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Isometries of a Subalgebra of $C^{(1)}[0,1]$

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ABSTRACT. By $C^{(1)}[0,1]$ we denote the Banach algebra of complex valued continuously differentiable functions on $[0,1]$ with norm given by

$$\|f\| = \sup_{x\in[0,1]} (|f(x)| + |f'(x)|) \text{ for } f \in C^{(1)}.$$

By A we denote the subalgebra of $C^{(1)}$ defined by

$$A = \{f \in C^{(1)} : f(0) = f(1) \text{ and } f'(0) = f'(1)\}.$$

By an isometry of A we mean a norm-preserving linear map of A onto itself.

The purpose of this article is to describe the isometries of A. More precisely, we show tht any isometry of A is induced by a point map of the interval $[0,1]$ onto itself.

The isometries of $C^{(1)}$ are determined by M. Cambern [**1**]. V.D. Pathak [**3**] have also determined the isometries of $C^{(n)}$, with norm given by

$$\|f\| = \sup_{x\in[0,1]} \sum_{r=0}^{n} \frac{|f^{(r)}(x)|}{r!}.$$

In the proof we shall follow the techniques of [**1**] and [**3**].

DEFINITION 1: We define a function d from $[0,1]\times[0,1]$ into $[-1/2, 1/2)$ by $d(x,y) = x-y-[x-y+1/2]$ and identify 0 with 1 then there exists a topology κ of $[0,1]$ induced by a metric $|d|$. Denote $[0,1]_\kappa$ for the topological space $[0,1]$ with the topology κ. For $x \in [0,1]_\kappa$ define $f'(x)$ by

$$f'(x) = \begin{cases} \lim_{d(y,x)\to 0} \dfrac{f(y)-f(x)}{d(y,x)} & \text{if } f:[0,1]_\kappa \to C \\ \lim_{d(y,x)\to 0} \dfrac{d(f(y)-f(x))}{d(y,x)} & \text{if } f:[0,1]_\kappa \to [0,1]_\kappa. \end{cases}$$

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By $C^{(1)}[0,1]_\kappa$ we denote the Banach algebra of complex valued continuously differentiable functions on $[0,1]_\tau$ into C with norm given by

$$\|f\| = \sup_{x\in[0,1]_\kappa} (|f(x)| + |f'(x)|) \quad \text{for} \quad f \in C^{(1)}[0,1]_\kappa.$$

By the identity function from $[0,1]$ onto $[0,1]_\kappa$ we can identity Banach algebra A with $C^{(1)}[0,1]_\kappa$. We prove the following propositions.

PROPOSITION 2. *Given $x \in [0,1]_\kappa$, $\theta \in [-\pi,\pi]$, then there exists $h \in C[0,1]_\kappa$ such that*

$$|h(x)| + |h'(x)| > |h(y)| + |h'(y)|$$

for $y \in [0,0]_\kappa$, $x \in [0,1]_\kappa$, $y \neq x$, with $|h(x)| = h(x) > 0$, $|h'(x)| = e^{i\theta}h(x) > 0$.

PROOF: Let f_0 be the real valued, nonnegative continuous function on $[0,1]_\kappa$ defined as follows

$$f_0(y) = \begin{cases} \frac{3}{4}d(y,x) + \frac{1}{8} & \cdots -\frac{1}{2} \le d(x,y) \le -\frac{1}{6} \\ 3d(y,x) + \frac{1}{2} & \cdots -\frac{1}{6} \le d(x,y) \le 0 \\ -3d(y,x) + \frac{1}{2} & \cdots 0 \le d(x,y) \le \frac{1}{6} \\ -\frac{3}{4}d(y,x) + \frac{1}{8} & \cdots \frac{1}{6} \le d(x,y) \le \frac{1}{2}. \end{cases}$$

Define $g(y) = \int_x^y f_0(t)\,dt$. It can be easily verified that $g(y)$ is as follows;

$$g(y) = \begin{cases} \frac{1}{32}(6d(y,x) - 1)(2d(y,x) + 1) & \cdots -\frac{1}{2} \le d(x,y) \le -\frac{1}{6} \\ \frac{1}{2}(3d(y,x) + 1)d(y,x) & \cdots -\frac{1}{6} \le d(x,y) \le 0 \\ -\frac{1}{2}(3d(y,x) - 1)d(y,x) & \cdots 0 \le d(x,y) \le \frac{1}{6} \\ -\frac{1}{32}(6d(y,x) + 1)(2d(y,x) - 1) & \cdots \frac{1}{6} \le d(x,y) \le \frac{1}{2}. \end{cases}$$

