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Limits and Colimits in Fibrewise Convergence Spaces

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ABSTRACT. In this paper, we introduce the concept of the fibrewise convergence space as a generalization of both the notion of fibrewise topology and that of convergence. Furthermore we observe the adjointness and Galois correspondence between the category of fibrewise topological spaces and the category of fibrewise convergence spaces. Finally we investigate the limit and colimit structures in these categories.

Introduction

Since I.M. James [**6**, **7**, **8**] introduced the notion of fibrewise topology, an extensive works on the theory of fibrewise topology have been carried out by many researchers. The fibrewise viewpoint is standard in the theory of fibre bundles. However, it has been recognized only recently that the same viewpoint is also of great value in other theories, such as general topology. Many of the familiar definitions and theorems of ordinary topology can be generalized, in a natural way, so that one can develop a theory of topology over a base.

In this paper, we introduce the concept of the fibrewise convergence space as a generalization of both the notion of fibrewise topology and that of convergence. Furthermore we observe the adjointness and Galois correspondence between the category of fibrewise topological spaces and the category of fibrewise convergence spaces. Finally we investigate the limit and colimit structures in these categories.

For general categorical background we refer to H. Herrlich and G.E. Strecker [**5**], for the fibrewise theory to I.M. James [**6**, **7**, **8**] and for the convergence space to E. Binz [**1**].

I. Preliminaries

In this section, we collect some basic definitions and known results about the convergence spaces from E. Binz [**1**] and fibrewise spaces from I.M. James [**6**, **7**, **8**] which we shall need in later sections.

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1.1. Convergence Spaces

For any set X, we denote by $F(X)$ the set of all filters on X, and by $P(F(X))$ the power set of $F(X)$.

DEFINITION 1.1: Let X be a set. A function $c : X \to P(F(X))$ is said to be a *convergence structure* if the following properties hold for any point $x \in X$:

1) for any $x \in X$, $\dot{x} \in c(x)$;
2) if $\mathcal{F} \in c(x)$ and $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G} \in c(x)$;
3) if $\mathcal{F}$, $\mathcal{G} \in c(x)$, then $\mathcal{F} \cap \mathcal{G} \in c(x)$.

Here $\dot{x}$ stands for the ultrafilter on X generated by $\{x\}$. The pair (X, c) is named a *convergence space.* The filters in $c(x)$ are said to be *convergent* to x. We usually write $\mathcal{F} \to x$ instead of $\mathcal{F} \in c(x)$.

DEFINITION 1.2: Let (X, c) and (Y, c') be the convergence spaces and $f : X \to Y$ a map. Then f is said to be *continuous at* $x \in X$ if for any $\mathcal{F} \in c(x)$, $f(\mathcal{F}) \in c'(f(x))$. And f is said to be *continuous* if f is continuous at each point x of X.

The class of all convergence spaces and continuous maps forms a category, which will be denoted by **Conv.**

The category of all sets and functions between them will be denoted by **Set**. The category of all topological spaces and continuous functions between them will be denoted by **Top**.

Let $(X, \mathcal{T})$ be a topological space. Define that a filter $\mathcal{F}$ on X converges to x with respect to $c_{\mathcal{T}}$ if the filter $\mathcal{F}$ contains the neighborhood filter $\mathcal{N}_x$ of x. Then it is easy to check that $c_{\mathcal{T}}$ is a convergence structure on X. This convergence structure $c_{\mathcal{T}}$ is called *the convergence structure generated by the topology* $\mathcal{T}$.

Let (X, c) be a convergence space. We call a subset A of X *open* if it belongs to every filter which converges to a point of A. The collection of all open sets of a convergence space X fulfills the axioms of a topology. This topology $\mathcal{T}_c$ is called *the topology associated to the convergence structure* c of X.

Thus we can define two functions E : **Top** $\to$ **Conv** by $E(X, \mathcal{T}) = (X, c_{\mathcal{T}})$ and G : **Conv** $\to$ **Top** by $G(X, c) = (X, \mathcal{T}_c)$. Then it is easy to check that E and G are functors.

