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A New Variational Inequality in Non-compact Sets and Its Application

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ABSTRACT. In this note, we shall prove a new variational inequality in non-compact sets and as an application, we prove a generalization of the Schauder-Tychonoff fixed point theorem.

Let E be a real Hausdorff topological vector space. Denote the dual space of E by E^* and the pairing between E^* and E by $\langle w, x\rangle$ for each $w \in E^*$ and $x \in E$. If A is a subset of E, we shall denote by 2^A the family of all subsets of A and by $cl\,A$ the closure of A in E, and $co\,A$ the convex hull of A.

The following Fan-Browder fixed point theorem [2] is essential in convex analysis and also the basic tool in proving many variational inequalities and intersection theorems in nonlinear functional analysis:

THEOREM[2]. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space and $T : X \to 2^X$ be a multimap satisfying the following:*

(1) *for each $x \in X$, $T(x)$ is non-empty convex,*
(2) *for each $y \in X$, $T^{-1}(y)$ is open.*

Then T has a fixed point $\hat{x} \in X$, i.e. $\hat{x} \in T(\hat{x})$.

The Fan-Browder theorem can be proved by using Brouwer's fixed point theorem or the KKM-theorem. Till now, there have been numerous generalizations and applications of this Theorem by several authors; e.g. see [3, 4] and references there.

In a recent paper [3], Ding-Kim-Tan further generalize the above result in non-compact locally convex spaces and the following is the special case of the fixed point version of their Theorem 1.

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LEMMA [3]. *Let X be a non-empty convex subset of a locally convex Hausdorff topological vector space and D be a non-empty compact subset of X. Let $T : X \to 2^D$ be a multimap satisfying the following:*

(1) *for each $x \in X$, $T(x)$ is non-empty and $co\,T(x) \subset D$,*
(2) *for each $y \in X$, $T^{-1}(y)$ is open.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in co\,T(\hat{x})$.

In this note, using this Lemma we shall prove a new variational inequality in non-compact sets and as an application, we prove a generalization of the Schauder-Tychonoff fixed point theorem.

First we prove the following variational inequality in non-compact sets.

THEOREM 1. *Let X be a non-empty bounded convex subset of a locally convex Hausdorff topological vector space E and D be a non-empty compact subset of X. Let $T : X \to E^*$ be a continuous mapping from the relative topology of X to the strong topology of E^* satisfying the following condition:*

$$(*) \qquad \text{for each } x \in X \setminus D, \quad \langle T(y), y - x \rangle \leq 0 \text{ for all } y \in X.$$

Then there exists a point $\hat{x} \in X$ such that

$$\langle T(\hat{x}), \hat{x} - x \rangle \leq 0 \text{ for all } x \in X.$$

PROOF: Suppose the contrary, i.e. for each $x \in X$ there exists a point $\tilde{x} \in X$ such that $\langle T(x), x - \tilde{x} \rangle > 0$. Then by the assumption $(*)$, $\tilde{x} \in D$. Now we define a multimap $P : X \to 2^D$ by

$$P(x) = \{y \in D : \langle T(x), x - y \rangle > 0\} \text{ for each } x \in X.$$

Then for each $x \in X$, $P(x)$ is non-empty. We now show that $co\,P(x) \subset D$. In fact, let $y_1, y_2 \in P(x)$; then $\langle T(x), x - y_1 \rangle > 0$ and $\langle T(x), x - y_2 \rangle > 0$, so that for each $t \in [0, 1]$ we have

$$\begin{aligned} &\langle T(x), x - (ty_1 + (1-t)y_2) \rangle \\ &= t\,\langle T(x), x - y_1 \rangle + (1-t)\,\langle T(x), x - y_2 \rangle > 0. \end{aligned}$$

