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A Property of Borel Subsets of Wiener Space*

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ABSTRACT. Wiener measure $m(\lambda B)$ can behave arbitrarily badly as a function of λ for Wiener measurable sets B. We show however that $m(\lambda B)$ is Borel measurable with respect to λ for any Borel subset B of $C_0[0,1]$.

1. Introduction

Let $C_0[0,1]$ denote the space of continuous functions on $[0,1]$ which vanish at 0. The space $C_0[0,1]$ equipped with Wiener measure m will be called Wiener space. Let $0 = t_0 < t_1 < \cdots < t_n \leq 1$, and let $-\infty \leq p_i \leq q_i \leq +\infty$ for $i = 1, 2, \ldots, n$; then the set

$$\{x \text{ in } C_0[0,1] \mid p_i \leq x(t_i) \leq q_i,\ i = 1, 2, \ldots, n\} \tag{1}$$

will be called an interval in $C_0[0,1]$. The smallest σ-algebra of sets containing the class of all intervals is the class of all Borel subsets of $C_0[0,1]$. We review some facts concerning scaling in Wiener space (see [**2**]).

Given a nested sequence of partitions $P_n : 0 = t_0 < t_1^{(n)} < t_2^{(n)} < \cdots < t_{k_n}^{(n)} = 1$ whose norm approaches zero and x in $C_0[0,1]$, let

$$S_{P_n}(x) = \sum_{j=1}^{k_n} \{x(t_j^{(n)}) - x(t_{j-1}^{(n)})\}^2. \tag{2}$$

For $\lambda \geq 0$, let $C_\lambda \equiv \{x \text{ in } C_0[0,1] \mid \lim_{n\to\infty} S_{P_n}(x) = \lambda^2\}$ and let $D \equiv \{x \text{ in } C_0[0,1] \mid \lim_{n\to\infty} S_{P_n}(x) \text{ fails to exists}\}$. Note that (i) $\lambda C_\mu = C_{\lambda\mu}$, (ii) D and C_λ, $\lambda \geq 0$, are all Borel subsets, (iii) $C_0[0,1]$ is the disjoint union of this family of sets and (iv) $m(C_1) = 1$ [**2**]. A

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subset E of $C_0[0,1]$ is said to be scale-invariant measurable provided λE is Wiener measurable for all $\lambda > 0$. Note that every Borel subset of $C_0[0,1]$ is scale-invariant measurable.

Wiener measure and Wiener measurability behave badly with respect to scaling. Let $f : (0,+\infty) \to [0,1]$ be an arbitrary (for example, nonmeasurable) function. It has long been known [**1**, **2**] that there exists a subset E of $C_0[0,1]$ such that λE is Wiener measurable for $\lambda > 0$ but

$$m(\lambda E) = f(\lambda). \tag{3}$$

Further, let $B = \bigcup_{\lambda \in Q} C_\lambda$ where Q is the set of all rationals. Then B is a Borel subset of $C_0[0,1]$ and

$$m(\lambda B) = \chi_Q(\lambda) \tag{4}$$

where $\chi_Q(\lambda)$ is the characteristic function of Q. Thus there exists a Borel subset B of $C_0[0,1]$ such that $m(\lambda B)$ is discontinuous everywhere as a function of λ.

In spite of the two negative results just discussed, we show that for every Borel subset B of $C_0[0,1]$, $m(\lambda B)$ is Borel measurable a function of λ.

2. Theorem

Now, we give the main result of this paper.

THEOREM. *For any Borel subset B of $C_0[0,1]$, $m(\lambda B)$ is a Borel measurable function of λ.*

PROOF: For a subset E of $\mathbf{R}^n$, we let $\underset{\sim}{E} = \bigcup_{\lambda>0}\{(\lambda, \lambda e_1, \lambda e_2, \ldots, \lambda e_n) \mid (e_1, e_2, \ldots, e_n) \text{ is in } E\} \subset (0,+\infty) \times \mathbf{R}^n$. Let $P = \{E \subset \mathbf{R}^n \mid \underset{\sim}{E} \text{ is a Borel subset of } (0,+\infty) \times \mathbf{R}^n\}$. It is easily seen that P is a σ-algebra and that every open subset of $\mathbf{R}^n$ is in P. Since every section of a Borel set is a Borel set, we have

(5) E is a Borel measurable subset of $\mathbf{R}^n$ if and only if $\underset{\sim}{E}$ is a Borel measurable subset of $(0,+\infty) \times \mathbf{R}^n$.

