Journal of the Korean
Statistical Society
Vol. 20, No. 1, 1991

A Class of Admissible Estimators in the One Parameter Exponential Family+

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ABSTRACT

This paper deals with the problem of estimating an arbitrary piecewise continuous function of the parameter under squared error loss in the one parameter exponential family. Using Blyth's(1951) method sufficient conditions are given for the admissibility of (possibly generalized Bayes) estimators. Also, some examples are provided for normal, binomial, and gamma distributions.

1. Introduction

Let X be a random variable with the density

$$f(x;\theta) = e^{\theta x - p(\theta)}, \quad x \in \mathbf{A},\ \theta \in \Theta \tag{1.1}$$

with respect to some σ-finite measure μ on $\mathbf{A}$, where $\mathbf{A}$ is an interval in the real line, and Θ is taken to be the natural parameter space

$$\Theta = \{\,\theta : e^{p(\theta)} = \int_{A} e^{\theta x}\, d\mu(x) < \infty\,\}.$$

From the convexity of the exponential function, Θ is an (possibly infinite) interval, $(\underline{\theta}, \bar{\theta})$ say, in the real line. It is well known that, in the interior of Θ, $E_\theta(X) = p'(\theta) \equiv \dfrac{dp(\theta)}{d\theta}$, where E_θ denotes the expectation of X with the density (1.1).

Consider the problem of estimating any (piecewise) continuous function $h(\theta)$ on Θ under squared error loss $L(\theta, d) = (h(\theta) - d)^2$, where $d \in \mathbf{D}$, the decision space.

+ Part of this article was addressed at the Conference of the 6th Korea and Japan of Statistics on July, 1989.

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Karlin(1958) investigated sufficient conditions for the admissibility of estimators of the form aX for estimating $h(\theta) = p'(\theta)$ under squared error loss. This result was generalized in several directions. Sufficient conditions for admissibility of aX+b for the same problem were later obtained by Ping(1964) using the Cramer-Rao Inequality and by Gupta(1966) following Karlin's argument. Zidek(1970) provided, using the formal Bayes approach, sufficient conditions for admissibility of X for estimating $h(\theta)$, not necessarily the mean $p'(\theta)$. For estimating $h(\theta)$ using Karlin's argument, the reader is referred to Ghosh and Meeden(1977), Ralescu and Ralescu(1981). On the other hand, Brown and Hwang(1982) have developed a simple and unified approach using Blyth's(1951) method for proving the admissibility of (possibly generalized Bayes) estimators of the mean vector of a multiparameter exponential family. The simplicity is achieved by using a single sequence of priors for all estimators. Das Gupta and Sinha(1984), using Brown and Hwang's technique which was in turn based on Blyth's method, gave sufficient conditions for the admissibility of (possibly generalized Bayes) estimators of $h(\theta)$, other than the mean $p'(\theta)$, under squared error loss.

In Chapter 2, we shall provide, using Blyth's method, sufficient conditions for the admissibility of (possibly generalized Bayes) estimators of $h(\theta)$ with the special form (see the equation (2.2)). This set of sufficient conditions differs from that given by Das Gupta and Sinha(1984). Also, we shall give, using Brown and Hwang's technique, sufficient conditions in cases when Θ is the real line or the proper subset of the real line. Finally, Chapter 3 contains examples for normal, binomial, and gamma distributions.

2. Sufficient conditions for admissibility

Let X be a random variable with the density (1.1).

Consider the problem of estimating an arbitrary(piecewise) continuous function $h(\theta)$ under squared error loss $L(\theta, d) = (h(\theta)-d)^2$.

The convexity of the loss function permits us to restrict attention only to nonrandomized estimators (see Berger, 1985, p41). Furthermore, there is no loss of generality in using a single observation X because of sufficiency.

Consider a prior distribution $G(\theta)$ on Θ with the differentiable density $g(\theta)$ with respect to Lebesgue measure.

