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Relationships Among Some Concepts of Multivariate Negative Dependence+

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ABSTRACT

In this article various notions of multivariate negative dependence for random variables are obtained. Various properties and interrelationships are also derived from these notions. Several counterexamples are given to illustrate that other implications may not hold.

1. Introduction

Lehmann(1966) has introduced various concepts of positive and negative dependence for two random variables. Stronger notions of bivariate positive and negative dependence were developed later by Esary and Proschan(1972). Multivariate generalizations of the notions of positive dependence were initiated by Harris(1970) and Brindley and Thompson(1972). Also Ebrahimi and Ghosh(1981), and Block, Savits and Shaked(1982) have extended these positive dependence concepts into the multivariate negative dependence analogs.

In this paper we derive some relationships among various concepts of multivariate negative dependence. In section 2, we introduce the notions of reverse rule of order 2(RR_2) in pairs, negatively likelihood ratio dependence(NLRD) and stochastically decreasing(SD) and show their relationships, that is, RR_2 in pairs $\Longleftrightarrow$ NLRD $\Longrightarrow$ SD. In section 3, various concepts of right corner set decreasing(RCSD) and left corner set increasing(LCSI) and preservation of LCSI are studied and the relation that LCSI implies left tail increasing in sequence(LTIS) is proved. In Section 4, concepts of negative upper orthant dependence(NUOD) and negative lower orthant dependence(NLOD), preservation of NLOD is also investigated. Counterexamples are given to illustrate no other implication holds among these concepts(right tail decreasing in sequence (RTDS) $\not\Longrightarrow$ NLOD, NLOD $\not\Longrightarrow$ LTIS).

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2. Reverse Rule of Order 2 and Negatively Likelihood Ratio Dependence

We start with the definition of reverse rule of order 2(RR_2) in pairs.

Definition 2.1(Karlin 1968). A function $f: R^n \to [0, \infty]$ is RR_2 in pairs if for any pair x_i, x_j, $f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n)$ viewed as a function of x_i, x_j with the other arguments held fixed satisfies for every $x_i \leq x_i'$, $x_j \leq x_j'(1 \leq i < j \leq n)$

$$\begin{vmatrix} f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) & f(x_1, \cdots, x_i', \cdots, x_j, \cdots, x_n) \\ f(x_1, \cdots, x_i, \cdots, x_j', \cdots, x_n) & f(x_1, \cdots, x_i', \cdots, x_j', \cdots, x_n) \end{vmatrix} \leq 0. \qquad (2.1)$$

If (2.1) holds for a probability densty function(pdf) $f(x_1, \cdots, x_n)$, then we say $(X_1, \cdots, X_n)$ or f is RR_2 in pairs($RR_2\{X_1, \cdots, X_n\}$). Dykstra, Hewett and Thompson(1973) have defined X_i is negatively likelihood ratio dependent($NLRD\{X_i \mid X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n\}$) on $X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n$ if for $x_i \leq x_i'$, $i = 1, 2, \cdots, n$,

$$f(x_1, \cdots, x_n) f(x_1', \cdots, x_n') \leq f(x_1, \cdots, x_{i-1}, x_i', x_{i+1}, \cdots, x_n) f(x_1', \cdots, x_{i-1}', x_i, x_{i+1}', \cdots, x_n') \qquad (2.2)$$

where f denotes the pdf of $X_1, \cdots, X_n$.

Definition 2.2(Barlow and Proschan, 1981). A random variable X_i is stochastically decreasing($SD\{X_i \mid X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n\}$) in random variables $X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n$ if $P(X_i > x_i \mid X_1 = x_1, \cdots, X_{i-1} = x_{i-1}, X_{i+1} = x_{i+1}, \cdots, X_n = x_n\}$ is decreasing in $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$.

The following example illustrates that if a joint probability mass function(pmf) f satisfies(2.2), then it does not necessarily imply that every marginal of f satisfies(2.2).

