JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 4, June 1991

Injective S-Systems and Regular Semigroups

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ABSTRACT. In this paper, we want to find some properties of subsystems of injective S-systems. Also we find a relationship between the regular semigroup S and the S-system M over S.

1. Introduction

The relation between an injective module M over R and a ring R can be explained by Baer criterion. The corresponding idea also can be applied to S-system M over semigroup S by congruence relation on S. But there are many different properties between R-modules and S-systems. So we need a new definition named weakly injective S-system. Also we want to study weakly injective S-systems and related semigroups.

Using the torsion free congruence of S-system defined by C.V. Hincle, Jr [4], we discover an equivalence condition of p-injective S-systems. Feller and Gantos [**3**] showed that a semigroup S is an inverse semigroup with zero and the set $E(S)$ of all idempotents of S is dually well-ordered if and only if every right and left S-system over S is injective. In this paper, we want to find a relationship between the regular semigroup S and the S-system M over S.

2. Notations and Preliminaries

DEFINITION 2.1: An element a in a semigroup S is called *regular* if there is an element x in S such that $axa = a$. If all elements of S are regular we call S a regular semigroup.

We note that if $axa = a$, then $e = ax$ is an idempotent element of S such that $ea = a$. Similarly, $f = xa$ is an idempotent such that $af =$

Received by the editors on October 30, 1990.

1980 *Mathematics subject classifications*: Primary 20M99; Secondary 16A52, 18B99.

Research supported by Korea Science and Engineering Foundation. (1988).

a. The regularity for ring was introduced first by J. Von Neumann for ring [**6**]. It plays an important role in semigroup theory.

DEFINITION 2.2: A non-empty set M is a (*right*) *S-system* over a semigroup S if there is a map $(x,s) \mapsto xs$ from $M \times S$ into M with the properties $(xs)t = x(st)$ for all $x \in M$, s, $t \in S$. If S has identity element 1, it satisfies $x1 = x$ for $x \in M$. By *centered S-system M* over a semigroup S, we mean an S-system containing an unique singleton element z such that $za = z$ for all $a \in S$. We call an such element z as *zero* and it is denoted by 0.

If M is centered and S has 0, then $x0$ is clearly a zero element of M. Since we can consider S-system M as unary algebra with operations as many as S, we can use the notation of universal algebra. For example, S-subsystem and homomorphism of S-systems are the same as those of universal algebra.

DEFINITION 2.3: A *congruence* K on an S-system M over S is an equivalence relation on M such that whenever $(m,n) \in K$ implies $(ms, ns) \in K$ for all $s \in S$. The identity congruence of M will be denoted by 1_M.

THEOREM 2.4. *If f is a homomorphism from S-system A into S-system B, then the relation* $\operatorname{Ker} f = \{(x,y) \mid f(x) = f(y)\}$ *is a congruence on A. If we define a function $g : A/\ker f \to B$ by $g(\bar{x}) = f(x)$ for all $\bar{x} \in A/\ker f$, then the following diagram commutes, where π is a natural projection.*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \pi}\downarrow & \nearrow_{g} & \\ A/\ker f & & \end{array}$$

PROOF: See [**7**, Theorem 5.3].

DEFINITION 2.5: An S-subsystem N of M is called *large* if for any homomorphism f from M into any S-system C, f itself is injective whenever the restriction of f on N is injective. Also we call M an *essential extension* of N.

THEOREM 2.6. *M is an essential extension of N if and only if for any congruence K of M, $K \cap (N \times N) = 1_N$ implies $K = 1_M$.*

PROOF: "If part". Let $f : M \to K$ be an homomorphism such that the restriction of f on N is injective. From the above Theorem 2.4, $\operatorname{Ker} f$ is a congruence on M. If $(x, y) \in \operatorname{Ker} f \cap (N \times N)$, then $f(x) = f(y)$ for all x, $y \in N$. Since f is injective, we have $x = y$. This means that $\operatorname{Ker} f \cap (N \times N) = 1_N$ and so $\operatorname{Ker} f = 1_M$. Thus f is injective.

"only if part". Conversely, assume that K is not identity congruence of M. Then there are x, y in M such that $(x, y) \in K$, $x \neq y$. The natural projection $\pi : M \to M/K$ by $\pi(m) = \{z \in M \mid (m, z) \in K\}$ is not one to one. Thus from the definition of essential extension, the restriction of π on N is not one to one. Hence there are a, b in N such that $\pi(a) = \pi(b)$, $a \neq b$. Since $\operatorname{Ker} \pi = K$, we have that $K \cap (N \times N) \neq 1_N$.