Now, define, for fixed $\alpha \in \mathbf{R}^1$ and $\eta(\neq -1) \in \mathbf{R}^1$,

$$I_x(t) = \int_{\Theta} t(\theta) \cdot \exp\{\theta(x+\alpha) - (\eta+1) \int_d^{\theta} h(t)dt\} d\theta, \quad x \in \mathbf{A},$$

where d is an interior point of Θ, and $\int_a^b h(t)dt$ exixts and is finite for every $[a, b] \subset \Theta$. Assume

$$I_x(\,|\,g_o'(\theta)\,|\,) < \infty \quad \text{for all } x, \tag{2.1}$$

where the prime denotes the differentiation of g_o with respect to θ, and

$$g_o(\theta) = g(\theta) \cdot \exp\{-\alpha\theta - p(\theta) + (\eta+1) \int_d^{\theta} h(t)\ dt\}$$

Let δ_g be defined by

$$\delta_g(x) = \frac{x+\alpha}{\eta+1} + \frac{I_x(g_o')}{(\eta+1)\, I_x(g_o)}, \quad x \in \mathbf{A}, \tag{2.2}$$

with the convention that $\frac{I_x(g_o')}{\infty} = 0$. Throughout this paper assume that $R(\theta, \delta_g) < \infty$ for all $\theta \in \Theta$ where $R(\theta, \delta_g)$ is the risk of δ_g.

Remark 2.1: Note that δ_g is the generalized Bayes estimator of $h(\theta)$ with respect to $g(\theta)$ under assumption (2.1) and the further assumptions; for each $x \in \mathbf{A}$,

$$I_x(g_o) < \infty \tag{2.3}$$

and

$$\lim_{\theta \to \underline{\theta}} g_o(\theta) \cdot \exp\{\theta(x+\alpha) - (\eta+1) \int_a^\theta h(t)dt\} = 0 = \lim_{\theta \to \overline{\theta}} g_o(\theta) \cdot \exp\{\theta(x+\alpha) - (\eta+1) \int_a^\theta h(t)\, dt\}. \tag{2.4}$$

This can be shown as follows: If the assumptions (2.1), (2.3), and (2.4) hold, then by integration by parts,

$$\begin{aligned} \delta_g(x) &= \frac{\int_\Theta h(\theta) \cdot e^{\theta x - p(\theta)} g(\theta)d\theta}{\int_\Theta e^{\theta x - p(\theta)}\, g(\theta)d\theta} \\ &= \frac{\int_\Theta e^{\theta(x+\alpha)}\, g_o(\theta)h(\theta) \cdot \exp\{-(\eta+1) \int_a^\theta h(t)dt\}d\theta}{\int_\Theta e^{\theta x}\, g_o(\theta) \cdot \exp\{\alpha\theta - (\eta+1) \int_a^\theta h(t)dt\}\, d\theta} \\ &= \frac{x+\alpha}{\eta+1} + \frac{I_x(g_o')}{(\eta+1)\, I_x(g_o)} \quad . \end{aligned}$$

In particular, if $\Theta = \mathbf{R}^1$, the real line, and (2.3) holds, then the assumption (2.4) is automatically satisfied, and hence δ_g is the appropriate generalized Bayes estimator of $h(\theta)$ under assumption (2.1). Also, if g vanishes outside a closed and bounded set in Θ, and (2.3) holds, then δ_g is again the appropriate generalized Bayes estimator of $h(\theta)$ under assumption (2.1).

Before providing sufficient conditions for admissibility of δ_g, we first give Blyth's(1951) method for proving the admissibility of estimators, stated below in the form appeared in Berger(1976, p345, Theorem 3). See also Stein(1955) and Berger(1985, p547).

Lemma 2.1: Let $\{h_n\}$ be a sequence of absolutely continuous functions defined on Θ such that

(I) for every $n \geq 1$, $\int_\Theta h_n^2(\theta) \cdot g(\theta)d\theta < \infty$;
(II) for every $n \geq 1$, there exists an $\mathbf{c} > 0$ such that $h_n(\theta) \geq \mathbf{c}$ for all θ in a set $\mathbf{s} \subseteq \Theta$ with $\int_s g(\theta)d\theta > 0$;
(III) $h_n(\theta) \to 1$ a.e. (Lebesgue measure) as $n \to \infty$.

Consider a sequence $\{g_n\}$ of prior densities with respect to Lebesgue measure such that $g_n(\theta) = h_n^2(\theta) \cdot g(\theta)$. Then, if

$$\triangle_n \equiv \int_\Theta \{R(\theta, \delta_{g_n}) - R(\theta, \delta_g)\} g_n(\theta)d\theta \to 0 \text{ as } n \to \infty,$$

δ_g is admissible for estimating $h(\theta)$, where δ_{g_n} is the proper Bayes estmator of $h(\theta)$ with respect to the prior density $g_n(\theta)$, and $R(\theta, \delta)$ is the risk of an estimator $\delta(X)$ of $h(\theta)$.

