# Model Reference Adaptive Control with System Parameter Constraints

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# 시스템 변수에 제한이 있는 경우의 간접적응제어

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요 약

시스템 변수에 선형제한이 있는 경우의 간접적응제어 문제가 고려되며 적응제어기의 수렴성과 안정성이 규명된다.

## 1. Introduction

In some practical case, known true constraints on the systems parameters are used to facilitate and improve estimation results. Constraint information is often available when a system model is developed based on experimental data or theoretical knowledge that specifies certain relationship among system parameters. For examples, there may be conservation rules, balance equations or other parametner constraints in chemical, biomedical, electrical, mechanical and other models, which should be used in parameter identification.

Chia and Chizeck[2] developed a recursive identification algorithms which estimate system parameters subject to known linear equality and/or inequality constraints.

In this paper we consider indirect model reference adaptive control problem for the case when the system parameters are constrained. Convergence conditions for the adaptive controller with parameter constraints are derived. Actually we extends the results of Goodwin and Sin[1] to the case of parameter constraints.

## 2. Indirect Model Reference Adaptive Control

Consider a linear discrete-time SISO system which is described by a deterministic autoregressive moving average

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(DARMA) model:

$$A(q^{-1})y(t)=B(q^{-1})u(t)$$
$$=q^{-d}B'(q^{-1})u(t) \cdots\cdots\cdots(1)$$

wher $q^{-1}$ is the backward shift operator. That is, $q^{-1}y(t) \triangleq y(t-1)$. Here $\{y(t)\}$, $\{u(t)\}$ denote the output and input respectively and $A(q^{-1})$, $B(q^{-1})$ are polynomials in $q^{-1}$:

$$A(q^{-1})=1+a_1q^{-1}+\cdots+a_nq^{-n} \cdots\cdots\cdots(2)$$
$$B(q^{-1})=q^{-d}(b_0+b_1q^{-1}+\cdots+b_mq^{-m}) \quad (3)$$

where $b_0 \neq 0$.

A DARMA model can be rewritten in regression form:

$$y(t)=\phi(t-1)^T\theta \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

wher

$$\phi(t-1)^T=[-y(t-1),\cdots,-y(t-n),$$
$$u(t-d),\cdots,u(t-d-m)] \cdots\cdots(5)$$
$$\theta=[a_1,\cdots,a_n, b_0,\cdots,b_m] \cdots\cdots\cdots\cdots(6)$$

It is assumed that the system considered is subjected to a known linear equality and /or inequality constraints on the system parameters:

$$L\theta=C \text{ or } L\theta \leq C \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(7)$$

where $L$ is a $l\times(n+m+1)$ matrix and rank$(L)=l$. $L$ and $C$ are known constant matrices.

Model reference control forces the output $\{y(t)\}$to track the output$\{y^*(t)\}$ of a reference model:

$$E(q^{-1})y^*(t)=q^{-d}H(q^{-1})r(t) \cdots\cdots\cdots(8)$$

where $r(t)$is a reference signal.

The control law which achieves the above objective is simply found by solving the prediction equality

$$E(q^{-1})=F(q^{-1})A(q^{-1})+q^{-d}G(q^{-1}) \quad (9)$$

The feedback control law is then given by

$$E(q^{-1})B'(q^{-1})u(t)+G(q^{-1})y(t)$$
$$=H(q^{-1})r(t) \cdots\cdots\cdots\cdots\cdots\cdots \quad (10)$$

To make the indirect adaptive control law, we then estimate the parameters in $A(q^{-1})$and $B(q^{-1})$ and replace the true parameters by estimated ones. Here we use the parameter-constrained recursive least squares algorithm[2)] for parameter estimation.

***Parameter-Constrained Least Squares Algorithm :***

$$\hat{\theta}(t+1)=\hat{\theta}(t)+\gamma(t+1)P(t)\phi(t+1)$$
$$\cdot[y(t+1)-\phi(t+1)^T\hat{\theta}(t)]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$
$$P(t+1)=P(t)-\gamma(t+1)P(t)\phi(t+1)$$
$$\cdot\phi(t+1)^TP(t) \cdots\cdots\cdots\cdots\cdots(12)$$
$$\gamma(t+1)=1/[1+\phi(t+1)^TP(t)\phi(t+1)]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(13)$$
$$\tilde{\theta}(t+1)=\hat{\theta}(t+1)-P(t+1)L^TH(t+1)$$
$$\cdot[L\hat{\theta}(t+1)-C] \cdots\cdots\cdots\cdots(14)$$
$$H(t+1)=H(t-1)+K(t)H(t)L\gamma(t+1)$$
$$\cdot P(t)\phi(t+1)\phi(t+1)^TP(t)L^T$$
$$\cdot H(t) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(15)$$

$$K(t+1)=1/[1-\phi(t+1)^T P(t) L^T \cdot H(t) L \gamma(t+1) P(t) \phi(t+1)] \quad \cdots\cdots (16)$$