Example 2.3. Let $X = (X_1, X_2, X_3)$ be a random vector given by the following joint pmf on $\{0,1\} \times \{0,1\} \times \{0,1\}$

		X_3 = 0		X_3 = 1	
		X_2		X_2	
		0	1	0	1
X_1	0	0	0.01	0.01	0.2
	1	0.01	0.56	0.01	0.2

(2.3)

It is easy to check from (2.3) that joint pmf $f(x_1, x_2, x_3)$ satisfies (2.2) and therefore X_1 is NLRD on X_2, X_3. Let $g(x_1, x_2)$ be the joint pmf of X_1 and X_2. Since $g(0,0) = 0.01$, $g(1,1) = 0.76$, $g(1,0) = 0.02$, $g(0,1) = 0.21$, g does not satisfy (2.2), so that X_1 is not NLRD on X_2. Moreover, $P[X_1 > 0 \mid X_2 = 0, X_3 = 0] = 0.01/0.01$, $P[X_1 > 0 \mid X_2 = 1, X_3 = 0] = 0.56/0.66$, $P[X_1 > 0 \mid X_2 = 0, X_3 = 1] = 0.01/0.02$, $P[X_1 > 0 \mid X_2 = 1, X_3 = 1] = 0.2/0.4$, so that $SD\{X_1 \mid X_2, X_3\}$. However, $P[X_1 > 0 \mid X_2 = 0] = 0.02/0.03$, $P[X_1 > 0 \mid X_2 = 1] = 0.76/0.97$ and therefore X_1 is not stochastically decreasing in X_2. This illustrates that if X_i is stochastically decreasing in $X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n$ then it does not necessarily imply that X_i is stochastically decreasing in a subvector of $X_1, \cdots, X_n$.

The following theorem gives interrelationships between (2.1) − (2.2).

Theorem 2.4. (a) $RR_2\{X_1, \cdots, X_n\} \Leftrightarrow NLRD\{X_i \mid X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n\}$,
(b) $NLRD\{X_i \mid X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n\} \Rightarrow SD\{X_i \mid X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n\}$.

Proof of (a). ($\Rightarrow$) For every choice of $x_i \le x_i'$, $i=1,\cdots,n$,

$$f(x_1,\cdots,x_n)f(x'_1,x_2,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_n) \le f(x'_1,x_2,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)f(x_1,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_n)$$
$$f(x_1',x_2,\cdots,x_n)f(x_1',x_2',x_3,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_n) \le f(x'_1,x_2',\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)f(x_1',x_2,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_n) \quad (2.4)$$
$$f(x'_1,\cdots,x_{i-1}',x_i,x_{i+1}',\cdots,x_{n-1}',x_n)f(x_1',\cdots,x_n') \le f(x'_1,x_2',\cdots,x'_{i-1},x_i',x'_{i+1},\cdots,x_{n-1}',x_n)f(x_1',\cdots,x_{i-1}',x_i,x_{i-1}',\cdots,x_n')$$

By multiplying (2.4) side by side and cancelling the common terms from both side, we obtain (2.2) and so X_i is NLRD on $X_1,\cdots,X_{i-1},X_{i+1},\cdots,X_n$.
($\Leftarrow$) In the definition of NLRD i.e. in (2.2) take any $x_i \le x_i'$, $x \le x_j'$ for $j \neq i$ and take $x_k = x_k'$ for $k=1,2,\cdots,n$ with $k \neq j$, $k \neq i$. Then we obtain (2.1).

Proof of (b). From (2.2) we have

$$\begin{vmatrix} f(x_1,\cdots,x_n) & f(x_1,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_n) \\ f(x_1',\cdots,x'_{i-1},x_i,x'_{i+1},\cdots,x_n') & f(x_1',\cdots,x_n') \end{vmatrix} \le 0$$

$$\Longrightarrow \begin{vmatrix} \int_t^\infty f(x_1',\cdots,x_i',\cdots,x_{n-1}',x_n')dx_i' & \int_t^\infty f(x_1,\cdots,x_{i-1},x_i',x_{i+1},\cdots,x_n)dx_i' \\ \int_{-\infty}^t f(x_1',\cdots,x_{i-1}',x_i,x_{i+1}',\cdots,x_n')dx_i & \int_{-\infty}^t f(x_1,\cdots,x_n)dx_i \end{vmatrix} \le 0$$

for $x_i \le x_i'$, $i=1,2,\cdots,n$. Adding the top row to the bottom row and converting to ratios, we obtain the following inequality

$$P[X_i > t \mid X_1 = x_1',\cdots,X_{i-1} = x_{i-1}',X_{i+1} = x_{i+1}',\cdots,X_n = x_n'] \le P[X_i > t \mid X_1 = x_1,\cdots,X_{i-1}=x_{i-1},\ X_{i+1}=x_{i+1},\cdots,X_n = x_n]$$

for $x_1 \le x_1',\cdots,x_{i-1} \le x_{i-1}',x_{i+1} \le x_{i+1}',\cdots,x_n \le x_n'$, which shows $SD\{X_i \mid X_1,\cdots,X_{i-1},X_{i+1},\cdots,X_n\}$.