Now, let $\{h_n\}$ be a sequence of absolutely continuous functions defined on Θ satisfying conditions (I), (II), and (III) of lemma 2.1. Consider a sequence $\{g_n\}$ of prior densities over Θ *with* respect to Lebesgue measure such that $g_n(\theta) = h_n^2(\theta) \cdot g(\theta)$. Then, the corresponding (proper) Bayes estimator δ_{g_n} is defined by

$$\delta_{g_n}(x) = \frac{x+\alpha}{\eta+1} + \frac{I_x(g_{on}')}{(\eta+1)\, I_x(g_{on})}, \quad x \in \mathbf{A},$$

where $g_{on}(\theta) = g_n(\theta) \cdot \exp\{-\alpha\theta - p(\theta) + (\eta+1)\int_a^\theta h(t)dt\}$,
under the assumptions

$$I_x(\,|\,g_{on}'\,|\,) < \infty \text{ for all } x \in \mathbf{A} \tag{2.5}$$

and

$$\lim_{\theta\to\underline{\theta}} g_{on}(\theta) \cdot \exp\{\theta(x+\alpha) - (\eta+1)\int_a^\theta h(t)dt\} = 0 = \lim_{\theta\to\bar{\theta}} g_{on}(\theta) \cdot \exp\{\theta(x+\alpha) - (\eta+1)\int_a^\theta h(t)\, dt\}. \tag{2.6}$$

We now provide a set of sufficient conditions for admissibility of δ_g in (2.2) for estimating $h(\theta)$ under squared error loss.

Theorem 2.1 : Let g be a differentiable prior density satisfying (2.1). Let $h(\theta)$ be an arbitrary (piecewise) continuous function on Θ. Assume that there exists a sequence $\{h_n\}$ of absolutely continuous functions defined on Θ satisfying (I), (II), and (III) of Lemma 2.1, and the conditions (2.5) and (2.6) such that

$$\int_\Theta g(\theta) \cdot [h_n'(\theta)]^2 d\theta \to 0 \text{ as } n \to \infty. \tag{2.7}$$

Also, assume that

$$\int_\Theta \frac{[g'(\theta)]^2}{g(\theta)} d\theta < \infty \tag{2.8}$$

and

$$\int_\Theta g(\theta) \cdot [\alpha + p'(\theta) - (\eta+1)h(\theta)]^2\, d\theta < \infty. \tag{2.9}$$

Then, δ_g in (2.2) is admissible for estimating $h(\theta)$ under squared error loss.

Proof : · By Lemma 2.1, it is enough to show that

$$\triangle_n \equiv \int_\Theta \{R(\theta,\delta_{g_n}) - R(\theta,\delta_g)\} g_n(\theta) d\theta \to 0 \text{ as } n \to \infty,$$

where $g_n(\theta) = h_n^2(\theta) g(\theta)$, $\theta \in \Theta$.
Now, applying Fubini's theorem yields

$$\begin{aligned}\triangle_n &= \int_\Theta \int_A [\{\delta_g(x) - h(\theta)\}^2 - \{\delta_{g_n}(x) - h(\theta)\}^2]\; f(x\,;\theta) g_n(\theta) d\mu(x) d\theta \\ &= \int_A \{[\delta_g(x)]^2 - [\delta_{g_n}(x)]^2\} \{\int_\Theta f(x\,;\theta) g_n(\theta) d\theta\}\, d\mu(x) \\ &\quad - 2\int_A [\delta_g(x) - \delta_{g_n}(x)][\int_\Theta f(x\,;\theta) h(\theta) g_n(\theta) d\theta]\, d\mu(x) \\ &= \int_A [\delta_g(x) - \delta_{g_n}(x)]^2 I_x^*(g_n) d\mu(x) \\ &= \frac{1}{(\eta+1)^2} \int_A \Big[\frac{I_x(g_o')}{I_x(g_o)} - \frac{I_x(g_{on}')}{I_x(g_{on})}\Big]^2 I_x^*(g_n) d\mu(x),\end{aligned} \tag{2.10}$$

where $I_x^*(t) = \int_\Theta t(\theta) f(x\,;\theta) d\theta$.
But, $I_x^*(g_n) = \int_\Theta g_n(\theta) \cdot e^{\theta x - p(\theta)}\, d\theta$

$$= \int_\Theta g_n(\theta) \cdot \exp\{-\alpha\theta - p(\theta) + (\eta+1)\int_a^\theta h(t)dt\} \cdot \exp\{\theta(x+\alpha) - (\eta+1)\int_a^\theta h(t)dt\}\, d\theta$$
$$= I_x(g_{on}).$$

