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Discontinuous Derivations on the Banach Algebras of Differentiable Functions

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ABSTRACT. This paper studies the structure and continuity of derivations of the Banach algebra $C^n(I)$ of n times continuously differentiable functions on an interval I into Banach $C^n(I)$-modules.

1. Introduction

Let $C^n(I)$, $I = [0,1]$, denote the algebra of all n times continuously differentiable complex valued functions on I. It is well known that $C^n(I)$ is a Banach algebra under the norm

$$||f||_n = \max_{t\in I} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}$$

whose structure space is I. We denote the space of bounded linear maps from a Banach space M into M by $B(M)$. A Banach $C^n(I)$-module is a Banach space M together with a continuous homomorphism $\rho : C^n(I) \longrightarrow B(M)$. A derivation, or a module derivation of $C^n(I)$ into M is a linear map $D : C^n(I) \longrightarrow M$ which satisfies the identity

$$D(fg) = \rho(f)D(g) + \rho(g)D(f), \quad f, g \in C^n(I).$$

To measure the discontinuity of a derivation D, one introduces the separating space $S(D)$. This is the subspace of M defined by

$$S(D) = \{m \in M \mid \text{ there exists } \{f_n\} \subset C^n(I), f_n \to 0 \text{ and } D(f_n) \to m\}$$

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It is easily checked that $S(D)$ is a closed submodule of M and the derivation D is continuous if and only if $S(D) = (0)$. The continuity ideal for a derivation $D : C^n(I) \longrightarrow M$ is

$$\Im(D) = \{f \in C^n(I) \mid \rho(f)S(D) = (0)\}.$$

Clearly $\Im(D)$ is a closed ideal in $C^n(I)$. It is proved in [**1**, Theorem 3.2], that

$$\Im(D) = \{f \mid D_f \text{ is continuous }\},$$

where for $f \in C^n(I)$, $D_f(\cdot) = \rho(f)D(\cdot)$. We use the notation

$$M_{n,k}(t_0) = \{f \in C^n(I) \mid f^{(j)}(t_0) = 0; j = 0, 1, \ldots, k\},$$
$$0 \leq k \leq n,\ t_0 \in I.$$

These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,0}(t_0)$ of functions vanishing at t_0.

In 1974 Bade and Curtis [**1**] proved that if D is a derivation from $C^n(I)$ with singularity set F, then F is finite, say $F = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and

$$\cap_{i=1}^m M_{n,n}(\lambda_i) \subset \Im(D) \subset \cap_{i=1}^m M_{n,0}(\lambda_i).$$

If $D : C^n(I) \longrightarrow M$ is a derivation, we have

$$D(p(z)) = \rho(p'(z))D(z), p \in P,$$

where P is the dense subalgebra of polynomials in z. If D is continuous, it is completely determined by this formula. Thus a continuous derivation D is uniquely determined by the vector $D(z)$ [**2**]. We need to define the notion of the k-differential subspace of a Banach $C^n(I)$-module, a concept first introduced by Kantorovitz who named it "semisimplicity manifold" [**3, 4**].

DEFINITON 1.1. *Let M be a Banach $C^n(I)$-module. The k-differential subspace is the set W_k ($k = 0, 1, \ldots, n$) of all vectors m such that the map*

$$p \longrightarrow \rho(p')m$$

is continuous for the $C^{n-k+1}(I)$ norm on P.

We quote the following results.

LEMMA 1.2 ([**6**, Lemma 3.2]). *Let M be a $C^n(I)$-module. A vector m lies in the k-differential subspace W_k if and only if the map*

$$p \longrightarrow \rho(p)m$$

is continuous for the $C^{n-k}(I)$ norm on P.

