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The Bourgain Property

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ABSTRACT. In this paper, we study the Bourgain property for real-valued functions, and give conditions for a family of real-valued functions to have the Bourgain property.

The family $\{f(\cdot)x : \|x\| \leq 1\}$ plays a strong role in determining Pettis integrability for a bounded weakly measurable function f from a measure space (Ω, Σ, μ) into a dual space X^*. Rather than viewing such families as subsets of $L_\infty(\mu)$, we now consider them simply as families of real-valued functions on Ω.

In this paper, we study the family $\{f(\cdot)x : \|x\| \leq 1\}$ for a bounded function $f : \Omega \to X^*$ and present some properties of the Bourgain property for real-valued functions on Ω.

The following property of real-valued functions was originally formulated by J. Bourgain.

DEFINITION 1: Let (Ω, Σ, μ) be a measure space. A family ψ of real-valued functions on Ω is said to have the *Bourgain property* if the following condition holds;

For each set A of positive measure and for each $\alpha > 0$, there is a finite collection F of subsets of positive measure of A such that for each $f \in \psi$, the inequality

$$\sup f(B) - \inf f(B) < \alpha$$

holds for some member B of F.

From the above definition, we have the following result.

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THEOREM 2. *If ψ has the Bourgain property, then every f in ψ is measurable.*

PROOF: Let f be any function in ψ and fix $n \in N$. Since ψ has the Bourgain property, for $A \in \Sigma$ with $\mu(A) > 0$ there exists a subset B of A with $\mu(B) > 0$ such that $\sup f(B) - \inf f(B) < \frac{1}{n}$.

By the Exhaustion lemma, there exists a sequence $(A_{n,k})_k$ of disjoint sets in Σ such that $\sup f(A_{n,k}) - \inf f(A_{n,}) < \frac{1}{n}$ for each k and $\Omega = \bigcup_k A_{n,k}$. Let $\varphi_n = \sum_{k=1}^{\infty} c_{n,k}\chi_{A_{n,k}}$, where $c_{n,k}$ is a real number satisfying $\inf f(A_{n,k}) \leq c_{n,k} \leq \sup f(A_{n,k})$.

Then φ_n converges to f uniformly a.e.. Hence f is measurable.

Studying functions $f : \Omega \to X^*$ into dual spaces, Riddle and Saab show that every bounded function $f : \Omega \to X^*$ which has the Bourgain property is Pettis integrable. Their proof relies on the following theorem by Bourgain.

THEOREM 3. [3] *If (Ω, Σ, μ) is a finite measure space and ψ is a family of real-valued functions on Ω having the Bourgain property, then*

(1) *the pointwise closure of ψ has the Bourgain property ;*
(2) *each element in the pointwise closure of ψ is measurable ;*
(3) *each element in the pointwise closure of ψ is the almost everywhere pointwise limit of a sequence in ψ.*

PROOF: The proof of (1) is completely straightforward. Towards verifying (2) and (3), take a function g belonging to the pointwise closure of ψ and an ultrafilter U on ψ that has g a cluster point. For A in Σ and $\alpha > 0$, let

$$\psi(A\,;\,\alpha) = \{f \in \psi\,;\, \sup f(A) - \inf f(A) < \alpha\}.$$

It follows from the definition of the Bourgain property that if A has positive measure and $\alpha > 0$, then there exists a subset B of A of positive measure with $\psi(B\,;\,\alpha)$ belonging to U. Now for each $\alpha > 0$, use Zorn's Lemma to find a maximal set P_α of mutually disjoint sets of positive measure such that $\psi(A\,;\,\alpha) \in U$ for each $A \in P_\alpha$. Note that each P_α is necessarily countable. Moreover,