Hence, since $g_{on}' = g_o'h_n^2 + 2g_oh_nh_n'$, (2.10) becomes

$$(\eta+1)^2 \triangle_n = \int_A \Big[\frac{I_x(g_o')}{I_x(g_o)} - \frac{I_x(g_{on}')}{I_x(g_{on})} \Big]^2 I_x(g_{on})\, d\mu(x)$$
$$= \int_A \Big[\frac{I_x(g_o')}{I_x(g_o)} - \frac{I_x(g_o'h_n^2)}{I_x(g_oh_n^2)} - 2\frac{I_x(g_oh_nh_n')}{I_x(g_oh_n^2)} \Big]^2 I_x(g_oh_n^2)\, d\mu(x)$$
$$\leq 8\int_A \frac{[I_x(g_oh_nh_n')]^2}{I_x(g_oh_n^2)}\, d\mu(x) + 2\int_A \Big[\frac{I_x(g_o')}{I_x(g_o)} - \frac{I_x(g_o'h_n^2)}{I_x(g_oh_n^2)} \Big]^2 I_x(g_oh_n^2)\, d\mu(x)$$
$$= 8A_n + 2B_n, \text{ say.} \qquad (2.11)$$

First, consider the term, A_n, in (2.11). Now for each $x \in \mathbf{A}$, using the Cauchy-Schwartz Inequality,

$$[I_x(g_oh_nh_n')]^2$$
$$= \Big[\int_{\Theta} g_o(\theta)h_n(\theta)h_n'(\theta) \cdot \exp\{\theta(x+\alpha)-(\eta+1)\textstyle\int_a^\theta h(t)dt\}\, d\theta\Big]^2$$
$$\leq \Big[\int_{\Theta} g_o(\theta)h_n^2(\theta) \cdot \exp\{\theta(x+\alpha)-(\eta+1)\textstyle\int_a^\theta h(t)dt\}\, d\theta\Big]$$
$$\cdot \Big[\int_{\Theta} g_o(\theta)\, \{h_n'(\theta)\}^2 \cdot \exp\{\theta(x+\alpha)-(\eta+1)\textstyle\int_a^\theta h(t)dt\}\, d\theta\Big]$$
$$= I_x(g_oh_n^2) \int_{\Theta} g(\theta)\, [h_n'(\theta)]^2 \cdot e^{\theta x - p(\theta)}\, d\theta$$
$$= I_x(g_oh_n^2)\; I_x^*(g \cdot h_n'^2). \qquad (2.12)$$

Substituting (2.12) into A_n yields, by Fubini's theorem and the condition (2.7) of Theorem 2.1,

$$A_n = \int_A \frac{[I_x(g_oh_nh_n')]^2}{I_x(g_oh_n^2)}\, d\mu(x)$$
$$\leq \int_A I_x^*(g \cdot h_n'^2)\, d\mu(x)$$
$$= \int_A \int_{\Theta} g(\theta)\, [h_n'(\theta)]^2 \cdot e^{\theta x - p(\theta)}\, d\mu(x)$$
$$= \int_{\Theta} g(\theta)\, [h_n'(\theta)]^2\, d\theta \to 0 \text{ as } n \to \infty. \qquad (2.13)$$

Next, consider the term, B_n, in (2.11). Then, using the Cauchy-Schwartz Inequality and the fact that for all $n \geq 1$, $h_n^2(\theta) \leq \mathbf{K} < \infty$ a.e. (Lebesgue) by (I) and (III) of Theorem 2.1,