Given $\tilde{\theta}(t)$, we from $\hat{A}(t, qi)$ and $\hat{B}'(t, q^{-1})$ from the coefficients as follows :

$$\hat{A}(t, q^{-1})=1+\tilde{a}_1(t)q^{-1}+\cdots+\tilde{a}_n(t)q^{-n} \quad \cdots\cdots (17)$$

$$\hat{B}'(t, q^{-1})=\tilde{b}_0(t)+\cdots+\tilde{b}_m(t)q^{-m} \cdots (18)$$

Next we determine $\hat{F}(t,q^{-1})$ and $\hat{G}(t,q^{-1})$ by solving the usual prediction equality

$$E(q^{-1})=\hat{F}(t,q^{-1})\hat{A}(t,q^{-1})+q^{-d}\hat{G}(t,q^{-1}) \quad \cdots\cdots (19)$$

Finally, the indirect model reference adaptive control law is given by

$$\hat{F}(t,q^{-1})\hat{B}'(t,q^{-1})u(t)+G(t,q^{-1})y(t) =E(q^{-1})y^*(t+d) \quad \cdots\cdots (20)$$

## 3. Convergence of Adaptive Control

We next investigate the convergence and stability attributes of parameter-constrained adaptive control for the SISO DARMA system. Without the parameter constraints, the asymptotic properties of this algorithm are known. In Goodwin and Sin[1], global convergence is proved. Here we extend these results to the parameter-constained case.

The following lemma is key to proving convergence of the adaptive control algorithm.

**Lemma 1** *For the alorithm(11) and (16) and subject to (4) and (7) it follows that*

(i)

$$\| \hat{\theta}(t)-\theta_o \|^2 \le \kappa_2 \| \hat{\theta}_2(0)-\theta_{2o} \|^2 ;$$

$$t \ge 1 \quad \cdots\cdots (21)$$

(ii)

$$\lim_{t\to\infty} \| \hat{\theta}(t)-\hat{\theta}(t-k) \| =0 \quad \cdots\cdots (22)$$

**Proof :** (i) We first decompose $\theta$ into two components. Since the linear constraints on parameters are consistent there exists an invertible matrix $L_1$ such that

$$[L_1 L_2] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = C \quad \cdots\cdots (23)$$

$$\theta_1=L_1^{-1}C-L_1^{-1}L_2\theta_2=\bar{L}_1-\bar{L}_2\theta_2 \quad (24)$$

Using the above equation $y(t)$ can be rewritten by

$$\bar{y}(t)=[\phi_1(t-1)^T \phi_2(t-1)^T] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} =\phi_1(t-1)^T\bar{L}_1 +(\phi_2(t-1)^T-\phi_1(t-1)^T\bar{L}_2)\theta_2 \quad \cdots\cdots (25)$$

Let

$$\bar{y}(t)=y(t)-\phi_1(t-1)^T\bar{L} \quad \cdots\cdots (26)$$

$$\bar{\phi}(t-1)^T=\phi_2(t-1)-\phi_1(t-1)^T\bar{L}_2 \quad (27)$$

Then we get a reduced unconstrained system model :

$$\bar{y}(t)=\bar{\phi}(t-1)^T\theta_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (28)$$

From the Lemma 3.3.6 in ref. 1)

$$\| \hat{\theta}_2(t)-\theta_{2o} \|^2 \leq \kappa_1 \| \hat{\theta}_2(0)-\theta_{2o} \|^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (29)$$

$$\| \hat{\theta}_1(t)-\theta_{1o} \|^2 = \| -\bar{L}_2(\hat{\theta}_2(t)-\theta_{2o} \|^2$$
$$\leq n \| l_{max} \|^2 \kappa_1 \| \hat{\theta}_2(0)-\theta_{2o} \|^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (30)$$

From (29) and (30)

$$\| \hat{\theta}(t)-\theta_o \|^2 \leq \| \hat{\theta}_1(t)-\theta_{1o} \|^2 + \| \hat{\theta}_2(t)-\theta_{2o} \|^2$$
$$\leq \kappa_1(1+n \| l_{max} \|^2) \| \hat{\theta}_2(0)-\theta_{2o} \|^2$$
$$\leq \kappa_2 \| \hat{\theta}_2(0)-\theta_{2o} \|^2 \quad \cdots\cdots\cdots\cdots (31)$$

This means that $\hat{\theta}(t)$ is a bounded sequence and proves (i).

(ii) From the Lemma 3.3.6 in ref. 1)

$$\lim_{t\to\infty} \| \hat{\theta}_2(t)-\hat{\theta}_2(t-k) \| =0 \quad \cdots\cdots\cdots (32)$$

From (30)

$$0 \leq \| \hat{\theta}_1(t)-\hat{\theta}_1(t-k) \|^2 \cdots\cdots\cdots (33)$$
$$\leq n \| l_{max} \|^2 \| \hat{\theta}_2(t)-\hat{\theta}_2(t-k) \|^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (34)$$

Take $t\to\infty$ on each side. Then

$$\lim_{t\to\infty} \| \hat{\theta}_1(t)-\hat{\theta}_1(t-k) \| =0 \quad \cdots\cdots\cdots (35)$$

Therefore

$$\lim_{t\to\infty} \| \hat{\theta}(t)-\hat{\theta}(t-k) \| =0 \quad \cdots\cdots\cdots (32)$$

This completes the proof.