3. Right Corner Set Decreasing and Left Corner Set Increasing

In the definitions of Ebrahimi and Ghosh(1981), the random vector $\mathbf{Y}$ is right tail decreasing in the vector $\mathbf{X}$(RTD$\{\mathbf{Y} \mid \mathbf{X}\}$) if $P\{\mathbf{Y} > \mathbf{y} \mid \mathbf{X} > \mathbf{x}\}$ is decreasing in $\mathbf{x}$ for all $\mathbf{y}$. Parallel to the RTD, the random vector $\mathbf{Y}$ is left tail increasing in the vector $\mathbf{X}$ (LTI$\{\mathbf{Y} \mid \mathbf{X}\}$) if $P(\mathbf{Y} \le \mathbf{y} \mid \mathbf{X} \le \mathbf{x})$ is increasing in $\mathbf{x}$ for all $\mathbf{y}$. Moreover, if for all i, $i=1,2,\cdots,n-1$, X_{i+1} is stochatically left tail increasing in $X_1,\cdots,X_i$, then $\{X_1,\cdots,X_n\}$ is called left tail increasing in sequence(LTIS$\{X_n \mid X_1,\cdots,X_{n-1}\}$).

Definition 3.1 (Brindley and Thompson, 1972). Random variables $X_1,\cdots,X_n$ are right corner set decreasing(RCSD$\{X_1,\cdots,X_n\}$) if

$$P\{X_1 > x_1',\cdots,X_n > x_n' \mid X_1 > x_1,\cdots,X_n > x_n\} \quad (3.1)$$

is decreasing in $\{x_i : x_i \ge x_i'\}$ for every choice of $x_1',\cdots,x_n'$. Similarly, random variables $X_1,\cdots,X_n$ are said to be left corner set increasing (LCSI$\{X_1,\cdots,X_n\}$) if for every choice of $x_1',\cdots,x_n'$

$$P\{X_1 \le x_1',\cdots,X_n \le x_n' \mid X_1 \le x_1,\cdots,X_n \le x_n\} \quad (3.2)$$

is increasing in $\{x_i : x_i \le x_i'\}$.

We obtain following LSCI example from the similar example which has RCSD property(Brindley et al. (1972)). Let X, Y, Z be uniformly distributed over the tetrahedron with vertices (0,0,0), (−1,0,0), (0,−1,0) and (0,0,−1).

Let $F(x,y,z) = F[\min(0,x), \min(0,y), \min(0,z)]$ and for $x,y,z \leq 0$, $F(x,y,z) = 0$, $x+y+z \leq -1$, $F(x,y,z) = (1+x+y+z)^3$, $-1 \leq x+y+z$.

If $-1 \leq x+y+z$ and $x \leq x'$ then $P\{X \leq x', Y \leq y', Z \leq z' \mid X \leq x, Y \leq y, Z \leq z\}$ is increasing function of x. Hence X, Y and Z are LSCI.

Theorem 3.2. If $\{X_1, \cdots, X_n\}$ is RCSD then any subset of $\{X_1, \cdots, X_n\}$ is RTD in any other subset of them.

Proof. For any subset of K $\{1,2,\cdots,n\}$ take $x_i' > x_i$ for $i \in K$ and $x_i' \leq x_i$ for $i \in \bar{K}$, where $\bar{K}$ denotes the complement of K. Then by the property of RCSD, $P\{\mathbf{X}_K > \mathbf{x}_K' \mid \mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}\}$ is decreasing in $\mathbf{x}_{\bar{K}}$ for all $\mathbf{x}_K'$.

Similarly, if $\{X_1, \cdots, X_n\}$ is LCSI then any subset of $\{X_1, \cdots, X_n\}$ is LTI in any other subset of them.

