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Some Results of Generalized Homomorphisms on Banach Algebras

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ABSTRACT. In the paper of Johnson[**4**], the results about continuity of homomorphisms were applied to those of generalized homomorphisms. This paper presents the results helpful to prove the continuity of generalized homomorphisms on Banach algebra and the continuity of generalized homomorphisms on it.

Throughout the paper all algebras and vector spaces are over $\mathbf{C}$. Usually $\mathfrak{A}$ and $\mathfrak{B}$ will be Banach algebras and T be a linear mapping from $\mathfrak{A}$ into $\mathfrak{B}$. We make great use of the *separating space* $\mathfrak{S}$.

$$\mathfrak{S} = \{s \in \mathfrak{B} : \text{there is a sequence } a_n \text{ in } \mathfrak{A} \text{ with } a_n \to 0 \text{ and } T(a_n) \to S\}.$$

We put $\mathfrak{S}^2$ for the *bilateral annihilator* of $\mathfrak{S}$, that is

$$\mathfrak{S}^2 = \{b : b \in \mathfrak{B}, bs = 0 = sb \text{ for all } s \in \mathfrak{S}\}$$

The following results are discussed in [**6**].

Let T be a linear operator from a Banach space $\mathfrak{A}$ into a Banach space $\mathfrak{B}$. Then (i) $\mathfrak{S}(T)$ is a closed linear subspace of $\mathfrak{B}$, (ii) T is continuous iff $\mathfrak{S}(T) = 0$, and (iii) if Q and R are continuous linear operators on $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and if $TQ = RT$, then $Q\mathfrak{S}(T) \subset \mathfrak{S}(T)$. Let R be a continuous linear mapping from $\mathfrak{B}$ into a Banach space $\mathfrak{C}$. Then (i) RT is continuous iff $R\mathfrak{S}(T) = \{0\}$, (ii) $(R\mathfrak{S}(T))^- = \mathfrak{S}(RT)$, and (iii) there is a constant M (independent of T and R) such that if RT is continuous, then $\|RT\| \leq M\|R\|$.

Let $\mathfrak{A}$ be a Banach algebra with unit and let $\mathfrak{B}$ be a left $\mathfrak{A}$-module. We shall regard the module action as implemented by a homomorphism ρ from $\mathfrak{A}$ to the algebra $B(\mathfrak{B})$ of bounded operators on $\mathfrak{B}$.

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The homomorphism ρ is not assumed to be continuous and the module action is commutative, that is, $\rho(ab)m = \rho(ba)m$, a, $b \in \mathfrak{A}$, $m \in \mathfrak{B}$.

Separating space $\mathfrak{S}$ is a submodule of $\mathfrak{B}$, and

$$I = I(T) = \{a \in \mathfrak{A} : \rho(a)\mathfrak{S}(T) = 0\}$$

is an ideal of $\mathfrak{A}$, which is called the *continuity ideal* for the operator T.

DEFINITION 1: Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebra and let T be a linear map of $\mathfrak{A}$ into $\mathfrak{B}$. Define the bilinear map $\hat{T}$ by $\hat{T}(a,b) = T(ab) - T(a)T(b)$ $(a,\ b \in \mathfrak{A})$. We say that T is a *generalized homomorphism* if $\hat{T}$ is continuous.

The following definition is due to Cusack [**3**].

DEFINITION 2: A closed ideal I of a Banach algebra $\mathfrak{A}$ is called a *separating ideal* if, for every sequence $\{a_n\}$ in $\mathfrak{A}$, there exists a natural number N such that

$$(Ia_n \ldots a_1)^- = (Ia_N \ldots a_1)^- \qquad (n \geq N).$$

The following lemma is due to Johnson [**4**].

LEMMA 3. *The space $\mathfrak{S}$ is a closed bi-ideal in the closed subalgebra $\mathfrak{B}_0$ generated by the range of T. The space $\mathfrak{S}^\perp$ contains all elements of the from $T(ab) - T(a)T(b)$. Also $\mathfrak{S}^\perp \cap \mathfrak{B}_0$ is a closed bi-ideal in $\mathfrak{B}_0$ and qT is a homomorphism, where q is the quotient map $\mathfrak{B}_0 \to \mathfrak{B}_0/(\mathfrak{S}^\perp \cap \mathfrak{B}_0)$.*

LEMMA 4. *If $T : \mathfrak{A} \to \mathfrak{B}$ is a generalized homomorphism where $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras, then I is an ideal in $\mathfrak{A}$ containing the kernel of ρ. The ideal I is closed if ρ is bounded from $\mathfrak{A}$ to $B(\mathfrak{B})$. Furthermore*

$$\begin{aligned} I &= \{a \in \mathfrak{A} \mid b \to T(ab) \text{ is bounded}\} \\ &= \{a \in \mathfrak{A} \mid b \to \rho(a)T(b) \text{ is bounded}\}. \end{aligned}$$