By the assumption $(*)$ again, $ty_1 + (1-t)y_2 \in D$ so that $co\,P(x) \subset D$. In order to apply Lemma to the multimap P, it remains to show that for each $y \in D$, $P^{-1}(y)$ is open in X. Let (x_α) be a net in $X \backslash P^{-1}(y)$ such that (x_α) converges to $x_0 \in X$. By Lemma 1 [**1**] (see also [**9**, Lemma1] where it was observed that the result holds for X being bounded instead of compact), for each $y \in D$, the mapping $x \to \langle T(x), x - y\rangle$ is continuous. Therefore we have $\langle T(x_0), x_0 - y\rangle = \lim_\alpha \langle T(x_\alpha), x_\alpha - y\rangle \le 0$, so that $x_0 \in X \backslash P^{-1}(y)$. Therefore for each $y \in D$, $P^{-1}(y)$ is open in X and the multimap P satisfies the whole assumptions of Lemma. Hence there exists a point $\hat{x} \in D$ such that $\hat{x} \in co\,P(\hat{x})$. Therefore there exist $\{y_i \in P(\hat{x}) \mid i = 1, \dots, n\}$ and $\{\lambda_i > 0 \mid i = 1, \dots, n\}$ such that $\hat{x} = \sum_{i=1}^n \lambda_i y_i$ and $\sum_{i=1}^n \lambda_i = 1$. Therefore we have

$$\begin{aligned} 0 &= \langle T(\hat{x}), \hat{x} - \hat{x}\rangle \\ &= \left\langle T(\hat{x}), \hat{x} - \sum_{i=1}^n \lambda_i y_i \right\rangle \\ &= \sum_{i=1}^n \lambda_i \langle T(\hat{x}), \hat{x} - y_i\rangle > 0 \end{aligned}$$

which is a contradiction. This completes the proof.

When $X = D$ is compact and convex, we obtain the following

COROLLARY [**2**]. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E and let $T : X \to E^*$ be a continuous mapping from the relative topology of X to the strong topology of E^*. Then there exists a point $\hat{x} \in X$ such that*

$$\langle T(\hat{x}), \hat{x} - x\rangle \le 0 \text{ for all } x \in X.$$

As in [**4**, **5**, **7**], we can further generalize Theorem 1 in more general settings, e.g. in complex locally convex spaces or to a multimap T with upper semicontinuity or lower semicontinuity.

As an application of Theorem 1, we prove a generalization of the Schauder-Tychonoff fixed point theorem.

THEOREM 2. *Let X be a paracompact bounded convex subset of a locally convex Hausdorff topological vector space E and D be a non-empty compact subset of X. Let $f : X \to E$ be a weakly continuous* *mapping (i.e. for each* $p \in E^*$, $p \circ f$ *is continuous) such that for each* $x \in X$, $f(x) \in cl\left[x + \bigcup_{\lambda>0} \lambda(X - x)\right]$. *Suppose further that for all* $x \in X \backslash D$ *and* $p \in E^*$,

$$(*) \qquad \text{if } p(y - f(y)) > 0 \text{ then } p(y - x) \leq 0.$$

Then there exists a point $\hat{x} \in X$ *such that* $f(\hat{x}) = \hat{x}$.

PROOF: Suppose that $x - f(x) \neq 0$ for all $x \in X$. Then for each $x \in X$, there exists at least one linear functional $p \in E^*$ such that $p(x - f(x)) > 0$. Now let $U(p) = \{x \in X \mid p(x - f(x)) > 0\}$ for each $p \in E^*$. Since f is weakly continuous, each $U(p)$ is an open subset of X and for each $x \in X$, $x \in U(p)$ for some $p \in E^*$. Therefore $\{U(p) \mid p \in E^*\}$ is an open covering of the paracompact set X, so that there exists an open locally finite refinement $\{V(p) \mid p \in E^*\}$ of $\{U(p) \mid p \in E^*\}$. Let $\{\beta_p \mid p \in E^*\}$ be th[illegible]ous partition of unity subordinated to this refinement.

Now we define a mapping $T : X \to E^*$ by

$$T(x) = \sum_{p \in E^*} \beta_p(x) p \quad \text{for each } x \in X.$$

Then for each $x \in X$, by the local finiteness of $\{V(p) \mid p \in E^*\}$, there exist $\{p_1, \ldots, p_n\} \subset E^*$ such that

$$(1) \qquad \begin{aligned} \langle T(x), x - f(x) \rangle &= \sum_{p \in E^*} \beta_p(x) p(x - f(x)) \\ &= \sum_{i=1}^{n} \beta_{p_i}(x - f(x)) > 0. \end{aligned}$$

