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Convexity of the Lagrangian for Set Functions

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ABSTRACT. We consider perturbation problems and Lagrangians for convex set function optimization problems. In particular, we prove that the Lagrangian $L(\Omega, y)$ is a convex set function in Ω for each y if the perturbation function is convex.

1. Introduction

To establish the setting for convex set function optimization, consider the following problem :

$$\inf F(\Omega) \quad \text{over } \Omega \in C$$

where C is a convex subfamily of a σ-algebra Σ, $F : \Sigma \to \overline{R} = R \cup \{+\infty\}$ is a convex set function. If F is redefined so that $F(\Omega) = +\infty$ for $\Omega \notin C$, then $\inf F(\Omega)$ over Σ is equivalent to the infimum of the new F over all of Σ. Thus, no generality is lost in our model if we restrict attention to the case $C = \Sigma$. In the next section, we consider perturbation problems and Lagrangians for convex set function optimization problems.

2. Perturbation Problems and Lagrangians

Let F be a set function of Σ into $\overline{R}$, and consider the minimization problem

$$\text{(P)} \qquad \inf_{\Omega \in \Sigma} F(\Omega).$$

Consider a family of perturbations of problem (P) obtained as follows : let U be an arbitrary linear space. Let K be a function on $\Sigma \times U$ into R such that $K(\Omega, 0) = F(\Omega)$; then we have an expression of F

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as the "envelope" of a collection of functions $\Omega \to K(\Omega, u)$, $u \in U$. For every $u \in U$, we shall consider the minimization problem :

$$\text{(Pu)} \qquad \inf_{\Omega \in \Sigma} K(\Omega, u).$$

Clearly, for $u = 0$, (Po) is none other than problem (P). The problems (Pu) will be said to be perturbation problems related to (K). To define the Lagrangian of the perturbation function K, let $U^* = Y$ be the dual space of U, with the bilinear pairing $\langle \ , \ \rangle$.

DEFINITION 1: Let $K : \Sigma \times U \to \overline{R}$ be a perturbation function. Let Y be the dual space of U with bilinear pairing $\langle \ , \ \rangle$. Then, the function $L : \Sigma \times Y \to \overline{R}$,

$$L(\Omega, y) = \inf\{K(\Omega, u) + \langle u, y \rangle : u \in U\}$$

is said to be the *Lagrangian* of the perturbation function K.

DEFINITION 2: [**6**] Let U be an arbitrary linear space and Y be its dual space. For a convex function $f : U \to \overline{R}$, the conjugate of f is the function $f^* : Y \to \overline{R}$ defined by

$$f^*(y) = \sup\{\langle u, y \rangle - f(u) : u \in U\}.$$

Similarly, the conjugate of a concave function $g : U \to \overline{R}$ is defined by

$$g^*(y) = \inf\{\langle u, y \rangle - g(u) : u \in U\}.$$

It is well known [**6**], [**7**] that if f is a convex (or concave) function, then the conjugate function f^* is a closed convex (or concave) function. Moreover, $(\mathrm{cl}\, f)^* = f^*$ and $f^{*}{*} = \mathrm{cl}\, f$, where $\mathrm{cl}\, f$ means the closure of f.

THEOREM 1. *Let Σ be a σ-algebra and U be any linear space with $U^* = Y$. Let $K : \Sigma \times U \to \overline{R}$ be a perturbation function of $F : \Sigma \to \overline{R}$. For a fixed $\Omega \in \Sigma$, define two functions $f : Y \to \overline{R}$ and $h : U \to \overline{R}$ by*

$$g(y) = L(\Omega, y) \quad \textit{and} \quad h(u) = -K(\Omega, u).$$

If $K(\Omega, \cdot)$ is a closed and convex function of $u \in U$ for each $\Omega \in \Sigma$, then the following hold :

(i) *h is closed and concave.*
(ii) *$h^*(y) = g(y)$ for each $y \in Y$.*
(iii) *g is concave.*
(iv) *$h = g^*$.*
(v) *$K(\Omega, u) = \sup\{L(\Omega, y) - \langle u, y\rangle : y \in Y\}$.*
(vi) *$F(\Omega) = \sup\{L(\Omega, y) : y \in Y\}$.*