**Theorem 1** *Provided that the parameter-constrained least squares algorithm is used to generate $\hat{\theta}(t)$ and provided the system is subject to known linear equality constraints and is stably invertible in the usual sense, then the indirect model reference adaptive control algorithm (11) to (20) is globally convergent in the sense that*

(i) $\{u(t)\}$, $\{y(t)\}$ *are bounded for all time*

(ii) $\lim_{t\to\infty}[y(t)-y^*(t)]=0$

**Proof** : As in the proof of Theorem 6.4.1 in ref. 1), the closed-loop system equations can be written by

$$\begin{bmatrix} E+[\hat{F}\cdot\bar{A}-\hat{F}\hat{A}] & -[\hat{F}\cdot\bar{B}'-\hat{F}\hat{B}'] \\ A\cdot[\hat{F}\cdot\bar{A}-\hat{F}\hat{A}] & EB'-A\cdot[\hat{F}\cdot\bar{B}'-\hat{F}\hat{B}'] \end{bmatrix} \cdot \begin{bmatrix} y(t+d) \\ u(t) \end{bmatrix} = \begin{bmatrix} Ey^*(t+d)+\hat{F}e(t+d) \\ EAy^*(t+d)+A\cdot\hat{F}e(t+d) \end{bmatrix} \quad \cdots\cdots (37)$$

(i) of Lemma 1 implies that $\hat{F}$ and $\hat{G}$ are bounded since the prediction equality (19) is solvable for any $\hat{A}$.

(ii) of Lemma 1 ensures that the model (37) is asymptotically time invariant and stable provided that $E^{-1}$ and $B^{-1}$ are both stable.

Thus, trom (37), $\{u(t-d)\}$ and $\{y(t)\}$ are asymptotically bounded by $\{e(t)\}$.

$$\begin{aligned}\bar{e}(t) &= \bar{y}-\phi(t-1)^{T}\hat{\theta}_2(t-1)\\ &= y(t)-(\phi_1(t-1)^{T}\bar{L}+\phi_2(t-1)^{T}\\ &\quad \theta_2(t-1)\phi_1(t-1)^{T}\bar{L}_2\hat{\theta}_2(t-1)\\ &= y(t)-(\phi_1(t-1)^{T}\theta_1(t-1)\\ &\quad +\phi_2(t-1)^{T}\hat{\theta}_2(t-1))\\ &= y(t)-\phi(t-1)^{T}\hat{\theta}(t-1)\\ &= e(t) \cdots\cdots (38)\end{aligned}$$

$$\lim_{t\to\infty}\frac{\bar{e}(t)^2}{c+\phi(t-1)^{T}\bar{\phi}(t-1)}=0 \quad (39)$$

From (38) and (39)

$$\lim_{t\to\infty}\frac{e(t)^2}{c+\bar{\phi}(t-1)^{T}\bar{\phi}(t-1)}=0 \quad (40)$$

Now we can apply the Lemma 6.2.1 in ref. 1) to show using (40) that $e(t)$ converge to zero and that $\{\phi\}$ is bounded. This establishes the theorem.

Next consider the case of inequality, that is $L\theta \leq C$. Let $\theta(t)$ be the unconstrained least squares estimation(LSE) and $\bar{\theta}(t)$ be the constrained LSE with equality constraints and $\tilde{\theta}(t)$ be the constrained LSE with inequality constraints. From Lemma 1

$$\lim_{t\to\infty} \| \hat{C}(t)-\hat{C}(t-k) \|^2 = 0 \cdots\cdots (41)$$

where $\hat{C}(t)=L\hat{\theta}(t)$ and $\hat{C}(t-k)=L\hat{\theta}(t-k)$. This implies that $\hat{\theta}(t)$ remains in the same region as $t\to\infty$, that is

$$\lim_{t\to\infty}\tilde{\theta}(t)=\hat{\theta}(t), \text{ if } \hat{C}(t)\leq C \cdots\cdots (42)$$

$$\lim_{t\to\infty}\tilde{\theta}(t)=\bar{\theta}(t), \text{ if } \hat{C}(t)\geq C \cdots\cdots (43)$$

As shown in the above, the indirect model reference adaptive control with the estimator $\hat{\theta}(t)$ and $\theta(t)$ are globally convergent. So the control with $\tilde{\theta}(t)$ is also globally convergent.

## 4. Conclusion

In this paper, we have investigated the convergence properties of a model reference adaptive controller, under the restriction that a known linear constraint on the system parameters.

## Refernces


1. Goodwin, G. C., and K. S., Sin (1984). Adaptive Filtering Prediction and Control, Prentice-Hall, New Jersey.
2. Chia, T. I., H. J. Chizeck (1989). "Recursive Parameter Identification of Constrained System", Submitted to Automatica for publication.