$$B_n = \int_A \Big[\frac{I_x(g_o')}{I_x(g_o)} - \frac{I_x(g_o' \cdot h_n^2)}{I_x(g_o \cdot h_n^2)} \Big]^2 I_x(g_o \cdot h_n^2)\, d\mu(x)$$
$$= \int_A \frac{[\, I_x(g_o')I_x(g_o \cdot h_n^2)/I_x(g_o) - I_x(g_o' \cdot h_n^2)\,]^2}{I_x(g_o \cdot h_n^2)}\, d\mu(x)$$
$$= \int_A \frac{[\, I_x\{g_o \cdot h_n^2(I_x(g_o')/I_x(g_o) - g_o' \cdot h_n^2/(g_o \cdot h_n^2))\}\,]^2}{I_x(g_o \cdot h_n^2)}\, d\mu(x)$$
$$\leq \int_A \frac{I_x(g_o \cdot h_n^2) \cdot I_x[g_oh_n^2(I_x(g_o')/I_x(g_o) - g_o'/g_o)^2]}{I_x(g_o \cdot h_n^2)}\, d\mu(x)$$
$$= \int_A I_x\Big[g_o \cdot h_n^2\Big\{ \frac{I_x(g_o')}{I_x(g_o)} - \frac{g_o'}{g_o} \Big\}^2 \Big]\, d\mu(x)$$
$$\leq K \int_A I_x\Big[g_o\Big\{ \frac{I_x(g_o')}{I_x(g_o)} - \frac{g_o'}{g_o} \Big\}^2 \Big]\, d\mu(x). \qquad (2.14)$$

Now, the integrand in (2.14) becomes, for each $x \in \mathbf{A}$,

$$
\begin{aligned}
& I_x[g_o \mid g_o'/g_o - I_x(g_o')/I_x(g_o) \mid ^2] \\
&= I_x[g_o'^2/g_o - 2g_o' \cdot I_x(g_o')/I_x(g_o) + g_o\{I_x(g_o')/I_x(g_o)\}^2] \\
&= I_x[g_o'^2/g_o] - [I_x(g_o')]^2/I_x(g_o) \le I_x[g_o'^2/g_o]. \qquad (2.15)
\end{aligned}
$$

Hence, substituting (2.15) into (2.14), we have, by Fubini's theorem,

$$
\begin{aligned}
B_n &\le K\int_A I_x(g_o'^2/g_o)\, d\mu(x) \\
&= K\int_A \{\int_\Theta \frac{[g_o'(\theta)]^2}{g_o(\theta)} \cdot \exp[\theta(x+\alpha) - (\eta+1)\int_a^\theta h(t)d(t)]\, d\theta\}\, d\mu(x) \\
&= K\int_\Theta \{\int_A e^{\theta x - p(\theta)} d\mu(x)\} \cdot \frac{[g_o'(\theta)]^2}{g_o(\theta)} \cdot \exp\{\alpha\theta + p(\theta) - (\eta+1)\int_a^\theta h(t)dt\}\, d\theta \\
&= K\int_\Theta \frac{[g_o'(\theta)]^2}{g_o(\theta)} \cdot \exp\{\alpha\theta + p(\theta) - (\eta+1)\int_a^\theta h(t)dt\} d\theta. \qquad (2.16)
\end{aligned}
$$

But,

$$
\begin{aligned}
g_o'(\theta) &= [\, g(\theta) \cdot \exp\{-\alpha\theta - p(\theta) + (\eta+1)\int_a^\theta h(t)dt\}\,]' \\
&= [\, g'(\theta) + g(\theta) \cdot \{-\alpha - p'(\theta) + (\eta+1)h(\theta)\}\,] \cdot \exp\{-\alpha\theta - p(\theta) + (\eta+1)\int_a^\theta h(t)dt\}.
\end{aligned}
$$

Hence, (2.16) becomes, using conditions (2.8) and (2.9) of Theorem 2.1,

$$
\begin{aligned}
B_n &\le K\int_\Theta \frac{[g_o'(\theta)]^2}{g_o(\theta)} \cdot \exp\{\alpha\theta + p(\theta) - (\eta+1)\int_a^\theta h(t)dt\}\, d\theta \\
&= K\int_\Theta \frac{\{g'(\theta) + g(\theta) \cdot [-\alpha - p'(\theta) + (\eta+1)h(\theta)]\}^2}{g(\theta) \cdot \exp\{-\alpha\theta - p(\theta) + (\eta+1)\int_a^\theta h(t)dt\}} \\
&\qquad \cdot \exp\{-\alpha\theta - p(\theta) + (\eta+1)\int_a^\theta h(t)dt\}\, d\theta \\
&= K\int_\Theta \frac{\{g'(\theta) + g(\theta) \cdot [-\alpha - p'(\theta) + (\eta+1)h(\theta)]\}^2}{g(\theta)}\, d\theta \\
&\le 2K\{\int_\Theta \frac{[g_o'(\theta)]^2}{g_o(\theta)}\, d\theta + \int_\Theta g(\theta) \cdot [-\alpha - p'(\theta) + (\eta+1)h(\theta)]^2 d\theta\} \\
&< \infty. \qquad (2.17)
\end{aligned}
$$