Theorem 3.3. The random vector $\mathbf{X} = (X_1, \cdots, X_n)$ is RCSD if and only if for every subset $K \subset \{1,2,\cdots,n\}$, $P\{\mathbf{X}_K > \mathbf{x}_K + \Delta_K \mid \mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}\}$ is decreasing in $\mathbf{x}_{\bar{K}}$ for all $\mathbf{x}_K$ and all $\Delta_K > 0$ where $\bar{\mathbf{K}}$ denotes the complement of $\mathbf{K}$).

Proof. For given subset $K \subset \{1,\cdots,n\}$

$$P\{\mathbf{X}_K > \mathbf{x}_K + \Delta_K \mid \mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}\} = P\{\mathbf{X} > \mathbf{x}' \mid \mathbf{X} > \mathbf{x}\}$$

where $x_i' = x_i + \Delta_i$, if $i \in K$ and $x_i' = -\infty$ if $i \in \bar{K}$. If $\mathbf{X}$ is RCSD, this probability is decreasing in $\mathbf{x}_{\bar{K}}$.

Now assume the converse hypothesis and let $\mathbf{x}'$ and $\mathbf{x}$ be given.

By taking $K = \{i : x_i' > x_i\}$ and $\bar{K} = \{i : x_i' \leq x_i\}$

$$\begin{aligned} P\{X > \mathbf{x}' \mid \mathbf{X} > \mathbf{x}\} &= P\{\mathbf{X}_K > \mathbf{x}_K', \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}' \mid \mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}\} \\ &= P\{\mathbf{X}_K > \mathbf{x}_K' \mid \mathbf{X}_K > \mathbf{x}_k, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}\} \\ &= P\{\mathbf{X}_K > \mathbf{x}_K + \Delta_K \mid \mathbf{X}_K > \mathbf{x}_K, \mathbf{X}_{\bar{K}} > \mathbf{x}_{\bar{K}}\}, \text{ by letting } \mathbf{x}_K' = \mathbf{x}_K + \Delta_K, \ \Delta_K > 0. \end{aligned}$$

By hypothesis this conditional probability is decreasing in $\mathbf{x}_{\bar{K}}$. Hence $\mathbf{X}$ is RCSD.

The following theorem exhibits a LCSI preservation property.

Theroem 3.4. Let the random variables $X_1, \cdots, X_m$ be LCSI and let $g_i : R \rightarrow R$ be a Borel measurable strictly increasing function for each $i=1,\cdots,m$. Define $Y_i = g_i(X_i), i=1,\cdots,m$. Then $Y_1, \cdots, Y_m$ are LCSI.

Proof. Put for $i=1,\cdots,m$, $y_i' = g_i(x_i')$ and, $y_i = g_i(x_i)$.

$\text{LCSI}\{X_1, \cdots, X_m\}$

$\Leftrightarrow P\{X_1 \leq x_1', \cdots, X_m \leq x_m' \mid X_1 \leq x_1, \cdots, X_m \leq x_m\}$ is increasing in $\{x_i : x_i \leq x_i'\}$

$\Leftrightarrow P\{g_1(X_1) \leq g_1(x_1'), \cdots, g_m(X_m) \leq g_m(x_m') \mid g_1(X_1) \leq g_1(x_1), \cdots, g_m(X_m) \leq g_m(x_m)\}$ is increasing in $\{g_i(x_i) : g_i(x_i) \leq g_i(x_i')\}$

$\Leftrightarrow P\{Y_1 \leq y_1', \cdots, Y_m \leq y_m' \mid Y_1 \leq y_1, \cdots, Y_m \leq y_m)$ is increasing in $\{y_i : y_i \leq y_i'\}$

$\Leftrightarrow \text{LCSI}\{Y_1, \cdots, Y_m\}$.

Theorem 3.5. $\text{LCSI}\{X_1, \cdots, X_n\} \Rightarrow \text{LTIS}\{X_n \mid X_1, \cdots, X_{n-1}\}$.

