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Some Properties of Regular Multiplication Rings

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ABSTRACT. Let R be a commutative ring with identity. A ring is said to be a regular multiplication ring if $A \subseteq B$, where A and B are ideals of R with B regular, implies that there exists an ideal C of R such that $A = BC$. We characterize such rings and study their properties.

1. Introduction

Throughout this paper, all rings will be commutative with identities and R will always denote a ring. An element of R which is not a zero-divisor is called *regular* and an ideal of R is called a *regular ideal* if it contains a regular element. A ring R is called a *regular multiplication ring* if $A \subseteq B$, where A and B are ideals of R with B regular implies that there exists an ideal C of R such that $A = BC$. A ring R is called *prüfer ring* if every finitely generated regular ideal of R is invertible.

The purpose of this paper is to give some characterizations of regular multiplication ring involving their regular ideals. A regular multiplication ring is a generalization of Dedekind domain, ZPI ring and multiplication.

A prüfer ring is a generalization of a regular multiplication ring. It is well known that a ring R is a regular multiplication ring if and only if every regular ideal of R is a unique product of powers of finitely many maximal ideals of R [4, theorem 17].

We also note that every regular prime ideal of a regular multiplication ring is maximal. In general, our terminology and notation from ring theory will follow that of Gilmer [3] and Larsen and McCarthy [5].

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2. Main Results

LEMMA 1. *A ring R is a regular multiplication ring if and only if every regular ideal is invertible.*

PROOF: Let B be a regular ideal of R. Then there is a regular element $b \in B$ and $(b) \subseteq B$. Since R is a regular multiplication ring, there is an ideal A of R such that $(b) = AB$. So B is a factor of a regular principal ideal, hence B is invertible.

Conversely, assume that every regular ideal is invertible. Let A and B be ideals of R with B regular and with $A \subseteq B$. There is an ideal C such that $BC = R$ by hypothesis. If $D = AC$, then $D \subseteq R$ and $BD = BAC = ABC = AR = A$. Therefore, R is a regular multiplication ring.

PROPOSITION 2. *Let R be a regular multiplication ring. If P is a regular prime ideal of R, then there are no ideals of R strictly between P and P^2.*

PROOF: Let P be a regular prime ideal of R such that $P^2 \subseteq A \subseteq P$. Then there are ideals C, D of R such that $A = PC$ and $P^2 = AD = PCD$. Since P is invertible, $P = CD$, so $C \subseteq P$ or $D \subseteq P$. If $C \subseteq P$, then $A = PC \subseteq P^2$, hence $P^2 = A$. If $D \subseteq P$, then $P^2 = AD \subseteq AP$ and $P \subseteq A$, hence $P = A$. Therefore $P^2 \subseteq A \subseteq P$ implies $A = P^2$ or $A = P$.

A ring R is called a *special primary ring* if R has a unique maximal ideal M and if each proper ideal of R is a power of M. By Proposition 2, we obtain the following corollary.

COROLLARY 3. *If R is a regular multiplication ring and P is a regular prime ideal, then for each positive integer n, R/P^n is a special primary ring.*

THEOREM 4. *A ring R is a regular multiplication ring if and only if localization R_P of R at prime ideal P is a regular multiplication ring for every maximal ideal P of R.*

PROOF: Let A' and B' be ideals of R_p such that $A' \subset B'$ with B' a regular ideal of R_P, $S = R \setminus P$. Then $A' = S^{-1}(A' \cap R)$, $B' = S^{-1}(B' \cap R)$, and so $S^{-1}(A' \cap R) \subseteq S^{-1}(B' \cap R)$ implies $A' \cap R \subseteq B' \cap R$. Since B' is a regular ideal of R_p, $B' \cap R$ is a regular ideal of R. There is an ideal C of R such that $(B' \cap R)C = A' \cap R$, so $S^{-1}\{(B' \cap R)C\} = S^{-1}(B' \cap R)S^{-1}C = S^{-1}(A' \cap R)$ and $A' = B'C'$, where $C' = S^{-1}C$. Hence R_p is a regular multiplication ring for every maximal ideal P of R.

Conversely, suppose R_p is a regular multiplication ring for every maximal ideal P of R. Let A, B be ideals of R with $A \subseteq B$, B regular. Since B is a regular ideal of R, BR_P is a regular ideal of R_p. Since $AR_P \subseteq BR_p$, there is an ideal CR_p of R_p such that $AR_p = BR_pCR_p = (BC)R_p$ for every maximal ideal P of R, and so $A = BC$ Hence R is a regular multiplication ring.