Now we shall show that T satisfies the whole hypotheses of Theorem 1. To show that T is continuous from the relative topology of X to the strong topology of E^*, let $(x_\alpha)_{\alpha \in \Gamma}$ be a net in X which converges to $x_0 \in X$. Since $\{V(p) \mid p \in E^*\}$ is locally finite, there exist an open neighborhood U of x_0 in X and finite members of $\{V(p) \mid p \in E^*\}$ such that $x_0 \in U \cap V(p_1) \cap \cdots \cap V(p_n) \neq \emptyset$. Then $U_0 := U \cap V(p_1) \cap \cdots \cap V(p_n)$ is an open neighborhood of x_0 in X. Since (x_α) converges to x_0, there exists $\alpha_0 \in \Gamma$ such that for any $\alpha \in \Gamma$ with $\alpha \geq \alpha_0$,

$x_\alpha \in U_0$. Therefore for any $\alpha \in \Gamma$ with $\alpha \geq \alpha_0$ and B any bounded subset of E,

$$\left|\sup_{y\in B}\langle T(x_\alpha)-T(x_0),y\rangle\right| = \left|\sup_{y\in B}\left[\sum_{i=1}^{n}(\beta_{p_i}(x_\alpha)-\beta_{p_i}(x_0))p_i(y)\right]\right|$$
$$\leq \sum_{i=1}^{n}\left|\sum_{p\in B}p_i(y)\right| |(\beta_{p_i}(x_\alpha)-\beta_{p_i}(x_0))|.$$

Since $p_i \in E^*$ and B is bounded, by Theorem 1.18 [6], there exists $M > 0$ such that $|\sup_{y\in B} p_i(y)| < M$ for all $i = 1,\ldots,n$. For any $\varepsilon > 0$, since each β_{p_i} is continuous, we can find $\alpha_1 \in \Gamma$ such that $\sum_{i=1}^{n}|(\beta_{p_i}(x_\alpha)-\beta_{p_i}(x_0))| < \frac{\varepsilon}{M}$ for all $\alpha \geq \alpha_1$. Let $\alpha_2 \geq \max\{\alpha_0,\alpha_1\}$. Then for all $\alpha \in \Gamma$ with $\alpha \geq \alpha_2$,

$$\left|\sup_{y\in B}\langle T(x_\alpha)-T(x_0),y\rangle\right| < \varepsilon,$$

so that $(T(x_\alpha))$ converges to $T(x_0)$ in the strong topology of E^*.

Finally, suppose that there exists $x_1 \in X\backslash D$ such that for some $y \in X$,

$$\langle T(y),y-x_1\rangle = \sum_{p\in E^*}\beta_p(y)p(y-x_1) > 0.$$

If $\beta_p(y) > 0$, then $p(y-f(y)) > 0$, so that by the assumption $(*)$, $p(y-x_1) \leq 0$. Hence we have $\sum_{p\in E^*}\beta_p(y)p(y-x_1) \leq 0$, which is a contradiction. Therefore for each $x \in X\backslash D$, $\langle T(y),y-x\rangle \leq 0$ for all $y \in X$. Hence by Theorem 1, there exists a point $\hat{x} \in X$ such that

$$\langle T(\hat{x}),\hat{x}-y\rangle \leq 0 \text{ for all } y \in X. \tag{2}$$

By the assumption, since $f(\hat{x}) \in cl\left[\hat{x}+\bigcup_{\lambda>0}(X-\hat{x})\right]$, there exists two nets $(y_\alpha) \subset X$, $(\lambda_\alpha) \subset R^+$ such that $(\hat{x}+\lambda_\alpha(y_\alpha-\hat{x}))$ converges to $f(\hat{x})$. Then by (2), we have

$$\begin{aligned}\langle T(\hat{x}),\hat{x}-f(\hat{x})\rangle &= \lim_{\alpha}\langle T(\hat{x}),\hat{x}-(\hat{x}+\lambda_\alpha(y_\alpha-\hat{x}))\rangle\\ &= \lim_{\alpha}\langle T(\hat{x}),\lambda_\alpha(\hat{x}-y_\alpha)\rangle\\ &= \lim_{\alpha}\lambda_\alpha\cdot\langle T(\hat{x}),\hat{x}-y_\alpha\rangle \leq 0,\end{aligned}$$

which contradicts to (1). This completes the proof.

Theorem 2 generalizes the classical Schauder-Tychonoff fixed point theorem and Halpern's generalization in several aspects.

Finally, as remarked before, we can also generalize Theorem 2 in more general settings (see [8]).

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