Proof. First we are to show that X_n is stochastically left tail increasing in $X_1, \cdots, X_{n-1}$. Take $x_i \leq x_i'$ and $x_i' = \infty$ for $i = 1,2,\cdots,n-1$, and $x_n' \leq x_n$. Then from (3.2) we have

$$P\{X_n \leq x_n' \mid X_1 \leq x_1, \cdots, X_{n-1} \leq x_{n-1},\ X_n \leq x_n\}. \tag{3.3}$$

In (3.3) by choosing $x_n \to \infty$ and $x_n' = t < \infty$ we obtain that $P\{X_n \leq t \mid X_1 \leq x_1, \cdots, X_{n-1} \leq x_{n-1}\}$ is increasing in $x_1, \cdots, x_{n-1}$. Thus by Brindley and Thompson(1972) we have that LCSI$\{X_1, \cdots, X_n\}$ implies LTIS$\{X_n \mid X_1, \cdots, X_{n-1}\}$.

4. Negative Upper(Lower) Orthant Dependence

Definition 4.1(Joag−Dev and Proschan, 1983). Random variables $X_1, \cdots, X_n$ are said to be negatively upper orthant dependent(NUOD$\{X_1, \cdots, X_n\}$) if for all real $x_1, \cdots, x_n$, $P(X_1 > x_1, \cdots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i)$ and they are negatively lower orthant dependent(NLOD$\{X_1, \cdots, X_n\}$) if for all real $x_1, \cdots, x_n$, $P(X_1 \leq x_1, \cdots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$. Furthermore, random variables $X_1, \cdots, X_n$ are negatively orthant dependent(NOD$\{X_1, \cdots, X_n\}$) if they are NLOD and NUOD.

Theorem 4.2. Let $\{G_i : 1 \leq i \leq n\}$ be a family of distributions of $X_1, \cdots, X_n$ which are NLOD and have same one dimensional marginal. If $G = \sum_{i=1}^{n} \alpha_i G_i$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$ then G is also a distribution of NLOD random variables $X_1, \cdots, X_n$.

Proof. By definition, the one dimensionals of G are the same as those of G_i, and so it can be easily proved.

Theorem 4.3. Let random variables $X_1, \cdots, X_n$ be NLOD, let $Y_1, \cdots, Y_m$ be conditionally independent given $X_1, \cdots, X_n$ and let Y_i be stochastically left tail increasing in $X_1, \cdots, X_n$ for all $i = 1, \cdots, m$. Then

(i) $X_1, \cdots, X_n, Y_1, \cdots, Y_m$ are NLOD, and (ii) $Y_1, \cdots, Y_m$ are NLOD.

Proof. (i). $P(X_1 \leq x_1, \cdots, X_n \leq x_n, Y_1 \leq y_1, \cdots, Y_m \leq y_m)$

$= P(Y_1 \leq y_1, \cdots, Y_m \leq y_m \mid X_1 \leq x_1, \cdots, X_n \leq x_n) P(X_1 \leq x_1, \cdots, X_n \leq x_n)$ By conditional independence

$= \prod_{i=1}^{m} P(Y_i \leq y_i \mid X_1 \leq x_1, \cdots, X_n \leq x_n) P(X_1 \leq x_1, \cdots, X_n \leq x_n)$ By assumption LTI$\{Y_i \mid X_1, \cdots, X_n\}$ for $i=1, \cdots, m$ and by the assumption NLOD$\{X_1, \cdots, X_n\}$

$\leq \prod_{i=1}^{m} P(Y_i \leq y_i) \prod_{j=1}^{n} P(X_j \leq x_j)$

$= P(X_1 \leq x_1) \cdots P(X_n \leq x_n) P(Y_1 \leq y_1) \cdots P(Y_m \leq y_m)$.

(ii). Taking $x_j \to \infty$ $(j=1, \cdots, n)$ in (i), (ii) follows.

The following counterexamples show that other implications may not hold.

According to Ebrahimi and Ghosh(1981) random variables $X_1, \cdots, X_n$ are said to be right tail decreasing in sequence(RTDS$\{X_n \mid X_1, \cdots, X_{n-1}\}$) if for all $i=2, \cdots, n$, X_i is stochastically right tail decreasing in $X_1, \cdots, X_{i-1}$.