Now, recall that, from (2.14),

$$B_n = \int_A b_n(x)\, d\mu(x) \text{ for } n \ge 1,$$

where, for each $x \in A$ and $n \ge 1$,

$$b_n(x) = [\, \frac{I_x(g_o')}{I_x(g_o)} - \frac{I_x(g_o' \cdot h_n^2)}{I_x(g_o \cdot h_n^2)}\,]^2 \cdot I_x(g_o \cdot h_n^2).$$

Then using the condition (III) of Theorem 2.1 yields, for all $x \in A$,

$$b_n(x) \to 0 \text{ as } n \to \infty. \qquad (2.18)$$

Hence, by the Dominated Convergence Theorem, (2.17) and (2.18) give

$$B_n \to 0 \text{ as } n \to \infty. \qquad (2.19)$$

Therefore, from(2.11), (2.13), and (2.19), we have

$$\Delta_n \to 0 \text{ as } n \to \infty.$$

When $\Theta = \mathbf{R}^1$, we use the sequence $\{h_n\}$ given in Brown and Hwang(1982) such that

$$h_1(\theta) = \begin{cases} 1 & , \ |\theta| \le 1 \\ 0 & , \ |\theta| > 1 \end{cases}$$

and

$$h_n(\theta) = \begin{cases} 1 & , \ |\theta| \le 1 \\ 1 - \dfrac{\ln(|\theta|)}{\ln(n)} & , \ 1 \le |\theta| \le n \\ 0 & , \ |\theta| \ge n \ , \ n=2,3,\cdots. \end{cases} \tag{2.20}$$

Also, when $\Theta \subset \mathbf{R}^1$, the proper subset, we take the sequence $\{h_n\}$ given in Brown and Hwang (1982) such that

$$h_1(\theta) = \begin{cases} 1 & , \ \wedge \le 1 \\ 0 & , \ \wedge > 1 \end{cases}$$

and

$$h_n(\theta) = \begin{cases} 1 & , \ \wedge \le 1 \\ 1 - \dfrac{\ln(\wedge)}{\ln(n)} & , \ 1 \le \wedge \le n \\ 0 & , \ \wedge \ge n \ , \ n=2,3,\cdots. \end{cases} \tag{2.20}$$

where $\wedge^2 = \wedge^2(\theta) = \ln^2 |\theta|$.

Note that the sequences (2.20) and (2.21) satisfy the conditions (I), (II), and (III) of Lemma 2.1. Then, we have the following corollaries as special cases of Theorem 2.1:

Corollary 2.1: Let $\Theta = \mathbf{R}^1$. If

$$\int_{R^1-S} \frac{g(\theta)}{|\theta|^2 \ln^2(|\theta| \vee 2)} d\theta < \infty \ , \tag{2.22}$$

where $S = \{\theta : |\theta| < 1\}$ and $a \vee b = \max(a, b)$, and the conditions (2.1), (2.8), and (2.9) of Theorem 2.1 are satisfied, then

$$\delta_g(X) = \frac{X+\alpha}{\eta+1} + \frac{I_x(g_o')}{(\eta+1) I_x(g_o)}$$

is admissible for estimating $h(\theta)$ under squared error loss.

Proof: It suffices to show that if (2.22) holds, then the condition (2.7) of Theorem 2.1 is satisfied. Now, for $h_n(\theta)$ in (2.20), we have

$$h_n'(\theta) = -\frac{\theta}{|\theta|^2 \ln(n)} \chi_{[1,n]}(|\theta|) \quad , \ n=2,3,\cdots$$

where $\chi_B(y) = \begin{cases} 1 , & y \in B \\ 0 , & y \notin B \end{cases}$.

Hence,

$$[h_n'(\theta)]^2 = \frac{\theta^2}{|\theta|^4 \ln^2(n)} \chi_{[1,n]}(|\theta|)$$

$$\leq \frac{1}{|\theta|^2 \ln^2(|\theta| \vee 2)} \chi_{[1,n]}(|\theta|)$$
$$\leq \frac{1}{|\theta|^2 \ln^2(|\theta| \vee 2)} \chi_{[1,\infty]}(|\theta|) \quad ,$$

and clearly $[h_n'(\theta)]^2 \to 0$ as $n \to \infty$. Therefore, by the Dominated Convergence Theorem, the condition (2.7) of Theorem 2.1 is satisfied.