Note that $(1_X, (X, \mathcal{T}_c))$ is the reflection for (X, c) with respect to the embedding functor E : **Top** $\to$ **Conv**. Moreover it is well known that the category **Top** is a bireflective subcategory of **Conv**.

1.2. Fibrewise Spaces

Given an object B of a category $\mathbf{C}$, the category $\mathbf{C}^B$ of objects under B is defined as follows. An *object under* B is a pair (u, X) consisting of an object X of $\mathbf{C}$ and a morphism $u : B \to X$ of $\mathbf{C}$, called the *insertion*. If X, Y are objects under B with insertions u, v then a morphism $f : X \to Y$ of $\mathbf{C}$ is a *morphism under* B if $f \circ u = v$. Composition in $\mathbf{C}^B$ is defined according to the composition in $\mathbf{C}$.

Again, given an object B of a category $\mathbf{C}$, the $\mathbf{C}_B$ of objects over B is defined as follows. An *object over* B is a pair (X, p) consisting of an object X of $\mathbf{C}$ and a morphism $p : X \to B$ of $\mathbf{C}$, called the *projection*. If X, Y are objects over B with projections p, q then a morphism $f : X \to Y$ of $\mathbf{C}$ is a *morphism over* B if $q \circ f = p$. Composition in $\mathbf{C}_B$ is defined according to the composition in $\mathbf{C}$.

Given an object B of a category $\mathbf{C}$, the category $\mathbf{C}_B^B$ of objects over and under B is defined as follows. An *object over and under* B is a triple (u, X, p) consisting of an object X of $\mathbf{C}$ and a pair of morphisms

$$B \xrightarrow{u} X \xrightarrow{p} B$$

of $\mathbf{C}$ such that $p \circ u = 1_B$. In particular B is regarded as an object over and under itself, taking 1_B to be both insertion and projection. If X, Y are objects over and under B, with projections p, q and insertions u, v, then a morphism $f : X \to Y$ in C is a *morphism over and under* B if $f \circ u = v$ and $q \circ f = p$.

In the category $\mathbf{Set}_B$, let X, Y be sets over B with projections p, q, respectively. A *fibre product* $X \times_B Y$ is the subset of $X \times Y$ consisting of pairs (x, y) such that $p(x) = q(y)$, with the projection r given by $r(x, y) = p(x) = q(y)$. In fact, $X \times_B Y$ is a product of X and Y in the category $\mathbf{Set}_B$.

2. Adjointness and Galois Correspondence

In this section, we will investigate the adjointness and Galois correspondence between the categories of fibrewise spaces.

Define $D : \mathbf{Set}_B \to \mathbf{Top}_B$ by $D(X, p) = ((X, \mathcal{D}), p)$ and $I : \mathbf{Set}_B \to \mathbf{Top}_B$ by $I(X, p) = ((X, \mathcal{I}), p)$, where $\mathcal{D}$ is the discrete topology on X

and $\mathcal{I}$ is the indiscrete topology on X. Then it is easy to check that D and I are functors.

PROPOSITION 2.1. *Let $D : \mathbf{Set}_B \to \mathbf{Top}_B$ be the discrete functor and $U : \mathbf{Top}_B \to \mathbf{Set}_B$ be the forgetful functor. Then D is a left adjoint of U and (D, U) is a Galois correspondence.*