Therefore $g \in C[0,1]_\kappa$ and $g' = f_0$. Thus

$$g(x) = 0, \quad g'(x) = \frac{1}{2}$$
$$|g(x)| + |g'(x)| = \frac{1}{2}$$

Now consider $|g(y)| + |g'(y)|$ for $y \in [0,1]_\kappa$ and $y \neq x$.

$$|g(y)| + |g'(y)|$$
$$= \begin{cases} -\frac{3}{8}d(y,x)^2 - \frac{7}{8}d(y,x) - \frac{3}{32} < \frac{1}{2} & \cdots -\frac{1}{2} \leq d(x,y) \leq -\frac{1}{6} \\ -\frac{3}{2}d(y,x)^2 + \frac{5}{2}d(y,x) + \frac{1}{2} < \frac{1}{2} & \cdots -\frac{1}{6} \leq d(x,y) < 0 \\ -\frac{3}{2}d(y,x)^2 - \frac{5}{2}d(y,x) + \frac{1}{2} < \frac{1}{2} & \cdots 0 < d(x,y) \leq \frac{1}{6} \\ -\frac{3}{8}d(y,x)^2 + \frac{7}{8}d(y,x) - \frac{3}{32} < \frac{1}{2} & \cdots \frac{1}{6} \leq d(x,y) \leq \frac{1}{2}. \end{cases}$$

From this it follows that the function $h \in C[0,1]_\kappa$ defined by $h(y) = 1 + e^{i\theta}g(y)$ has the desired properties.

If X is any compact Hausdorff space, we will denote by $C(X)$ the Banach algebra of continuous complex functions defined on X with the norm $\| \ \|_\infty$ determined by $\|g\|_\infty = \sup_{x\in X} |g(x)|$ for $g \in C(X)$. Now let W denote the compact space $[0,1]\times[-\pi,\pi]$, given $f \in A$, we define $\tilde{f} \in C(W)$ by

$$\tilde{f}(x,\theta) = f(x) + e^{i\theta}f'(x) \qquad (x,\theta) \in W.$$

The following lemma is then obvious.

LEMMA 3. *The mapping $f \to \tilde{f}$ establishes a linear and norm preserving correspondence between A and the closed subspace S of $C(W)$, $S = \{\tilde{f} : f \in A\}$.*

Next given $(x,\theta) \in W$, we define a continuous linear functional $L_{(x,\theta)}$ on A by

$$L_{(x,\theta)}(f) = \tilde{f}(x,\theta), \qquad f \in A.$$

In view of Proposition 2 the proof of the following lemma is analogous to the proof of Lemma 1.2 in [**1**].

LEMMA 4. *If an element f^* of A^* is an extreme point of the unit ball U^* of A^* then f^* is of the form $e^{i\eta}L_{(x,\theta)}$ for some $\eta \in [-\pi,\pi]$, $(x,\theta) \in W$.*

We now suppose that T is an isometry of A. The adjoint T^* is then an isometry of A^*, and thus carries extreme points of U^* onto itself.

LEMMA 5. *The image by T of the constant function 1 of A is a constant function $e^{i\lambda}$, $\lambda \in [-\pi,\pi]$.*

PROOF: For each extreme point $e^{i\eta}L_{(x,\theta)}$ of U^*,

$$|(e^{i\eta}L_{(x,\theta)})(1)| = 1.$$

Thus for each extreme point $T^*(e^{i\eta}L_{(x,\theta)})(1)| = 1$. Therefore, $|(L_{(x,\theta)})(T(1))| = 1$. Thus for a fixed x, $|(T(1))(x)+e^{i\theta}(T(1))'(x)| = 1$ for all $\theta \in [-\pi,\pi]$. Choosing θ so that

$$\arg((T(1))(x)) = \arg(e^{i\theta}(T(1))'(x))$$

we get

$$|(T(1))(x)| + |(T(1))'(x)| = 1.$$

Again by choosing θ, so that

$$\arg((T(1))(x)) = \pi + \arg(e^{i\theta}(T(1))'(x))$$

we get

$$|\,|(T(1))(x)| - |(T(1))'(x)|\,| = 1.$$

Thus either

$$\{|(T(1))(x)| = 1 \text{ and } |(T(1))'(x)| = 0\}.$$

or

$$\{|(T(1))(x)| = 0 \text{ and } |(T(1))'(x)| = 1\}. \tag{1}$$

Therefore, for any $x \in [0,1]$, $|(T(1))(x)| = 1$ or $|(T(1))(x)| = 0$. But since $|(T(1))|$ is a continuous function on $[0,1]$ we have