THEOREM 1.3 (cf. [**2**, Theorem 4.4]). *Let M be a $C^n(I)$ module with k-differential subspace W_k. For $m \in W_k$ $(0 \leq k \leq n)$, we define $|||m|||_k = \sup\{||\rho(p)m|| \mid ||p||_{n-k} \leq 1\}$. Then*

(1) $||m|| \leq |||m|||_0 \leq |||m|||_1 \leq \cdots \leq |||m|||_k$, $m \in W_k$,

(2) *W_k is a Banach space under the norm $|||\cdot|||_k$,*

(3) *W_k is a $C^n(I)$-module and there exists a unique continuous homomorphism*

$$\gamma_k : C^{n-k}(I) \longrightarrow B(W_k)$$

such that $\gamma_k(p)m = \rho(p)m$, $m \in W_k$, $p \in P$,

(4) *If $S \in B(W_k)$ and $S\rho(z) = \rho(z)S$, then $SW_k \subset W_k$ and $|||S|||_k \leq ||S||$, where $|||S|||_k$ is norm of S in $B(W_k)$.*

Recall that the ascent of an eigenvalue λ for a linear operator T is the smallest integer k such that $(T-\lambda I)^{k+1}m = 0$ implies $(T-\lambda I)^k m = 0$.

THEOREM 1.4 (cf. [**2**, Theorem 3.1]). *If $\gamma_k : C^{n-k}(I) \longrightarrow B(W_k)$ is a continuous homomorphism, every eigenvalue which lies in I for $\gamma_k(z)$ has ascent at most $n-k+1$.*

LEMMA 1.5 ([**5**, Lemma 5.4]). *Let $\lambda \in I$ and suppose $f \in M_{n,k-1}(\lambda)$ for some $k = 1, 2, ..., n$. Then*

$$\left|\frac{f^{(k-m)}(t)}{(t-\lambda)^m}\right| \leq ||f^{(k)}||_\infty$$

for every $t \in I - \{\lambda\}(m = 0, 1, ..., k)$.

THEOREM 1.6 ([**2**, Theorem 3.2]). *Let $D : C^n(I) \longrightarrow M$ be a discontinuous derivation. Then*

$$\cap\{M_{n,n-1}(\lambda) \mid \lambda \in hull(\Im(D))\} \subset \Im(D).$$

A nontrivial derivation $D : C^n(I) \longrightarrow M$ is called singular if D vanishes on P (equivalently $D(z) = 0$). A singular derivation is necessarily discontinuous. A derivation D is decomposable if D can be expressed in the form $D = E + F$, where E is continuous and F is singular. Such a splitting is unique. It was shown in **[2]** that a derivation $D : C^n(I) \longrightarrow M$ is decomposable if and only if $D(z) \in W_1$.

2. Main Results

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be the hull of a discontinuous derivation D from $C^n(I)$ into a Banach $C^n(I)$-module M. Choose $e_k \in C^n(I)$, $1 \le k \le m$, such that $e_k(\lambda) = 1$ in the neighborhood U_k of λ_k, $e_i = 0\,(i \ne k)$ in the neighborhood V_k of λ_k such that $V_k^- \subset U_k$ and $\sum_{i=1}^m e_i(\lambda) \ne 0$ for all $\lambda \in I$. Let $e_0 = 1 - \sum_{i=1}^m e_i$. Then

$$e_0 \in \cap_{i=1}^m M_{n,n}(\lambda_i) \subset \Im(D)$$

and

$$D(f) = \sum_{i=0}^{m} \rho(e_i) D(f), \quad f \in C^n(I).$$

Let $D_i(\cdot) = \rho(e_i)D(\cdot)$. Then D_0 is continuous and D_i $(i = 1, 2, \ldots, m)$ is discontinuous. We have $\text{hull}(\Im(D_i)) = \{\lambda_i\}(i = 1, 2, \ldots, m)$ and

$$\Im(D) = \cap_{i=1}^m \Im(D_i).$$

LEMMA 2.1. *Let $D : C^n(I) \longrightarrow M$ be a discontinuous derivation with hull $(\Im(D)) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and let $1 \le k \le n$. Then $D(z) \in W_k$ and $\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$ if and only if $D_i(z) \in W_k$ and $M_{n,n-k}(\lambda_i) \subset \Im(D_i)$ for each i, $1 \le i \le m$.*

PROOF: Suppose $D(z) \in W_k$ and $\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$. By Theorem 1.3, $D_i(z) = \rho(e_i)D(z) \in W_k$. If $g \in M_{n,n-k}(\lambda_i)$, then $\rho(g)D_i(\cdot)$ is continuous. Thus $M_{n,n-k}(\lambda_i) \subset \Im(D_i)$.