PROOF: For any (X,p) in $\mathbf{Set}_B$, there exist $D(X,p) = ((X,\mathcal{D}),p) \in \mathbf{Top}_B$ and a map $1_X : (X,p) \to U((X,\mathcal{D}),p)$. Consider $((Y,\mathcal{T}),q) \in \mathbf{Top}_B$ and a map $f : (X,p) \to U((Y,\mathcal{T}),q) = (Y,q)$. Since $f : (X,\mathcal{D}) \to (Y,\mathcal{T})$ is a continuous map and $q \circ f = p$, there exists a unique continuous map $f : ((X,\mathcal{D}),p) \to ((Y,\mathcal{T}),q)$ such that $U(f) \circ 1_X = f$. Thus 1_X is a U-universal map for (X,p) in $\mathbf{Set}_B$. Hence D is a left adjoint of U. Clearly $1_{\mathbf{Set}_B} = U \circ D$. For any $((X,\mathcal{T}),p)$ in $\mathbf{Top}_B$, there exist $D \circ U((X,\mathcal{T}),p) \in \mathbf{Top}_B$ and a continuous map $1_X : D \circ U((X,\mathcal{T}),p) \to ((X,\mathcal{T}),p)$. Thus $D \circ U \leq 1_{\mathbf{Top}_B}$. Hence (D,u) is a Galois correspondence.

Similarly, we have analogous results in the categories under B. The forgetful functor $U : \mathbf{Top}^B \to \mathbf{Set}^B$ is a left adjoint of the indiscrete functor $I : \mathbf{Set}^B \to \mathbf{Top}^B$. Also (U, I) is a Galois correspondence.

Define $D : \mathbf{Set}_B \to \mathbf{Conv}_B$ by $D(X,p) = ((X,c^*),p)$ and $I : \mathbf{Set}_B \to \mathbf{Conv}_B$ by $I(X,p) = ((X,c_*),p)$, where c^* is the discrete convergence structure and c_* is the indiscrete convergence structure. Then it is easy to check that D and I are functors.

PROPOSITION 2.2. *Let $D : \mathbf{Set}_B \to \mathbf{Conv}_B$ be the discrete functor and $U : \mathbf{Conv}_B \to \mathbf{Set}_B$ be the forgetful functor. Then D is a left adjoint of U and (D, U) is a Galois correspondence.*

Similarly, we have analogous results in the categories under B. The forgetful functor $U : \mathbf{Conv}^B \to \mathbf{Set}^B$ is a left adjoint of the indiscrete functor $I : \mathbf{Set}^B \to \mathbf{Conv}^B$. Also (U, I) is a Galois correspondence.

Next, we investigate the relation of the category of fibrewise topological spaces and the category of fibrewise convergence spaces. Define $E : \mathbf{Top}_B \to \mathbf{Conv}_B$ by $E((X,\mathcal{T}),p) = ((X,c_{\mathcal{T}}),p)$ and $G : \mathbf{Conv}_B \to \mathbf{Top}_B$ by $G((X,c),p) = ((X,\mathcal{T}_c),p)$. Then it is easy to check that E and G are functors.

PROPOSITION 2.3. *For a topological space B, let $G : \mathbf{Conv}_B \to \mathbf{Top}_B$ be the associated functor and $E : \mathbf{Top}_B \to \mathbf{Conv}_B$ be the embedding functor. Then G is a left adjoint of E and (G, E) is a Galois correspondence.*

PROOF: For any space $((X,c),p)$ in $\mathbf{Conv}_B$, there exist $G((X,c),p) = ((X,\mathcal{T}_c),p) \in \mathbf{Top}_B$ and a continuous map $1_X : ((X,c),p) \to E((X,\mathcal{T}),p)$. Consider a topological space $((Y,\mathcal{T}'),q)$ over B and a continuous map $f : ((X,c),p) \to E((Y,\mathcal{T}'),q)$ over B. Then there exists a unique continuous map $f : ((X,\mathcal{T}_c),p) \to ((Y,\mathcal{T}'),q)$ such that $E(f) \circ 1_X = f$. Thus 1_X is a E-universal map for $((X,c),p)$ in $\mathbf{Conv}_B$. Hence G is a left adjoint of E. Clearly $G \circ E = 1_{\mathbf{Top}_B}$. For any $((X,c),p)$ in $\mathbf{Conv}_B$, $E \circ E((X,c),p) \in \mathbf{Conv}_B$ since B is a topological space. Then $1_X : ((X,c),p) \to E \circ G((X,c),p)$ is a continuous map. Thus $1_{\mathbf{Conv}_B} \leq E \circ G$. Hence (G,E) is a Galois correspondence.