Corollary 2.2 : Let Θ $\mathbf{R}^1$, the proper subset, If

$$\int_{\mathbf{R}^1 - S'} \frac{g(\theta)}{|\theta|^2 \wedge^2(\theta) \ln^2(\wedge(\theta) \vee 2)} d\theta < \infty \ , \qquad (2.23)$$

where $S' = \{\theta : \wedge(\theta) < 1\}$, and the conditions (2.1), (2.8), and (2.9) of Theorem 2.1 are satisfied, then

$$\delta_g(X) = \frac{X+\alpha}{\eta+1} + \frac{I_x(g_o')}{(\eta+1)\, I_x(g_o)}$$

is admissible for estimating $h(\theta)$ under squared error loss.

The proof of this Corollary is similar to that of Corollary 2.1.

3. Examples

In the following examples we use, respectively, Corollaries 2.1 and 2.2 for $\Theta = \mathbf{R}^1$ and $\Theta \subsetneq \mathbf{R}^1$. For simplicity we consider prior distributions with densities of the form $g(\theta) = \exp\{\alpha\theta + p(\theta) - (\eta+1)\int_d^\theta h(t)dt\}$ for which the conditions (2.8) and (2.9) of Corollaries 2.1 and 2.2 are equivalent.

Example 3.1 : Let $X \sim N(\theta, 1)$, $\theta \in \Theta = \mathbf{R}^1$. Then, $p(\theta) = (1/2)\theta^2$. We wish to estimate $h(\theta) = \theta^k$ under squared error loss where k is an odd positive integer. In this case, $g(\theta) = \exp\{\alpha\theta + \frac{1}{2}\theta^2 - \frac{\eta+1}{k+1}\theta^{k+1}\}$, where, without loss of generality, we take $d=0$. Since $g_o(\theta) \equiv 1$, $\delta_g(X) = \frac{X+\alpha}{\eta+1}$, $\alpha \in \mathbf{R}^1$, $\eta(\neq 1) \in \mathbf{R}^1$. It can be easily shown that for $x \in \mathbf{R}^1$,

$$I_x(g_o) = \int_{-\infty}^{\infty} \exp\{\theta(x+\alpha) - \frac{\eta+1}{k+1}\theta^{k+1}\}\, d\theta < \infty$$

if α $\mathbf{R}^1$ and $\eta > -1$ for all positive odd k's, and hence $\delta_g(X) = (X+\alpha)/(\eta+1)$ is the generalized Bayes estimator of θ^k, k a positive odd integer, if $\alpha \in \mathbf{R}^1$ and $\eta > -1$. Now, look at the conditions (2.9) and (2.22) of Corollary 2.1. Simple algebra shows that (2.9) is satisfied for either $\alpha=0$, $\eta=0$ or $\alpha \in \mathbf{R}^1$, $\eta > 0$ when $k=1$, or $\alpha \in \mathbf{R}^1$ and $\eta > -1$ when $k \geq 3$. In order to check condition (2.22) it suffices to show the finiteness of

$$\int_{|\theta|>2} \frac{\exp\{\alpha\theta + (1/2)\theta^2 - [(\eta+1)/(k+1)]\theta^{k+1}\}}{|\theta|^2 \ln^2|\theta|} d\theta.$$

But, it is easy to show that the above integral is finite if either $\alpha=0$, $\eta=0$ or $\alpha \in \mathbf{R}^1$, $\eta > 0$ for $k=1$, or $\alpha \in \mathbf{R}^1$, $\eta > -1$ for $k \geq 3$. Therefore, $\delta_g(X) = (X+\alpha)/(\eta+1)$ is admissible for estimating θ^k, k a positive odd integer if either $\alpha=0$, $\eta=0$ or $\alpha \in \mathbf{R}^1$, $\eta>0$ for $k=1$, or $\alpha \in \mathbf{R}^1$, $\eta>-1$ for $k\geq 3$. These results confirm those of Gupta(1966) for $k=1$, and Ghosh and Meeder (1977) for $k\geq 3$, respectively.