DEFINITION 2.7: An S-system I is *injective* if for any one to one homomorphism $f : A \to B$ and for any homorphism $h : A \to I$, there is a homomorphism $g : B \to I$ such that $gf = h$ for any S-systems A, B.

We can easily prove that an S-system I is injective if and only if I has no proper essential extension [**1**]. In ring and module theory, the following theorem is very useful for checking if given module M is injective or not.

THEOREM 2.8(Baer). *An R-module M is injective if and only if for any right ideal K of a ring R and for every $f \in \operatorname{Hom}_R(K, M)$, there exists an element m in M such that $f(k) = mk$ for all $k \in K$.*

PROOF: See [**6**, Lemma 1].

If we exchange a ring R into a semigroup S, then the module M becomes an S-system. Also the above Bare criterion was supposed to hold on S-system. But C.V. Hinkle, Jr [**4**] found that the above Theorem 2.8 does not hold for S-system except some special case. So he defined a weakly injective S-system. Since we want to find a relationship between S-system and regular semigroup S, we need the following definition.

DEFINITION 2.9: Let M be an S-system and $a \in M$. Then the set $aS = \{ax \mid x \in S\}$ is called a *cyclic S-system* of M. If there exists an element m in M such that $M = mS$, then we call M a *cyclic S-system* and such element m is called a *generator* of M. Also we define the set $aS^1 = \{ax \mid x \in S\} \cup \{a\}$. If S has an identity element 1, then $aS^1 = aS$.

DEFINITION 2.10: An S-system M is called *c-injective* if and only if for every principal right ideal K of S and for any homomorphism $f : K \to M$, there is an element m in M such that $f(s) = ms$ for all $s \in K$.

If we put S itself as an S-system M, the cyclic S-subsystem aS becomes a principal right ideal of S.

For any S-system M, we denote the sets

$$\begin{aligned} P(V) &= \{s \in S \mid us = vs \text{ for all } (u,v) \in V \subset M \times M\}, \\ P(N) &= \{(u,v) \in S \times S \mid yu = yv \text{ for all } y \in N \subset M\}, \\ Y(U) &= \{m \in M \mid mu = mv \text{ for all } (u,v) \in U \subset S \times S\}, \\ Y(T) &= \{(u,v) \in M \times M \mid ut = vt \text{ for all } t \in T \subset S\}. \end{aligned}$$

DEFINITION 2.11([4]): For a centered S-system M over a semigroup S with 0. An S-subsystem A of M is called *meet-large* if the intersection of A with any non-zero S-subsystem N of M is always non-zero.

In ring theory, there is a one to one correspondence between the set of ideals and the set of congruences. But for an S-system M over a semigroup S with 0, the above property does not hold in general. It is clear that B is a meet-large S-system of A if and only if $bS \cap B \neq 0$ for all nonzero b of A.

THEOREM 2.12. *Let M be a centered S-system over a semigroup S with 0. If M is an essential extension of N, then $xS^1 \cap N \neq 0$ for all nonzero $x \in M$.*

PROOF: Let $K = (xS^1 \times xS^1) \cup 1_M$. Then K is a congruence relation of M and $K \neq 1_M$, since $(x, 0) \in K$. Hence from Theorem 2.6, we have $K \cap (N \times N) \neq 1_N$. So there is $(a, b) \in K \cap (N \times N)$ satisfying $a \neq b$. Since $K \cap (N \times N) = [(xS^1 \times xS^1) \cup 1_M] \cap (N \times N) = [(xS^1 \times xS^1) \cap (N \times N)] \cup 1_N = [(xS^1 \cap N) \times (xS^1 \cap N)] \cup 1_N$, we have $|xS^1 \cap N| \geq 2$. Therefore $xS^1 \cap N \neq 0$.

COROLLARY 2.13. *Every large S-subsystem N of M is meet-large in M.*

DEFINITION 2.14: For a centered S-system M over a semigroup S with 0, the set $Z(M) = \{(m,n) \in M \times M \mid ms = ns$ for all s in some meet-large right ideal of $S\}$ is called a *singular relation* of M. M is called *non-singular* if $Z(M) = 1_M$, M is called singular if $Z(M) = M \times M$. Semigroup S is called *right non-singular* if S itself is non-singular when we consider S as S-system.