$$|(T(1))(x)| \equiv 0 \quad \text{or} \quad |(T(1))'(x)| \equiv 1.$$

Now $|(T(1))(x)| \equiv 0$ implies that $(T(1))(x) \equiv (T(1))'(x) \equiv 0$ which contradicts (1).

Hence $|(T(1))(x)| \equiv 1$ from which it follows that $(T(1))'(x) \equiv 0$ and hence

$$T(1) \equiv e^{i\lambda} \text{ for some fixed } \lambda \in [-\pi, \pi].$$

We denote $T^*(L_{(x,\theta)})$ by

$$e^{i\lambda(x,\theta)} L_{(y(x,\theta),\psi_{(x,\theta)})}.$$

The above Lemma 4 shows that $\lambda(x,\theta) \equiv \lambda$ for all $\theta \in [-\pi, \pi]$. For

$$T^*(L_{(x,\theta)})(1) = e^{i\lambda(x,\theta)} L_{(y(x,\theta),\psi(x,\theta))}(1),$$

so that $L_{(x,\theta)}(T(1)) = e^{i\lambda(x,\theta)}$ and thus $L_{(x,\theta)}(e^{i\lambda}) = e^{i\lambda(x,\theta)}$. Hence $\lambda(x,\theta) = \lambda$.

LEMMA 6. *If $x \in [0,1]_\kappa$, then for all $\theta \in [-\pi, \pi]$,*

$$y_{(x,\theta)} = y_{(x,0)}.$$

PROOF: For fixed $x \in [0,1]_\kappa$, we consider the map $\rho : [-\pi, \pi] \to [0,1]_\kappa$ given by

$$\rho(\theta) = y_{(x,\theta)}.$$

Consider the function h of the Proposition 2 constructed for $(y_{(x,\theta)}, \psi_{(x,\theta)})$, then

$$\begin{aligned}
&\frac{3}{2}|\theta - \theta_1| \\
&\geq |(e^{i\theta} - e^{i\theta_1})| \, \|T(h)\| \\
&\geq |(e^{i\theta} - e^{i\theta_1})(T(h)'(x))| \\
&= |T(h)(x) + e^{i\theta}T(h)'(x) - T(h)(x) - e^{i\theta_1}T(h)'(x)| \\
&= |L_{(x,\theta)}(T(h)) - L_{(x,\theta_1)}(T(h))| \\
&= |T^*(L_{(x,\theta)} - L_{(x,\theta_1)})(h)| \\
&= |h(y_{(x,\theta)}) + e^{i\psi(x,\theta)}h'(y_{(x,\theta)}) - h(y_{(x,\theta_1)}) - e^{i\psi(x,\theta_1)}h'(y_{(x,\theta_1)})| \\
&\geq \big| |h(y_{(x,\theta)}) + e^{i\psi(x,\theta)}h'(y_{(x,\theta)})| - |h(y_{(x,\theta_1)}) + e^{i\psi(x,\theta_1)}h'(y_{(x,\theta_1)})| \big| \\
&\geq \big| \|h\| - \frac{3}{2} + \frac{1}{2}d^2(y_{(x,\theta)}, y_{(x,\theta_1)}) + \frac{3}{2}|d(y_{(x,\theta)}, y_{(x,\theta_1)})| \big| \\
&= \left| \frac{1}{2}d^2(y_{(x,\theta)}, y_{(x,\theta_1)}) + \frac{3}{2}|d(y_{(x,\theta)}, y_{(x,\theta_1)})| \right|
\end{aligned}$$

for sufficiently small $|\theta - \theta_1|$. Therefore ρ is continuous. Hence the image of $[-\pi, \pi]$ in $[0,1]_\kappa$ is connected subset of $[0,1]_\kappa$. It is in fact singleton. For otherwise we could find g in A such that $g \equiv g' \equiv 0$ on an open interval $I \in \rho([-\pi,\pi])$ while for some $y_{(x,\phi)} \notin [0,1]_\kappa$,