Example 3.2 : Let $X \sim \text{Binomial}[n, \frac{e^\theta}{1+e^\theta}], \theta \in \Theta = \mathbf{R}^1$. Here, $p(\theta) = n\ \ln(1+e^\theta)$. It is desired to estimate $h(\theta) = (\frac{e^\theta}{1+e^\theta})^2$ under squard error loss. In this case,

$$g(\theta) = (1+e^\theta)^{n-(\eta+1)} \cdot \exp\{\alpha\theta + \frac{(\eta+1)e^\theta}{1+e^\theta}\},$$

where, without loss of generality, we take $d=0$. Since $g_o(\theta)=1$, $\delta_g(X) = (X+\alpha)/(\eta+1)$. Note that for all $x = 0, 1, 2, \cdots, n$,

$$\begin{aligned} I_x(g_o) &= \int_{-\infty}^{\infty} (1+e^\theta)^{n-(\eta+1)} \cdot \exp\{\theta(x+\alpha) + \frac{(\eta+1)e^\theta}{(1+e^\theta)}\}\, d\theta \\ &= \int_0^1 y^{x+\alpha-1} (1-y)^{\eta-(x+\alpha)} e^{(\eta+1)y}\, dy \\ &< \infty \qquad \text{if } \alpha>0 \text{ and } \eta>n+\alpha-1. \end{aligned}$$

Hence, $\delta_g(X)$ is the generalized Bayes estimator of $h(\theta)$ with respect to g if $\alpha>0$ and $\eta>n+\alpha-1$. Now, consider the conditions (2.9) and (2.22) of Corollary 2.1. Then, simple algebra shows that both (2.9) and (2.22) are satisfied if either $\alpha=0$, $\eta\geq n-1$, or $\alpha>0, \eta>n+\alpha-1$. Hence, $\delta_g(X) = (X+\alpha)/(\eta+1)$ is admissible for estimating $h(\theta) = (\frac{e^\theta}{1+e^\theta})^2$ under squared error loss if either $\alpha=0$, $\eta\geq n-1$, or $\alpha>0$ and $\eta>n+\alpha-1$. This result contains admissibility of X/n, a result by Ghosh and Meeden(1981). It is interesting to note that for $\alpha=0$ and $\eta=n-1$, $\delta_g(X) = X/n$ is not the generalized Bayes estimator with respect to g since for $\alpha=0$ or $\eta=n+\alpha-1$, $I_x(g_o)=\infty$ for $x=0$ or n, respectively and hence condition (2.3) fails.

Example 3.3 : Let $X \sim \text{Gamma}(\beta, \theta)$ with the density

$$f(x\,;\theta) = \frac{(-\theta)^\beta}{\Gamma(\beta)} x^{\beta-1} e^{\theta x},\ \beta>0(\text{known}),\ x>0\ ,\ \theta \in \Theta = (-\infty, 0) \subsetneq \mathbf{R}^1.$$

In this case $p(\theta) = -\beta \cdot \ln(-\theta)$. We want to estimate $h(\theta) = -1/\theta$ under squared error loss. Take $g(\theta) = c(d)\cdot(-\theta)^{\eta-\beta+1}\cdot e^{\alpha\theta}$, $\eta \neq -1$, $\alpha \in \mathbf{R}^1$, where $-\infty<d<0$ and $c(d)$ is a constant depending on d. Since $g_o(\theta)\equiv 1$, $\delta_g(X) = (X+\alpha)/(\eta+1)$. Here, $I_x(g_o)<\infty$ for all $x>0$ if $\alpha\geq 0$ and $\eta>-2(\eta\neq -1)$. Hence, $\delta_g(X)$ is the generalized Bayes estimator of $-1/\theta$ with respect to g if $\alpha\geq 0$ and $\eta>-2(\eta\neq-1)$. Now, we want to check the conditions (2.9) and (2.23) of Corollary 2.2. First, condition (2.9) is satisfied if either $\alpha=0$, $\eta=\beta-1$ or $\alpha>0$, $\eta>\beta$. In order to check (2.23) it suffices to consider the finiteness of

$$\int_{\{\theta\in\Theta : \Lambda(\theta)\geq 2\}} \frac{(-\theta)^{\eta-\beta+1}\cdot e^{\alpha\theta}}{|\theta|^2 \Lambda^2(\theta) \ln^2(\Lambda(\theta))}\, d\theta \quad .$$

where $\Lambda^2(\theta) = \ln^2(-\theta)$. After simple algebra we can show that the above integral is finite if either $\alpha=0$, $\eta=\beta$, or $\alpha>0$, $\eta\geq\beta$. Hence, $\delta_g(X) = (X+\alpha)/(\eta+1)$ is admissible for estimating $h(\theta) = -1/\theta$ under squared error loss if $\alpha>0$ and $\eta>\beta$.

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