PROOF: Note that for each a, b in $\mathfrak{A}$, $\hat{T}(a,b)\mathfrak{S} = (T(ab) - T(a)T(b))\mathfrak{S} = \{0\}$ by Lemma 3. Since $\rho(ab)\mathfrak{S} = T(ab)\mathfrak{S} = \hat{T}(a,b)\mathfrak{S} + T(a)T(b)\mathfrak{S} = \rho(a)\rho(b)\mathfrak{S}$, I is an ideal in $\mathfrak{A}$. Since $\rho(a)T$ is bounded if and only if $\rho(a)\mathfrak{S} = \{0\}$, and since $L(a,b)$ is always bounded in b for each fixed a, $T(ab)$ is bounded in b if and only if $\rho(a)T(b)$ is bounded and so we are done.

LEMMA 5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. If $T : \mathfrak{A} \to \mathfrak{B}$ is generalized homomorphism with associated continuity ideal I which is closed, then there exists a constant M such that for each a, $b \in I$*

$$\|T(ab)\| \le M\|a\|\,\|b\|.$$

PROOF: If $a \in I$, then by Lemma 4, there exists a constant M_a such that

$$\|T(ab)\| \le M_a\|b\|, \qquad b \in \mathfrak{A}.$$

If in addition $b \in I$, then since $T(ab) = T(ba)$, there exists a constant M_b such that

$$\|T(ab)\| \le M_b\|a\|, \qquad a \in I.$$

Since I is closed, then a standard application of the uniform boundedness principle asserts the existence of a constant M satisfying

$$\|T(ab)\| \le M\|a\|\,\|b\|, \qquad a, b \in I.$$

THEOREM 6. *Let $T : \mathfrak{A} \to \mathfrak{B}$ be a generalized homomorphism where $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras. Assume that the continuity ideal I is closed. Then I^2 is closed and T is bounded on I^2 (and also on I) if*

(a) *I has a bounded approximate identity or*
(b) *I^2 is finitely generated in I and has finite codimension in I.*

PROOF: If I has a bounded approximate identity, then by the Cohen Factorization Theorem, $I^2 = I$, and furthermore if $\{z_n\} \subset I$ and $z_n \to 0$, then $z_n = ab_n$; a, $b_n \in I$ and $b_n \to 0$. The boundedness of T on $I^2 = I$ now follows by Lemma 5. If $I^2 = \sum_{i=1}^{n} a_i I$, $a_i \in I$, and if there exists a finite dimensional subspace E of I such that $I = I^2 \oplus E$, then an application of the open mapping theorem yields that I^2 is closed. Again by the open mapping theorem, if $u \in I$, we may choose $u_i \in I$ satisfying $\|u_i\| \le K\|u\|$ and $u = \sum_{i=1}^{n} a_i u_i$. By Lemma 5,

$$\|T(u)\| \le MK\|u\| \sum_{i=1}^{n} \|a_i\|.$$

Since I^2 has finite codimension in I, T is bounded on I.

The following lemma is due to Loy [**5**].

LEMMA 7. *Let $\mathfrak{A}$ be a separable Banach algebra such that $\mathfrak{A}^2$ is closed in $\mathfrak{A}$. Then there is a constant K and an integer m such that if $a \in \mathfrak{A}^2$ there exist a_i, $b_i \in \mathfrak{A}$, $1 \leq i \leq m$, satisfying*

$$\text{(i)} \quad a = \sum_{i=1}^{m} a_i b_i \qquad\qquad \text{(ii)} \quad \sum_{i=1}^{m} \|a_i\| \, \|b\| \leq K\|a\|.$$

THEOREM 8. *Let $T : \mathfrak{A} \to \mathfrak{B}$ be a generalized homomorphism, where $\mathfrak{A}$ is a separable Banach algebra. Then T is continuous on I^2 if I and I^2 are closed.*

PROOF: By Lemma 7, there exists a constant K such that for $z \in I^2$

$$\inf\{\sum_{i=1}^{n} \|a_i\| \, \|b_i\| : \sum_{i=1}^{n} a_i b_i = z; a_i, b_i \in I\} \leq K\|z\|.$$

By Lemma 5, if $z = \sum_{i=1}^{n} a_i b_i$, a_i, $b_i \in I$, then $\|T(z)\| \leq \sum_{i=1}^{n} \|T(a_i b_i)\| \leq \sum_{i=1}^{n} M\|a_i\| \, \|b_i\|$ if I is closed. Therefore $\|T(z)\| \leq MK\|z\|$.