THEOREM 2.15. *For a centered S-system M over a semigroup S with 0, the set $\{(m,n) \in M \times M \mid P(m,n)$ is a meet-large right ideal of $S\}$ is a singular relation of M.*

PROOF: If $P(m,n)$ is a meet-large right ideal of S, then we can consider $P(m,n)$ is a meet-large subsystem of S-system S itself. So $mx = nx$ for all $x \in P(m,n)$. Since $P(m,n)$ is meet-large, we get $(m,n) \in Z(M)$ by definition of $Z(M)$. Conversely, if $(m,n) \in Z(M)$, then $(m,n) \in Y(K)$ for some meet-large right ideal K of S. So $K \subset P(Y(K)) \subset P(m,n)$. Since K is a meet-large right ideal of S. By Theorem 2.12 and Corollary 2.13, we can easily prove that $P(m,n)$ is also a meet-large subsystem of the S-system S.

THEOREM 2.16. *Let M be a centered S-system over a semigroup S with 0, 1. Then M is non-singular if and only if for any relation T of M. $P(T)$ is not proper meet-large in S.*

PROOF: Let $P(T)$ be a meet-large subsystem of N, $N \subset S$. For any $n \in N$, define a map $f : S \to N$ by $f(x) = nx$ for all $x \in S$. Then $f^{-1}(P(T))$ is also meet-large in S, since the preimage of a meet-large set is also meet-large [4]. For any $x \in f^{-1}(P(T))$, we have $anx = bnx$ for all $(a,b) \in T$, and hence $f^{-1}(P(T)) \subset P(an,bn)$ for all $(a,b) \in T$. So $P(an,bn)$ is a meet-large right ideal of S. By Theorem 2.15, we get $an = bn$ and so $n \in P(T)$. This means that $N = P(T)$.

Conversely, if $P(a,b)$ is a meet-large right ideal of S, then $ax = bx$ for all x. Hence $a = b$ and so M is non-singular.

3. Main Results

THEOREM 3.1. *Every right ideal of a semigroup S is generated by idempotent element of S if and only if for any S-system M, any right* *ideal I of S and for any homomorphism $f : I \to M$, there is an element m in M such that $f(s) = ms$ for all $s \in I$.*

PROOF: "if part". Let I be any right ideal of S. If we define a function f as an identity function of I, then there exists m in I such that $mx = x$ for all $x \in I$. Therefore if we put $x = m$, then we have $m = mm$. Consequently we have that $I = mS$.

"only if part". For any S-system M and for any right ideal I of S, $I = eS$ for some idempotent e of S. Let $f : I \to M$ be any homomorphism. If we put $f(e) = m$, then $f(ex) = f(eex) = f(e)ex = mex$ for all $x \in S$.

COROLLARY 3.2. *If every right ideal of a semigroup S is generated by idempotent element, then any S-system M is c-injective.*

THEOREM 3.3. *A semigroup S is regular if and only if every S-system M over S is c-injective.*

PROOF: Let a be any element of S. Then aS^1 is a principal right ideal of S. For the identity map $f : aS^1 \to aS^1$, there is an element m in aS such that $as = mas$ for all $s \in S^1$. Thus $a = ma = (ax)a$ for some $x \in S^1$.

Conversely, for any S-system M and for any principal right ideal aS^1 of S, we have $aS^1 = eS$ for some idempotent e of S, since S is regular ([**2**], Lemma 1.13). For any homomorphism $f : eS \to M$, if we put $f(e) = m$, then $f(es) = f(ees) = mes$ for all $s \in S$. Thus M is a c-injective S-system.

It is easy to show that if e is an idempotent element of S, then the principal right ideal eS is equal to $P(T)$ for some subset T of $S \times S$. In fact $eS = P(Y(e))$. But for any subset T of $S \times S$, $P(T)$ is not of the form eS. But there are many semigroup having the property that for any relation T of S, $P(T)$ is a principal right ideal generated by an idempotent element of S.

THEOREM 3.4. *An S-system M is a c-injective if and only if $Mx = Y(P(x))$ for all $x \in S$.*

PROOF: "if part". Let $u \in Y(P(x))$. Define a map $f : xS \to M$ by $f(xs) = us$ for all s of S. Then the map f is well defined, since $xa = xb$ implies $(a, b) \in P(x)$. By the fact M is c-injective, there is an element m in M such that $f(xs) = m(xs)$ for all $s \in S$. Thus $u = u1 = f(x1) = mx \in Mx$. Converse inclusion $Mx \subset Y(P(x))$ can be proved easily. So we have $Mx = Y(P(x))$ for all $x \in S$.