We also obtain following similar results about the categories $\mathbf{Conv}^B$ and $\mathbf{Conv}^B_B$.

PROPOSITION 2.4. *Let $G : \mathbf{Conv}^B \to \mathbf{Top}^B$ be the associated functor and $E : \mathbf{Top}^B \to \mathbf{Conv}^B$ be the embedding functor. Then G is a left adjoint of E and (G, E) is a Galois correspondence.*

PROPOSITION 2.5. *For a topological space B, let $G : \mathbf{Conv}^B_B \to \mathbf{Top}^B_B$ be the associated functor and $E : \mathbf{Top}^B_B \to \mathbf{Conv}^B_B$ be the embedding functor. Then G is a left adjoint of E and (G, E) is a Galois correspondence.*

For other cases, we have adjointness and Galois correspondences only when they are trivial cases.

3. Limits and Colimits in Fibrewise Convergence Spaces

3.1. Limits in Fibrewise Convergence Spaces

In this section, we will extend the notions of limits in the category $\mathbf{Conv}$ to the categories $\mathbf{Conv}_B$, $\mathbf{Conv}^B$ and $\mathbf{Conv}^B_B$.

First, we recall the limit structures in the category $\mathbf{Conv}$.

Let X, Y, Z be convergence spaces. Then the set $X \times Y$ with the initial convergence structure with respect to the projection maps $\{pr_1 : X \times Y \to X, pr_2 : X \times Y \to Y\}$ is the product of X and Y in the category **Conv**. And the set $E = \{x \in X : f(x) = g(x)\}$ with the initial convergence structure with respect to $e : E \to X$ is the equalizer of f, $g : X \to Y$ in the category **Conv**. Also the set $X \times_Z Y$ with the initial convergence structure with respect to the projection maps $\{pr_1 : X \times_Z Y \to X, pr_2 : X \times_Z Y \to Y\}$ is the pullback of the triad $X \xrightarrow{f} Z \xleftarrow{g} Y$ in the category **Conv**. Let X_i be subspaces of the convergence space Y. Then the set $\cap X_i$ with the initial convergence structure with respect to $d : \cap X_i \to Y$ is the intersection of X_i in the category **Conv**.

Next, we investigate the limit structures in the category $\mathbf{Conv}_B$.

PROPOSITION 3.1. *For convergence spaces (X, p), (Y, q) over B, let the fibre product $X \times_B Y$ be a subspace of $X \times Y$. Then $X \times_B Y$ is the product of X and Y in the category $\mathbf{Conv}_B$.*

LEMMA 3.2. *Equalizers in any category $\mathbf{C}$ are also equalizers in the category $\mathbf{C}_B$ over B.*

PROOF: For spaces (X, p), (Y, q) over B, consider f, $g : X \to Y$ in $\mathbf{C}^B$. Let (E, e) be an equalizer of f and g in $\mathbf{C}$. Consider $(E', r') \in \mathbf{C}_B$ and a morphism $e' : E' \to X \in \mathbf{C}_B$ such that $f \circ e' = g \circ e'$. Then there exists a unique morphism $\bar{e} : E' \to E$ such that $e' = e \circ \bar{e}$, since (E, e) is an equalizer of f and g in $\mathbf{C}$. Since $p \circ e \circ \bar{e} = p \circ e' = r'$, $\bar{e} \in \mathbf{C}_B$. And clearly $f \circ e = g \circ e$. Thus $((E, p \circ e), e)$ is an equalizer of f and g in $\mathbf{C}_B$.

By the above lemma, we obtain the following result.