THEOREM 9. *Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{B}$ a semisimple Banach algebra. Let T be a generalized homomorphism from $\mathfrak{A}$ onto a dense subalgebra of $\mathfrak{B}$ such that spectrum $\sigma(a) = 0$ for each $a \in \mathfrak{S}(T)$. Then T is continuous.*

PROOF: Let $a \in \mathfrak{S}(T)$. Since $\mathfrak{S}(T)$ is a bi-ideal of $\mathfrak{B}_0$ generated by the range of T, then $ba \in \mathfrak{S}(T)$ for each b in $\mathfrak{B}_0$. By the hypothesis $\sigma(ba) = 0$, so the spectral radius of ba is zero. Hence $ba \in q-\text{Inv}(\mathfrak{B}_0)$ and so $a \in \text{rad}(\mathfrak{B}_0)$. Since $\mathfrak{B}$ is a semisimple, $a = 0$. Hence $\mathfrak{S}(T) = 0$. Then the theorem holds.

REMARK: If the homomorphism from a Banach algebra $\mathfrak{A}$ into a semisimple Banach algebra $\mathfrak{B}$ is continuous, then every generalized homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ is continuous.

PROOF: Consider that $\mathfrak{B}$ is primitive. Suppose $\mathfrak{B}$ acts irreducibly on the Banach space $\mathfrak{A}$. If s, t are nonzero elements of $\mathfrak{S}$ and $\mathfrak{S}^{\perp}$ respectively, let ξ, $\eta \in \mathfrak{A}$ with $s\xi \neq 0$ and $t\eta \neq 0$. Let $b \in \mathfrak{B}$ with

$bt\eta = \xi$. Then $sbt = 0$ but $sbt\eta \neq 0$. Thus either $\mathfrak{S}^{\perp} = \{0\}$ or $\mathfrak{S} = \{0\}$. In the first case, by Lemma 3, T is a homomorphism. By the assumption T is continuous. In the second case, T is clearly continuous. Let $\mathfrak{B}$ be any semisimple Banach algebra, and let P be a primitive ideal of $\mathfrak{B}$. We define the quotient map $q : \mathfrak{B} \to \mathfrak{B}/P$, then $\mathfrak{B}/P$ is primitive. Hence the above proof requires that qT is continuous. We have $0 = \mathfrak{S}(qT) = (q\mathfrak{S}(T))^{-}$, so $\mathfrak{S}(T) \subset P$. Since $\mathfrak{B}$ is semisimple, $\mathfrak{S}(T) = 0$. Therefore T is continuous.

The next result which is due to Thomas [7] is called the stability lemma.

LEMMA 10. *Let X and Y be Banach spaces, S a linear transformation from X to Y with separating space $\mathfrak{S}$. Let $\{T_n\}$, $\{R_n\}$ be sequences of bounded operators in X and Y respectively. Then $\mathfrak{S}(ST_1 \ldots T_{n+1}) \subseteq \mathfrak{S}(ST_1 \ldots T_1)$ for each integer N. Furthermore there exists an integer N such that for all $n \geq N$*

$$\mathfrak{S}(ST_1 \ldots T_1) = \mathfrak{S}(ST_1 \ldots T_N).$$

If for each n, $ST_n - R_nS$ is a bounded operator from X to Y, then

$$(R_1 \ldots R_n\mathfrak{s})^{-} = (R_1 \ldots R_N\mathfrak{S})^{-}, \qquad n \geq N.$$

THEOREM 11. *Let $\mathfrak{A}$ be a Banach algebra and let $\mathfrak{B}$ be a Banach algebra which does not have a nonzero separating ideal. Then every generalized homomorphism T from $\mathfrak{A}$ into $\mathfrak{B}$ is continuous.*

PROOF: Let $T : \mathfrak{A} \to \mathfrak{B}$ be a generalized homomorphism. By Lemma 8, $\mathfrak{S}(T) = \mathfrak{S}$ is closed bi-ideal in closed subalgebra $\mathfrak{B}_0$ generated by the range of T. For every sequence $\{b_n\}$ in $\mathfrak{B}$, let $T_n(a) = aa_n$ and $R_n(b) = bb_n$ where $T(a_n) = b_n$ for each n ($a \in \mathfrak{A}$, $b \in \mathfrak{B}$). Then $T_n : \mathfrak{A} \to \mathfrak{A}$ and $R_n : \mathfrak{B} \to \mathfrak{B}$ are continuous linear operators. Since

$$\begin{aligned}(TT_n - R_nT)(x) &= T(xa_n) - T(x)b_n \\ &= \hat{T}(x, a_n) + T(x)T(a_n) - T(x)b_n \\ &= \hat{T}(x, a_n) \text{ for each } n,\end{aligned}$$

$TT_n - R_nT$ is continuous for each n. By Lemma 10, there exists N such that for all $n \geq N$,

$$(R_1 \ldots R_n\mathfrak{S})^{-} = (R_1 \ldots R_N\mathfrak{S})^{-} \qquad (n \geq N).$$

Since $R_1 \ldots R_n\mathfrak{S}(T) = \mathfrak{S}(T)b_n \ldots b_1$, $\mathfrak{S}(T)$ is a separating ideal, hence T is continuous.

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