"only if part". Let $f : xS \to M$ be any homomorphism, $x \in S$. Then for any $(s,t) \in P(x)$, $f(x)s = f(xs) = f(x)t$. Thus $f(x) \in Y(P(x)) = Mx$ and so $f(x) = mx$ for some $m \in M$.

COROLLARY 3.5. *A semigroup S is regular if and only if for any S-system M, $Mx = Y(P(x))$ for all $x \in S$.*

THEOREM 3.6. *Let B be a subsystem of an S-system A. If K is a maximal element of the set $\{C \mid C$ is a congruence of A such that $C \cap (B \times B) = 1_B\}$, then B/K is a large subsystem of A/K and isomorphic to B.*

PROOF: See [**1**, Theorem 8].

THEOREM 3.7. *If M is a subsystem of an injective S-system I, then the followings are pairwise equivalent:*

1) M is injective.

2) There is a congruence relation K on I such that $M \cong I/K$ and the $\pi|_M$ is an isomorphism, where $\pi : I \to I/K$ is the natural projection.

3) For some S-system B, there is a homorphism $h : I \to B$ such that M is maximal in the set $\{A \mid h|_A$ is one to one$\}$.

PROOF: 1) implies 2). By Theorem 3.6, there is a congruence relation K of I such that $M \cong M/K$ and M/K is a large subsystem of I/K. Let $g = (\pi|_M)^{-1} : M/K \to M$ be an isomorphism. Since M is injective, there is a homomorphism $f : I/K \to M$ such that the restriction of f on set M/K is equal to g. Since M/K is a large subsystem of I/K, f is one to one. For any $x \in I/K$, there is an element m in M/K such that $f(x) = g(m) = f(m)$, since $g = f$ on the set M/K. Thus $I/K = M/K$ and the natural projection $\pi : I \to I/K$ has the property that the restriction of π on set M is an isomorphism.

2) implies 3). Let $i : M \to I/K$ be an isomorphism. If we define $f : M \to I$ by $f(m) = (\pi|_M)^{-1}(i(m))$, then it is obvious that f is an isomorphism and $\pi \circ f = i$. Since I is injective, there is a homomorphism $f' : I \to I$ such that $f'|_M = f$. Let $h = \pi \circ f'$. If $h(a) = h(b)$ for a, $b \in M$, we have $i(a) = \pi \circ f'(a) = \pi \circ f(b) = i(b)$.

Since i is one to one, $a = b$. Thus $h|_M$ is also one to one. If N is any subsystem of I containing M properly, then there is an element n in N such that n is not in M. Since $h(n) \in I/K$, there is m in M such that $h(n) = i(m)$. Since $i(m) = \pi \circ f(m) = \pi \circ f'(m) = h(m)$, $h|_N$ is not one to one. Thus M is a maximal element of the set $\{A \mid h|_A \text{ is one to one}\}$.

3) implies 1). Suppose that there is a homomorphism $h : I \to B$ such that M is maximal in the set $\{A \mid h|_A \text{ is one to one}\}$. Let M' be the injective hull of M. Since I is injective, we have that $M' \subset I$. Let $g = h|_{M'} : M' \to B$. Since M' is an essential extension of M, g is also one to one. From the maximality of M, we have $M = M'$ and so M is injective.

COROLLARY 3.8. *Let a semigroup S be self injective. A right ideal I of S is injective if and only if $I = eS$ for some idempotent e of S.*

PROOF: For any element a of S, the map $f_a : S \to S$ by $f_a(x) = ax$ is called a left translation of S corresponding a. Let e be an idempotent element of S. For any x, $y \in eS$, $f_e(x) = f_e(y)$ implies $x = y$ and so $f_e|_{eS}$ is one to one. If f is one to one and $eS \subset M \subset S$, then for any $m \in M$, $f_e(m) = em = eem = f_e(em)$. Since $em \in eS \subset M$, $e = em$. So we have that eS is maximal in the set $\{f_e|_M \text{ is one to one}\}$. By Theorem 3.7, eS is injective.

Conversely, we know that S has a fixed element x, since every injective S-system has a fixed element ([**9**], Proposition 4.4). Since I is injective, there is a homomorphism $h : S \to I$ such that $h|_I$ is the identity map of I. If we put $h(x) = e$, then e is an idempotent element and so $I = eS$.

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