$$|g(y_{(x,\phi)})| < |g'(y_{(x,\phi)})|.$$

For instance, one may take

$$g(y) = \begin{cases} 0 & y \le y_1 \\ (y-y_1)^2 & y_1 \le y \le \dfrac{1+3y_1}{4} \\ -\left(y - \dfrac{1+y_1}{2}\right)^2 & \dfrac{1+3y_1}{4} \le y \le \dfrac{3+y_1}{4} \\ (y-1)^2 & \dfrac{3+y_1}{4} \le y \le 1 \end{cases}$$

where y_1 is least upper bound of I and $y_{(x,\phi)}$ sufficiently near to y_1. Thsu for an infinite number of $\theta \in [-\pi,\pi]$ with $y_{(x,\theta)} \in I$,

$$L_{(x,\theta)}(T(g)) = T^*(L_{(x,\theta)})(g)| = e^{i\lambda} L_{(y(x,\theta),\psi(x,\theta))}(g) = 0$$

while

$$L_{(x,\phi)}(T(g)) = e^{i\lambda} L_{(y(x,\phi),\psi(x,\phi))}(g) \ne 0.$$

Since ρ is continuous, $\rho^{-1}(I)$ is in $[-\pi,\pi]$ and therefore there exist an infinite number of θ's such that

$$(2) \qquad L_{(x,\theta)}(T(g)) = 0 \quad \text{while} \quad L_{(x,\phi)}(T(g)) \ne 0.$$

Therefore $(T(g))(x) + e^{i\theta}(T(g))'(x) = 0$.

Varying θ we can see that $(T(g))'(x) = 0$. Thus $L_{(x,\phi)}(T(g)) = 0$ which contradict (2).

Hence $y_{(x,\theta)} = y_{(x,0)}$ for all $\theta \in [-\pi,\pi]$.

Finally we define a point map τ of $[0,1]_\kappa$ to $[0,1]_\kappa$ by

$$\tau(x) = y_{(x,0)}.$$

Consideration of $(T^{-1})^*$ shows that τ is onto, and, applying Lemma 6, one-one.

THEOREM 7. *Let T be an isometry of A. Then, for $f \in A$,*

$$(T(f))(x) = e^{i\lambda} f(\tau(x))$$

with $e^{i\lambda} = T(1)$. Moreover, τ is one of the functions F, $1-F$ where F is the mapping of $[0,1]_\kappa$ onto $[0,1]_\kappa$ defined by $F(x) = x + c - [x+c]$, $x \in [0,1]_\kappa$, $c \in [0,1]_\kappa$.

PROOF: Given $x \in [0,1]_\kappa$ and $\theta \in [-\pi, \pi]$, consider the function g of the Proposition 2 constructed for (x, θ). Clearly, g does not depend on θ; $g(x) = 1$; $g'(x)$ is positive real and $g'(x) > |g'(y)|$ for all $y \in [0,1]_\kappa$, $y \neq x$. Therefore,

$$\begin{aligned} \|g\| &= g'(x) \\ &= e^{i\theta} L_{(x,\theta)}(g) \\ &= e^{i\theta} T^* L_{(x,\theta)}(T^{-1}(g)) \\ &= e^{i(\lambda-\theta)} L_{(\tau(x),\psi_{(x,\theta)})}(T^{-1}(g)). \end{aligned}$$

Thus we have for all $\theta \in [-\pi, \pi]$

$$\|g\| = e^{i(\lambda-\theta)}(T^{-1}(g))(\tau(x)) + e^{i\psi_{(x,\theta)}}(T^{-1}(g))'(\tau(x)). \tag{3}$$

Since

$$\|g\| = \|T^{-1}(g)\| = \sup_{y \in [0,1]_\kappa} |(T^{-1}(g))(y)|$$

by (3) we have

$$\|g\| = |(T^{-1}(g))(\tau(x))| + |(T^{-1}(g))'(\tau(x))|.$$

Again since g is independent of θ,

$$(T^{-1}(g))(\tau(x)), (T^{-1}(g))'(\tau(x))$$

are independent of θ but

$$A(\theta) = \{e^{\psi_{(x,\theta)}}(T^{-1}(g))'(\tau(x))\}$$

depend on θ for otherwise (3) cannot be true. In other words, $A(\theta)$ is not constant. Now by (3) $A(\theta)$ must be on a circle with center as $\{(T^{-1}(g))(\tau(x))\}$ and radius equal to $\|g\|$.