PROPOSITION 3.3. *For convergence spaces (X, p), (Y, q) over B, consider continuous maps f, $g : X \to Y$ in the category* **Conv**. *Let (E, e) be the equalizer of f and g in the category* **Conv**. *Then (E, e) is also an equalizer of f and g in the category $\mathbf{Conv}_B$.*

LEMMA 3.4. *Intersections in any category $\mathbf{C}$ are also intersections in the category $\mathbf{C}_B$ over B.*

PROOF: For spaces (X_i, p_i), (Y, q) over B, consider morphisms $m_i : X_i \to Y$ in $\mathbf{C}_B$. Let (X, d) be an intersection of (X_i, m_i) in $\mathbf{C}$. And let $q \circ d : X \to B$, then clearly $d \in \mathbf{C}_B$. Since $p_i \circ d_i = q \circ m_i \circ d_i = q \circ d$, $d_i \in \mathbf{C}_B$. Suppose that $(Z, r) \in \mathbf{C}_B$ and $g : Z \to Y$, $g_i : Z \to X_i$ such that $m_i \circ g_i = g$ for all $i \in I$. Then there exists a unique morphism

$h : Z \to X$ such that $d \circ h = g$, since $(X,d)$is an intersection of (X_i, m_i) in $\mathbf{C}$. And since $q \circ d \circ h = q \circ g = r$, $h \in \mathbf{C}_B$. Thus $((X, q \circ d), d)$ is an intersection of $((X_i, p_i), m_i)$ in $\mathbf{C}_B$.

By the above lemma, we obtain the following result.

PROPOSITION 3.5. *For convergence space (Y,q), consider a family of subspaces $\{(X_i,p_i)\}_{i\in I}$ of Y. Let $(\cap X_i, d)$ be an intersection of $\{X_i\}_{i\in I}$ in the category* **Conv**. *Then $(\cap X_i, d)$ is also an intersection of $\{X_i\}_{i\in I}$ in the category* $\mathbf{Conv}_B$.

LEMMA 3.6. *Pullbacks in any category* $\mathbf{C}$ *are also pullbacks in the category* $\mathbf{C}_B$ *over* B.

PROOF: For spaces (X,p), (Y,q), (Z,r) over B, consider a triad $X \xrightarrow{f} Z \xleftarrow{g} Y$. Let $(X \times_Z Y, pr_1, pr_2)$ be a pullback of the triad in the category $\mathbf{C}$. And let $t = p \circ pr_1 = q \circ pr_2$. Consider $(W,s) \in \mathbf{C}_B$ and morphisms $f' : W \to X$, $g' : W \to Y \in \mathbf{C}_B$ such that $f \circ f' = g \circ g'$. Then there exists a unique morphism $h : W \to X \times_Z Y$ such that $pr_1 \circ h = f'$ and $pr_2 \circ h = g'$, since $(X \times_Z Y, pr_1, pr_2)$ is the pullback of the triad in the category $\mathbf{C}$. And since $t \circ h = p \circ pr_1 \circ h = p \circ f' = s$, $h \in \mathbf{C}_B$. Thus $((X \times_Z Y, t), pr_1, pr_2)$ is a pullback of the triad in the category $\mathbf{C}_B$.

By the above lemma, we obtain the following result.

PROPOSITION 3.7. *For convergence spaces (X,p), (Y,q), (Z,r) over B, consider the triad $X \xrightarrow{f} Z \xleftarrow{g} Y$. Let $X \times_Z Y$ be a pullback of the triad in the category* **Conv**. *Then $X \times_Z Y$ is also a pullback of the triad in the category* $\mathbf{Conv}_B$.

Next, we investigate the limit structures in the category $\mathbf{Conv}^B$.

THEOREM 3.8. *For a functor $D : \mathbf{I} \to \mathbf{C}^B$, let a natural source $(L,(l_i))$ be a limit of $\overline{D} : \mathbf{I} \to \mathbf{C}$. Then $(L,(l_i))$ is also a limit of $D : \mathbf{I} \to \mathbf{C}^B$.*

PROOF: Let morphisms $u_i : B \to Di$ be the insertion for each $i \in I$. Then (B, u_i) is a natural source for $\overline{D} : \mathbf{I} \to \mathbf{C}$. since $(L,(l_i))$ is a limit of $\overline{D} : \mathbf{I} \to \mathbf{C}$, there exists a unique morphism $r : B \to L$ such that $l_i \circ r = u_i$ for each $i \in I$. Then $(r, L) \in \mathbf{C}^B$. Thus $(r,(L,(l_i)))$ is a natural source for $D : \mathbf{I} \to \mathbf{C}^B$. Consider any natural source

