{array}{ccc}
\sigma(X, x_0, G) & \xrightarrow{\lambda_*} & \sigma(X, y_0, G) \\
 & {\scriptstyle (\lambda\rho)_* \cdot \lambda_*} \searrow & \Big\downarrow (\lambda\rho)_* \\
 & & \sigma(X, x_0, G)
\end{array}
$$

which is commutative. Then $(\lambda\rho)_* \circ \lambda_*$ is the identity map of $\sigma(X, x_0, G)$. It can be shown that $\lambda_* \circ (\lambda\rho)_*$ is the identity map of $\sigma(X, y_0, G)$. Hence the map $\lambda_*$ is a homeomorphism of $\sigma(X, x_0, G)$ to $\sigma(X, y_0, G)$.

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