LEMMA 5. *Let R be a prüfer ring and let T be an overring of R. If M is a maximal ideal of T, then $T_M = R_N$, where $N = M \cap R$.*

PROOF: Exercise X–15 [5, P. 249].

COROLLARY 6. *Every overring of a regular multiplication ring is a regular multiplication ring.*

PROOF: Let R be a regular multiplication ring and T an overring of R such that $R \subset T \subset K$, where K is a total quotient ring of R. Let M be a maximal ideal of T. Since a regular multiplication ring is a prüfer, R is a prüfer ring. By Lemma 5, $T_M = R_{M\cap R}$. Thus $R_{M\cap R}$ is not a field, and so $M \cap R \neq 0$. Hence $M \cap R$ is a maximal ideal of R. Since R is a regular multiplication ring, $R_{M\cap R}$ is a regular multiplication ring.

PROPOSITION 7. *If a ring R is a quasi-local regular multiplication ring, then the set of regular ideals of R is totally ordered by inclusion.*

PROOF: Since every regular ideal is a power of only maximal ideal of R, the set of regular ideals of R is totally ordered by inclusion.

THEOREM 8. *Let $\{R_i \mid i = 1, \ldots, n\}$ be a finite collection of rings with identities and let $R = R_1 \oplus \cdots \oplus R_n$ be the direct sum of these rings. Then R is a regular multiplication ring if and only if each R_i is a regular multiplication.*

PROOF: Let A_i be a regular ideal of R_i, $1 \leq i \leq n$. Then $R_1 \oplus \cdots \oplus R_{i-1} \oplus A_i \oplus R_{i+1} \oplus \cdots \oplus R_n$ is a regular ideal of R. Since R is a regular multiplication ring. $R_1 \oplus \cdots \oplus R_{i-1} \oplus A_i \oplus R_{i+1} \oplus \cdots \oplus R_n = M_1^{v_1} \cdots M_m^{v_m}$, where $M_1, \ldots, M_m$ are maximal ideals of R, M_k can be written $M_k = R_1 \oplus \cdots \oplus B_{jk} \oplus \cdots \oplus R_n$ for some maximal ideal B_{jk} of R_j. Since $R_1 \oplus \cdots \oplus R_{i-1} \oplus A_i \oplus R_{i+1} \oplus \cdots \oplus R_n \subseteq M_k$ for each k, then $i = jk$, for each k. Therefore we may assume that $M_k = R_1 \oplus \cdots \oplus R_{i-1} \oplus B_k \oplus R_{i+1} \oplus \cdots \oplus R_n$ for each k, where B_k is a maximal ideal of R_i. Then

$$M_k^{v_k} = R_1 \oplus \cdots \oplus R_{i-1} \oplus B_k^{v_k} \oplus R_{i+1} \oplus \cdots \oplus R_n,$$
$$M_1^{v_1} \cdots M_m^{v_m} = R_1 \oplus \cdots \oplus B_1^{v_1} \cdots B_m^{v_m} \oplus \cdots \oplus R_n.$$

Hence $A_i = B_1^{v_1} \cdots B_m^{v_m}$, where B_j is a maximal ideal of R_i for $1 \leq j \leq m$. Therefore, R_i is a regular multiplication ring.

Conversely, suppose each R_i is a regular multiplication ring. Let A be a regular ideal of R. Then A can be written uniquely in the form $A = A_1 \oplus \cdots \oplus A_n$. where A_i is a unique product of powers of finitely many maximal ideals of R_i for $1 \leq i \leq n$ i.e., $A_i = M_{i1}^{v_{i1}} \cdots M_{ik}^{v_{ik}}$, where $M_{i1}, \ldots, M_{ik}$ are miximal ideals of R_i and $v_{i1}, \ldots, v_{ik}$ are positive integers. Moreover $P_{ij} = R_1 \oplus \cdots \oplus R_{i-1} \oplus M_{ij} \oplus R_{i+1} \oplus \cdots \oplus R_n$ is a maximal ideal of R for $1 \leq i \leq n, 1 \leq j \leq k$. Hence A is a product of powers of finitely many maximal ideals of R. Therefore, R is a regular multiplication ring.

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