Conversely, suppose $M_{n,n-k}(\lambda_i) \subset \Im(D_i)$ and $D_i(z) \in W_k$. From $\Im(D) = \cap_{i=1}^m \Im(D_i)$, it follows that

$$\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D).$$

Since $\rho(1 - e_0)(\lambda) \ne 0$ for all $\lambda \in I$, we have $(1 - e_0)^{-1} \in C^n(I)$. By Theorem 1.3, $D(z) \in W_k$.

THEOREM 2.2. *Let $D : C^n(I) \longrightarrow M$ be a discontinuous derivation and let $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be the hull of $\Im(D)$, $1 \leq k \leq n$. Then $S(D) \subset W_k$ if and only if $\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$.*

PROOF: Suppose $S(D) \subset W_k$. Then by Theorem 3.2 of [2], for $x \in S(D)$,

$$\rho((z-\lambda_1)^n(z-\lambda_2)^n \cdots (z-\lambda_m)^n)x = 0.$$

Put $y = \rho((z-\lambda_2)^n(z-\lambda_3)^n \cdots (z-\lambda_m)^n)x$. By Theorem 1.3, we see $y \in W_k$. Using Theorem 1.4, we have $\rho(z-\lambda_1)^{n-k+1}y = 0$. If we continue this process for $\lambda_2, \ldots, \lambda_m$, then

$$\rho((z-\lambda_1)^{n-k+1}(z-\lambda_2)^{n-k+1} \cdots (z-\lambda_m)^{n-k+1})x = 0.$$

So $(z-\lambda_1)^{n-k+1}(z-\lambda_2)^{n-k+1} \cdots (z-\lambda_m)^{n-k+1} \in \Im(D)$. Since $\Im(D)$ is a closed ideal,

$$\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D).$$

Conversely, suppose $\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$. By Lemma 2.1, $M_{n,n-k}(\lambda_i) \subset \Im(D_i)$, for each i, $1 \leq i \leq m$. Since $S(D) \subset S(D_1) + S(D_2) + \cdots + S(D_m)$, we may assume that D has the point zero for its singularity set. If $x \in S(D)$, then $\rho(z^{n-k+1})x = 0$. For any polynomial p,

$$\begin{aligned}\|\rho(p)x\| &= \|\rho(p(0) + p'(0)z + \cdots + \frac{p^{(n-k)}(0)}{(n-k)!}z^{n-k})x\| \\ &\leq C\|p\|_{n-k}, C > 0.\end{aligned}$$

Hence $x \in W_k$.

COROLLARY 2.3. *Let $D : C^n(I) \longrightarrow M$ be a derivation. Then $S(D) \subset W_1$.*

PROOF: It is easily proved by Theorem 1.6.

THEOREM 2.4. *Let $D : C^n(I) \longrightarrow M$ be a derivation with hull $(\Im(D)) = \{\lambda_1, \lambda_2, ..., \lambda_m\}$. If $\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$, $k =$*

$1, 2, ..., n$,
then $D|_{C^{2n-k+1}(I)}$ *is continuous for the* $C^{2n-k+1}(I)$ *norm.*

PROOF: We may assume that D has a singleton singularity set $\{0\}$ and $M_{n,n-k}(0) \subset \Im(D)$. Let $f \in M_{2n-k+1,2n-k+1}(0)$ and define $g = z^{-n+k-1}f$. Then $g \in M_{n,n}(0)$. By Lemma 1.5,

$$\begin{aligned} ||g||_n &= \sup_{t\neq 0} \sum_{i=0}^{n} \frac{1}{i!} |(\frac{f}{z^{n-k+1}})^{(i)}(t)| \\ &\leq \sup_{t\neq 0} \sum_{i=0}^{n} \sum_{j=0}^{i} C_{i,j} |\frac{f^{(i-j)}(t)}{t^{n-k+1+j}}| \, (C_{i,j} > 0) \\ &\leq \sup_{t\neq 0} \sum_{i=0}^{n} \sum_{j=0}^{i} C_{i,j} ||f^{n-k+1+i}||_\infty \\ &\leq C||f||_{2n-k+1}, \quad C > 0. \end{aligned}$$