$(r', (L', l'_i))$ for $D : \mathbf{I} \to \mathbf{C}^B$. Since $(L, (l_i))$ is a limit of $\overline{D} : \mathbf{I} \to \mathbf{C}$, there exists a unique morphism $h : L' \to L$ such that $l_i \circ h = l'_i$ for each $i \in I$. But $l_i \circ h \circ r' = l'_i \circ r' = u_i = l_i \circ r$. Then $h \circ r' = r$ since (l_i) is mono-source. Thus $h \in \mathbf{C}^B$. Hence the result follows.

Note that products, equalizers, pullbacks and intersections are limits. Hence by the above theorem products, equalizers, pullbacks and intersections in the category **Conv** is also products, equalizers, pullbacks and intersections respectively in the category $\mathbf{Conv}^B$.

From the results obtained in the categories $\mathbf{Conv}_B$ and $\mathbf{Conv}^B$, we obtain that equalizers, pullbacks and intersections in the category **Conv** are also equalizers, pullbacks and intersections respectively in the category $\mathbf{Conv}_B^B$. And products in the category $\mathbf{Conv}_B$ are products in the category $\mathbf{Conv}_B^B$.

3.2. Colimits in Fibrewise Convergence Spaces

In the section, we will extend the notions of colimits in the category **Conv** to the categories $\mathbf{Conv}_B$, $\mathbf{Conv}^B$ and $\mathbf{Conv}_B^B$.

First, we recall the colimit structures in the category **Conv**.

Let X, Y, Z be convergence spaces, then the set $X + Y$ with the final convergence structure with respect to the inclusion maps $\{i : X \to X + Y, j : Y \to X + Y\}$ is the coproduct of X, Y in the category **Conv**. And the set Y/Q with the final convergence structure with respect to the natural map $\natural : Y \to Y/Q$ is the coequalizer of f, $g : X \to Y$ in the category **Conv**. Also the set $X +^Z Y$ with the final convergence structure with respect to the inclusion maps $\{i : X \to X +^Z Y, j : Y \to X +^Z Y\}$ is the pushout of the cotriad $X \xleftarrow{f} Z \xrightarrow{g} Y$ in the category **Conv**.

Next, we investigate the colimit structures in the category $\mathbf{Conv}_B$.

THEOREM 3.9. *For a functor $D : \mathbf{I} \to \mathbf{C}_B$, let a natural sink (k_i, K) be a colimit of $\overline{D} : \mathbf{I} \to \mathbf{C}$. Then (k_i, K) is also a colimit of $D : \mathbf{I} \to \mathbf{C}_B$.*

PROOF: Let morphism $p_i : Di \to B$ be the projection for each $i \in I$. Then (p_i, B) is a natural sink for $\overline{D} : \mathbf{I} \to \mathbf{C}$. Since (k_i, K) is a colimit of $\overline{D} : \mathbf{I} \to \mathbf{C}$, there exists a unique morphism $p : K \to B$ such that $p \circ k_i = p_i$ for each $i \in I$. Then $(K, p) \in \mathbf{C}_B$. Thus $((k_i, K), p)$ is a natural sink for $D : \mathbf{I} \to \mathbf{C_B}$. Consider any natural sink $(k_i', K'), p')$ for $D : \mathbf{I} \to \mathbf{C_B}$. Since (k_i, K) is a colimit of $\overline{D} : \mathbf{I} \to \mathbf{C}$, there exists a unique morphism $h : K \to K'$ such that $h \circ k_i = k_i'$ for each $i \in I$. But $p' \circ h \circ k_i = p' \circ k_i' = p_i = p \circ k_i$. Then $p' \circ h = p$ since (k_i) is epi-sink. Thus $h \in \mathbf{C}_B$. Hence the result follows.

Note that coproducts, coequalizers and pushouts are colimits. Hence by the above theorem coproducts, coequalizers and pushouts in the category **Conv** are also coproducts, coequalizers and pushouts respectively in the category $\mathbf{Conv}_B$.