PROOF: (i), (iii) are straightfoward.
(ii)

$$\begin{aligned} h^*(y) &= \inf\{\langle y, u\rangle - h(u) : u \in U\} \\ &= \inf\{\langle y, u\rangle + K(\Omega, u) : u \in U\} \\ &= L(\Omega, y) = g(y). \end{aligned}$$

(iv) By (ii), $(h^*)^* = g^*$. Since h is closed, $(h^*)^* = h$.
(v) $g^*(u) = \inf\{\langle u, y\rangle - L(\Omega, y) : y \in Y\}$. So, by (iv) and the definition of h, the result is obtained.
(vi)

$$\begin{aligned} F(\Omega) = K(\Omega, 0) &= \sup\{L(\Omega, y) - \langle 0, y\rangle : y \in Y\} \\ &= \sup\{L(\Omega, y) : y \in Y\}. \end{aligned}$$

THEOREM 2. *The Lagrangian $L(\Omega, y)$ is concave in y for each $\Omega \in \Sigma$, and if the perturbation function K is convex on $\Sigma \times U$, then $L(\Omega, y)$ is a convex set function in Ω for each $y \in Y$.*

PROOF: The first part is clear by the definition of the Lagrangian. To prove the second part, let λ, Ω, Λ be given and let $\{\Gamma_n\}$ be a Morris-sequence associated with $\langle \lambda, \Omega, \Lambda\rangle$. For any given $\varepsilon > 0$, there exist u^*, $v^* \in U$ such that

$$\inf\{K(\Omega, u) + \langle u, y\rangle : u \in U\} \geq K(\Omega, u^*) + \langle u^*, y\rangle - \varepsilon$$

and

$$\inf\{K(\Lambda, u) + \langle u, y\rangle : u \in U\} \geq K(\Lambda, v^*) + \langle v^*, y\rangle - \varepsilon.$$

Let $K(\Omega, u^*) = r$, $K(\Lambda, v^*) = s$. Then, $(\Omega, u^*, r), (\Lambda, v^*, s) \in [K\,;\Sigma \times U]$. Since K is convex on $\Sigma \times U$, there exist a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ and a sequence $\{t_k\}$ with $t_k \to \lambda r + (1-\lambda)s$ such that

$$K(\Gamma_{n_k}, \lambda u^* + (1-\lambda)v^*) \le t_k.$$

Since $t_k \to \lambda r + (1-\lambda)s$, for given $\varepsilon > 0$, we have (if necessary, by taking a subsequence),

$$K(\Gamma_{n_k}, \lambda u^* + (1-\lambda)v^*) \le \lambda r + (1-\lambda)s, \quad k = 1, 2, \dots .$$

Hence,

$$\begin{aligned} &K(\Gamma_{n_k}, \lambda u^* + (1-\lambda)v^*) + \langle \lambda u^* + (1-\lambda)v^*, y\rangle \\ &\le \lambda K(\Omega, u^*) + (1-\lambda)K(\Lambda, v^*) + \lambda\langle u^*, y\rangle + (1-\lambda)\langle v^*, y\rangle + \varepsilon \\ &\le \lambda\{\inf_{u\in U}\{K(\Omega, u) + \langle u, y\rangle\} + \varepsilon\} \\ &\qquad + (1-\lambda)\{\inf_{u\in U}\{K(\Lambda, v) + \langle u, y\rangle\} + \varepsilon\} + \varepsilon \\ &= \lambda L(\Omega, y) + (1-\lambda)L(\Lambda, y) + 2\varepsilon. \end{aligned}$$

Also,

$$\begin{aligned} L(\Gamma_{n_k}, y) &= \inf\{K(\Gamma_{n_k}, u) + \langle u, y\rangle : u \in U\} \\ &\le K(\Gamma_{n_k}, \lambda u^* + (1-\lambda)v^*) + \langle \lambda u^* + (1-\lambda)v^*, y\rangle. \end{aligned}$$

Combining the above two inequalities, we have

$$L(\Gamma_{n_k}, y) \le \lambda L(\Omega, y) + (1-\lambda)L(\Lambda, y) + 2\varepsilon, \qquad k = 1, 2, \dots .$$

Since $\varepsilon > 0$ is arbitrary, this gives a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that

$$\liminf_{k\to\infty} L(\Gamma_{n_k}, y) \le \lambda L(\Omega, y) + (1-\lambda)L(\Lambda, y).$$

Therefore, the proof is complete.

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