(a) the set $\Omega - \bigcup P_\alpha$ has measure 0 for each $\alpha > 0$ and

(b) if F is a finite subset of positive reals and Q_α is a finite subset of P_α for each α in F, then g belongs to the pointwise closure of $\bigcap_{\alpha\in F}\bigcap_{A\in Q_\alpha}\psi(A;\alpha)$. The maximality of P_α yields condition (a) and (b) follows because g is a cluster point of U. Now let $(A_{m,n})_n$ be an enumeration of $P_{\frac{1}{m}}$, and set $B=\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty}A_{m,n}$. By condition (a), we have $\mu(\Omega-B)=0$. Pick some point $w_{m,n}$ in each set $A_{m,n}$ and define

$$f_m=\sum_{n=1}^{\infty}g(w_{m,n})\chi_{A_{m,n}}.$$

Each f_m is measurable and a quick computation using (b) shows that the sequence (f_m) converges to g uniformly on B. Therefore g is measurable. Unfortunately, the functions f_m may not belong to ψ. To establish (3), use condition (b) to pick for each integer m a function h_m belonging to $\bigcap_{i=1}^{\infty}\bigcap_{n=1}^{m}\psi(A_{i,n};\frac{1}{i})$ such that $|h_m(w_{i,n})-g(w_{i,n})|<\frac{1}{i}$ for each $1\le i,\ n\le m$. The triangle inequality now insures that $(h_m(w))$ congerges to $g(w)$ for each w in B. This completes the proof.

We can make the following general definition of the Bourgain property for functions into Banach spaces.

DEFINITION 4: A function $f:\Omega\to X$ is said to have the *Bourgain property* if the set $\{x^*f;\ \|x^*\|\le 1\}$ has the Bourgain property.

If in the above definition the space X is a dual space, then the following result holds.

THEOREM 5. *Let $f:\Omega\to X^*$ be a function into a dual space. Then $\{f(\cdot)x:\|x\|\le 1\}$ has the Bourgain property if and only if $\{x^{**}f:\|x\|^{**}\le 1\}$ has the Bourgain property.*

PROOF: If $\{x^{**}f:\|x\|^{**}\le 1\}$ has the Bourgain property, then it is obvious that $\{f(\cdot)x:\|x\|\le 1\}$ has the Bourgain property.

Conversely, suppose $\{f(\cdot)x:\|x\|\le 1\}$ has the Bourgain property. For any $x^{**}\in X^{**}$ with $\|x\|^{**}\le 1$, there exists a net (x_α) in X such that (x_α) weak*-converges to x^{**}. Hence $x^{**}f$ belongs to the pointwise closure of $\{f(\cdot)x:\|x\|\le 1\}$ and $\{x^{**}f|\|x\|^{**}\le 1\}$ is contained in the pointwise closure of $\{f(\cdot)x:\|x\|\le 1\}$. Since $\{f(\cdot)x:\|x\|\le 1\}$ has the Bourgain property, so has its pointwise closure by theorem 3.

Hence $\{x^{**}f : \|x\|^{**} \leq 1\}$ has the Bourgain property.

For two functions $f : \Omega \to X$ and $g : \Omega \to X$ such that $x^*f = x^*g$a.e. for all $x^* \in X^*$, we have the following result.

THEOREM 6. *Let $f : \Omega \to X$ and $g : \Omega \to X$ be functions such that $x^*f = x^*g$ a.e. for all $x^* \in X^*$. Then f has the Bourgain property if and only if g has the Bourgain property.*

PROOF: Since $x^*f = x^*g$ a.e. for all $x^* \in X^*$, there exists a null set N such that $x^*f(w) = x^*g(w)$ for all $w \in \Omega - N$.

Since both x^*f and x^*g have the same supremum and infimum on the set $A - N$ for any set A of positive measure, the conclusion now follows immediately.

COROLLARY 7. *Let $f : \Omega \to X^*$ and $g : \Omega \to X^*$ be functions such that $f(\cdot)x = g(\cdot)x$ for all $x \in X$. Then f has the Bourgain property if and only if g has the Bourgain property.*

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