Next, we investigate the colimit structures in the category $\mathbf{Conv}^B$.

PROPOSITION 3.10. *For a family $\{(u_i, X_i)\}_{i \in I}$ of convergence spaces under B, let fibre-wedge sum $\coprod^B X_i$ be a quotient space of $\coprod X_i$. Then $\coprod^B X_i$ is a coproduct of the family $\{X_i\}_{i \in I}$ in the category $\mathbf{Conv}^B$.*

LEMMA 3.11. *Coequalizers in any category $\mathbf{C}$ are also coequalizers in the category $\mathbf{C}^B$.*

PROOF: For spaces (u, X), (v, Y) under B, consider morphisms f, $g : X \to Y$ in $\mathbf{C}^B$. Let $(\natural, Z)$ be a coequalizer of f and g in $\mathbf{C}$. Consider $(r', Z') \in \mathbf{C}^B$ and a morphism $\alpha : Y \to Z' \in \mathbf{C}^B$ such that $\alpha \circ f = \alpha \circ g$. Then there exists a unique morphism $h : Z \to Z'$ such that $h \circ \natural = \alpha$, since $(\natural, Z)$ is a coequalizer of f and g in $\mathbf{C}$. Since $h \circ \natural \circ v = \alpha \circ v = r'$, $h \in \mathbf{C}^B$. And clearly $\natural \circ f = \natural \circ g$. Hence $(\natural(\natural \circ v, Z))$ is a coequalizer of f and g in $\mathbf{C}^B$.

By the above lemma, we obtain the following result.

PROPOSITION 3.12. *For convergence spaces (u, X), (v, Y) under B, consider continuous maps f, $g : X \to Y$ in the category $\mathbf{Conv}^B$. Let $(\natural, Y/Q)$ be a coequalizer of f and g in the category **Conv**. Then $(\natural, Y/Q)$ is also a coequalizer of f and g in the category $\mathbf{Conv}^B$.*

LEMMA 3.13. *Pushouts in any category $\mathbf{C}$ are also pushouts in the category $\mathbf{C}^B$.*

PROOF: For spaces (u, X), (v, Y), (w, Z) under B, consider a cotriad $X \stackrel{f}{\leftarrow} Z \stackrel{g}{\rightarrow} Y$. Let $(i, j, X +^Z Y)$ be a pushout of the cotriad in the category $\mathbf{C}$. And let $t = i \circ u = j \circ v$. Consider $(s, W) \in \mathbf{C}^B$ and morphisms $f' : X \to W$, $g' : Y \to W \in \mathbf{C}^B$ such that $f' \circ f = g' \circ g$. Then there exists a unique morphism $k : X +^Z Y \to W$ such that $k \circ i = f'$ and $k \circ j = g'$, since $(i, j, X +^Z Y)$ is the pushout of the cotriad in the category $\mathbf{C}$. And since $k \circ t = k \circ i \circ u = f' \circ u = s$, $k \in \mathbf{C}^B$. Thus $(i, j, (t, X +^Z Y))$ is a pushout of the cotriad in the category $\mathbf{C}^B$.

By the above lemma, we obtain the following result.

PROPOSITION 3.14. *For convergence spaces* (u, X), (v, Y), (w, Z) *under* B, *consider a cotriad* $X \xleftarrow{f} Z \xrightarrow{g} Y$. *Let* $X +^Z Y$ *be a pushout of the cotriad in the category* **Conv**. *Then* $X +^Z Y$ *is also a pushout of the cotriad in the category* $\mathbf{Conv}^B$.

From the results obtained in the categories $\mathbf{Conv}_B$ and $\mathbf{Conv}^B$, we obtain that coequalizers, pushouts in the category **Conv** are also coequalizers, pushouts respectively in the category $\mathbf{Conv}_B^B$. And coproducts in the category $\mathbf{Conv}^B$ are coproducts in the category $\mathbf{Conv}_B^B$.

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