Example 4.4. Let the trivariate discrete random vector $\mathbf{X} = (X_1, X_2, X_3)$ take values (1,1,1), (1,2,2), (2,1,2) and (2,2,1) each with probability 1/4. Then $P(X_1 = 1) = P(X_2 = 1) = P(X_3 = 1) = P(X_1 = 2) = P(X_2 = 2) = P(X_3 = 2) = 1/2$. Since $P\{X_3 > 0 \mid X_2 > 0, X_1 > 0\} = 1$, P

$\{X_3 > 0 \mid X_2 > 0, X_1 > 1\} = 1$, $P\{X_3 > 0 \mid X_2 > 1, X_1 > 1\} = 1$, $P\{X_3 > 1 \mid X_2 > 1, X_1 > 0\} = 1/2$, $P\{X_3 > 1 \mid X_2 > 0, X_1 > 1\} = 1/2$, $P\{X_3 > 1 \mid X_2 > 1, X_1 > 1\} = 0$, so that X_1, X_2 and X_3 are RTDS by the symmetry of X_1, X_2, X_3. Since $P\{X_1 \leq 1, X_2 \leq 1, X_3 \leq 1\} = 11/4 > 11/8 = P\{X_1 \leq 1\}P\{X_2 \leq 1\}P\{X_3 \leq 1\}$, X_1, X_2 and X_3 are not NLOD.

Johnson and Kotz(1975) have provided necessary and sufficient conditions for the NLOD property for the following Farlie-Gumbel-Morgenstern(FGM) system.

Consider the case n=3. An explicit form the three-dimensional FGM system is

$$F(x_1, x_2, x_3) = F_1(x_1)F_2(x_2)F_3(x_3)[1 + \alpha_{12}G_1(x_1)G_2(x_2) + \alpha_{13}G_1(x_1)G_3(x_3) + \alpha_{23}G_2(x_2)G_3(x_3) + \alpha_{123}G_1(X_1)G_2(x_2)G_3(x_3)]$$

and

$$G(x_1, x_2, x_3) = P[X_1 > x_1,\ X_2 > x_2,\ X_3 > x_3] = G_1(x_1)G_2(x_2)G_3(x_3)[1 + \alpha_{12}F_1(x_1)F_2(x_2) + \alpha_{13}F_1(x_1)F_3(x_3) + \alpha_{23}F_2(x_2)F_3(x_3) - \alpha_{123}F_1(x_1)F_2(x_2)F_3(x_3)]$$

where $F_j(x_j) = P(X_j \leq x_j)$ and $G_j(x_j) = 1 - F_j(x_j)$, $j=1,2,3$.

They have shown that (X_1, X_2, X_3) is NLOD if and only if $\alpha_{ij} \leq 0 (1 \leq i < j \leq 3)$,

$$\alpha_{12} + \alpha_{13} + \alpha_{23} + \alpha_{123} \leq 0,\quad \alpha_{12} + \alpha_{13} + \alpha_{23} - \alpha_{123} \leq 0 \tag{4.1}$$

We use this fact to explain that the random variables X_1, X_2, X_3 are not necessarily LTIS, when (4.1) holds.

Example 4.5. Let (X_1, X_2, X_3) be NLOD and satisfy FGM system. Then

$$P[X_3 \leq x_3 \mid X_1 \leq x_1,\ X_2 \leq x_2] =$$

$$P[X_3 \leq x_3] \{\frac{1 + (\alpha_{13}G_1(x_1)G_3(x_3) + \alpha_{23}G_2(x_2)G_3(x_3) + \alpha_{123}G_1(x_1)G_2(x_2)G_3(x_3))}{1 + \alpha_{12}G_1(x_1)G_2(x_2)}\}. \tag{4.2}$$

Assume that each G_j is a continuous function and choose x_1, x_3 such that $G_1(x_1) = 1/2$, $G_3(x_3) < 1$. Also let $\alpha_{12} = -0.3$, $\alpha_{13} = -0.1$, $\alpha_{23} = -0.1$ and $\alpha_{123} = 0.4$ so that (4.1) is satisfied. Now with the choice $x_2' < x_2''$ such that $G_2(x_2') = 1/2$, $G_2(x_2'') = 1/4$ it follows that the expression in the bracket of (4.2) with $x_2 = x_2'$ is 1 while the expression in the bracket of (4.2) with $x_2 = x_2'$ is less than 1. This shows that the LTIS property does not necessarily hold.

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