On the other hand $A(\theta)$ must be on or within the circle with center as origin and radius equal to $\rho = |(T^{-1}(g))'(\tau(x))| = \|g\| - |(T^{-1}(g))(\tau(x))|$. This implies that $(T^{-1}(g))(\tau(x)) = 0$ for otherwise $A(\theta)$ has to be a constant which is false.

Also by (3)

$$\|g\| = e^{i(\lambda-\theta+\psi_{(x,\theta)})}(T^{-1}(g))(\tau(x)).$$

Since the left hand side is independent of θ, we have

$$\lambda - \theta + \psi_{(x,\theta)} = \lambda + \psi_{(x,0)}.$$

Hence for all $\theta \in [-\pi, \pi]$,

$$\psi_{(x,\theta)} = \psi_{(x,0)} + \theta.$$

Now f be any element of A such that $f(x) = 0$ then for all $\theta \in [-\pi, \pi]$

$$\begin{aligned} f'(x) &= e^{i\theta}L_{(x,\theta)}(f) \\ &= e^{i\theta}T^{*}L_{(x,\theta)}(T^{-1}(f)) \\ &= e^{i(\lambda-\theta)}L_{(\tau(x),\psi_{(x,\theta)})}(T^{-1}(f)) \\ &= e^{i(\lambda-\theta)}[(T^{-1}(f))(\tau(x)) + e^{i\psi_{(x,\theta)}}(T^{-1}(f))'(\tau(x))] \\ &= e^{i\lambda}[e^{-i\theta}(T^{-1}(f))(\tau(x)) + e^{i\psi_{(x,0)}}(T^{-1}(f))'(\tau(x))] \end{aligned}$$

so that $(T^{-1}(f))(\tau(x)) = 0$. For an arbitrary $f \in A$, define $g(y) = f(y) - f(x)$, $y \in [0,1]$ then $g(x) = 0$ and so

$$\begin{aligned} 0 = (T^{-1}(g))(\tau(x)) &= (T^{-1}(f))(\tau(x)) - f(x)(T^{-1}(1))(\tau(x)) \\ &= (T^{-1}(f))(\tau(x)) - e^{-i\lambda}f(x). \end{aligned}$$

Thus, replacing f by $T(f)$, it follows that for all $x \in [0,1]_K$ and $f \in C^{(1)}[0,1]_K$,

$$(T(f))(x) = e^{i\lambda}f(\tau(x)).$$

Now F_0 is the mapping of $[0,1]_\kappa$ onto itself given by

$$F_0(y) = \begin{cases} 2\left(2y-\tau(x)+\frac{1}{2}\right)^3 - \left(2y-\tau(x)+\frac{1}{2}\right) \\ \qquad + \dfrac{2\tau(x)+1}{4} & \cdots 0 \le y \le \frac{\tau(x)}{2} \\ y & \cdots \frac{\tau(x)}{2} \le y \le \frac{\tau(x)+}{2} \\ 2\left(2y-\tau(x)-\frac{3}{2}\right)^3 - \left(2y-\tau(x)-\frac{3}{2}\right) \\ \qquad + \dfrac{2\tau(x)+1}{4} & \cdots \frac{\tau(x)+1}{2} \le y \le 1. \end{cases}$$

Therefore

$$\begin{aligned}(T(F_0))(x) + (T(F_0))'(x) &= L_{(x,0)}(T(F_0)) \\ &= T^* L_{(x,0)}(F_0) \\ &= e^{i\lambda} L_{(\tau(x),\psi_{(x,0)})}(F_0) \\ &= e^{i\lambda}[F_0(\tau(x)) + e^{i\psi_{(x,0)}}(F_0)'(\tau(x))] \\ &= e^{i\lambda}[\tau(x) + e^{i\psi_{(x,0)}}].\end{aligned}$$

therefore

$$\begin{aligned}(T(F_0))'(x) &= e^{i(\lambda+\psi_{(x,0)})}, \\ \tau(x)' &= e^{i\psi_{(x,0)}}.\end{aligned}$$

But since $\tau(x)'$ is real valued and $\tau(x)$ is one to one we have $\tau(x)' \equiv 1$ or $\tau(x)' \equiv -1$ and, therefore $\tau(x) = x + c - [x+c]$ or $\tau(x) = -x + c - [x+c]$.

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