For a subset K of $C_0[0,1]$, we let $\widetilde{K} = \bigcup_{\lambda>0}\{(\lambda, \lambda x) \mid x \text{ is in } K\} \subset (0,+\infty) \times C_0[0,1]$. Let $S = \{K \subset C_0[0,1] \mid \widetilde{K}$ is a Borel subset of $(0, +\infty) \times C_0[0,1]\}$. Q is a σ-algebra as is easily shown. Given $0 < t_1 < t_2 < \cdots < t_n \leq 1$, let J_t be the function from $C_0[0,1]$ into $\mathbf{R}^n$ defined by $J_t(x) = (x(t_1), x(t_2), \ldots, x(t_n))$ and let F be the function from $(0,+\infty) \times C_0[0,1]$ into $(0,+\infty) \times \mathbf{R}^n$ defined by $F(\lambda, x) = (\lambda, x(t_1), x(t_2), \ldots, x(t_n))$. Then F is continuous and for any subset E of $\mathbf{R}^n$,

(6)
$$
\begin{aligned}
&F^{-1}(\underset{\sim}{E}) \\
&= \bigcup_{\lambda>0} F^{-1}(\{(\lambda, \lambda e_1, \lambda e_2, \ldots, \lambda e_n) \mid (e_1, e_2, \ldots, e_n) \text{ is in } E\}) \\
&= \bigcup_{\lambda>0} \{(\lambda, \lambda x) \mid J_t(x) \text{ is in } E\} = \widetilde{J_t^{-1}}(E).
\end{aligned}
$$

Hence, if E is a Borel subset, then by (5), $\underset{\sim}{E}$ is a Borel subset. Then since F is continuous, $F^{-1}(\underset{\sim}{E})$ is a Borel subset, and so, by (6), $\widetilde{J_t^{-1}(E)}$ is a Borel subset; that is, $J_t^{-1}(E)$ is in S. Thus S is a σ-algebra containing all intervals. Since every section of a Borel set is a Borel set, we have

(7) $\qquad B$ is a Borel subset of $C_0[0,1]$ if and only if
$\qquad \widetilde{B}$ is a Borel subset of $(0,+\infty) \times C_0[0,1]$.

Now, let B is a Borel subset of $C_0[0,1]$. By (7), $\widetilde{B}$ is a Borel subset of $(0,+\infty) \times C_0[0,1]$ and for all $\lambda > 0$, $(\widetilde{B})^{(\lambda)} = \lambda B$. From the first part of the Fubini theorem, $m(\lambda B) = m((\widetilde{B})^{(\lambda)})$ is a Borel measurable function of λ. Thus, we have proved the theorem.

PROPOSITION. *There exists a scale-invariant and non-Borel measurable subset V of $C_0[0,1]$ such that $m(\lambda V) = 1/2$ for every $\lambda > 0$. Thus $m(\lambda V)$ is Borel measurable as a function of λ although V is not Borel subset of $C_0[0,1]$.*

PROOF: Let A be a non-Borel subset of $(0,+\infty)$ and let J be a functional on $C_0[0,1]$ with $J(x) = x(1)$. Let $V = \{(\bigcup_{\mu\in A} C_{1/\mu}) \cap J^{-1}(0,+\infty)\} \cup \{(\bigcup_{\mu\notin A} C_{1/\mu}) \cap J^{-1}(-\infty,0)\}$.

Then by [**2**, Theorem 5], V is a scale-invariant measurable subset of $C_0[0,1]$ and $m(\lambda V) = 1/2$ for all $\lambda > 0$. Since $m(\lambda[J^{-1}(0,+\infty)\cap V) = \frac{1}{2}\chi_A(\lambda)$, from the contrapositive of Theorem, $J^{-1}(0,+\infty)\cap V$ is non-Borel measurable, that is, V is non-Borel measurable.

References

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