Hence

$$\begin{aligned} ||D(f)|| &= ||\rho(z)^{n-k+1} D(g) + \rho(g) D(z^{n-k+1})|| \\ &\leq C'||g||_n \\ &\leq CC'||f||_{2n-k+1}, \quad C' > 0. \end{aligned}$$

COROLLARY 2.5. *Let* $D : C^n(I) \longrightarrow M$ *be a derivation. Then* $D|_{C^{2n}(I)}$ *is continuous for the* $C^{2n}(I)$ *norm.*

THEOREM 2.6. *Let* $D : C^n(I) \longrightarrow M$ *be a derivation and let* $1 \leq k \leq n$. *Then* $D(z) \in W_k$, $S(D) \subset W_k$ *if and only if the image of* D *is contained in* W_k.

PROOF: Suppose $D(z) \in W_k$ and $S(D) \subset W_k$. For $f \in C^n(I)$, there exists $\{p_\ell\}$ in P such that $p_\ell \to f$. Since $p_\ell - f \to 0$,

$$\lim_{\ell\to\infty} D(p_\ell - f) = \lim_{\ell\to\infty} D(p_\ell) - D(f) \in W_k.$$

By Theorem 1.3, we have

$$\lim_{\ell\to\infty} D(p_\ell) = \lim_{\ell\to\infty} \rho(p'_\ell) D(z) = \gamma_1(f') D(z) \in W_k.$$

So $D(f) \in W_k$.

Conversely, suppose $D(C^n(I)) \subset W_k$. Since $D(z) \in W_1$, $D = E+F$ where E is continuous and F is singular. For $f \in C^n(I)$,

$$\rho((z-\lambda_1)^n(z-\lambda_2)^n \cdots (z-\lambda_m)^n)F(f) = 0$$

where $\{\lambda_1, \lambda_2, \ldots \lambda_m\} = hull(\Im(D))$. Since $F(f) \in W_k$,

$$\rho((z-\lambda_1)^{n-k+1}(z-\lambda_2)^{n-k+1} \cdots (z-\lambda_m)^{n-k+1})F(f) = 0.$$

Thus $\cap_{i=1}^m M_{n,n-k}(\lambda_i) \subset \Im(D)$. If $x \in S(D)$, then there exists $\{f_\ell\}$ in $C^n(I)$ such that $f_\ell \to 0$ and $Df_\ell \to x$. Thus $F(f_\ell) \to x$ and

$$\rho((z-\lambda_1)^{n-k+1}(z-\lambda_2)^{n-k+1} \cdots (z-\lambda_m)^{n-k+1})x = 0.$$

So $x \in W_k$.

References

[1] W. G. Bade and P. C. Curtis,Jr., *The continuity of derivations of Banach algebra*, J. Funct. Anal. **16** (1974), 372–387.

[2] W. G. Bade and P. C. Curtis,Jr., *The structure of module derivations of Banach algebras of differentiable functions*, J. Funct. Anal. **28** (1978), 226–247.

[3] S. Kantorovitz, *The semi-simplicity manifold of arbitrary operators*, Trans. Amer. Math. Soc. **123** (1966), 241–252.

[4] S. Kantorovitz, "Spectral theory of Banach space operators," Springer-Verlag and New York, 1983.

[5] K. B. Laursen, *A structure theorem for discontinuous derivations of Banach algebras of differential functions*, Proc. Amer. Math. Soc. **102** (1988), 507–513.

[6] Kil-Woung Jun, Young-Whan Lee and Dal-Won Park, *The continuity of derivations and module homomorphism*, Bull. Korean Math. Soc. **27** (1990).

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