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A Parametric Empirical Bayesian Method for Multiple Comparisons[+]

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ABSTRACT

For all pairwise comparisons of treatments, Bayesian simultaneous confidence intervals are proposed and studied. First Bayesian solutions are obtained for a fixed prior, and then prior parameters are estimated by a parametric empirical Bayesian method. The nominal confidence level is shown to be controlled asymptotically. An extension to the unbalanced design is also considered.

1. Introduction

Let $\bar{y}_1, \cdots \bar{y}_k$ denote treatment means based on n replications each, such that $\bar{y}_i \sim N(\theta_i, \sigma^2/n)$, $i=1, \cdots k$; independently, and let s^2 be an estimate of the error variance σ^2 such that $rs^2/\sigma^2 \sim \chi_r^2$, independently of $\bar{y}_1, \cdots, \bar{y}_k$. This setting often represents a reduction of many balanced designs for the comparison of treatments.

Tukey(1953) proposed simultaneous confidence intervals for all pairwise differences $\theta_i - \theta_j$ such that

$$\theta_i - \theta_j \in [\bar{y}_i - \bar{y}_j \pm Q_{k,r}^{(\alpha)} s/\sqrt{n}], \quad (1 \leq i < j \leq k) \tag{1.1}$$

where $Q_{k,r}^{(\alpha)}$ denotes the upper α quantile of the Studentized range distribution with parameters k and r.

This procedure, known as T-procedure, has been the prototype of multiple comparison procedures. Following the T-procedure, various formulations including stepwise multiple tests have been proposed for multiple comparison. See Hochberg and Tamhane(1987) for discussions about diverse multiple comparison methods.

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Bayesian approach to multiple comparison has also been attempted, and Waller and Duncan (1969) was the first to provide a Bayesian solution to this problem. Dixon and Duncan(1975) derived minimum Bayes risk interval estimates for pairwise differences, and Duncan and Godbold (1979) extended the results to the unbalanced case. In all these Bayesian studies, an additive loss was assumed and therefore the procedures given by them have the nature of contorlling *per-comparison* error rate.

The purpose of this article is to propose a Bayesian multiple comparison procedure which controls *experimentwise* error rate. First we consider Bayesian simultaneous confidence intervals with respect to a fixed prior, and then prior parameters are estimated by a parametric empirical Bayesian method. It is shown that the nominal confidence level is controlled asymptotically for the proposed procedure. Finally, the extension to the unbalanced case is also given.

2. A Parametric Empirical Bayes Procedure

The prior distribution of $\theta_1, \cdots \theta_k$ for given σ^2 is taken to be

$$\theta_i \mid \sigma^2 \sim N(\psi,\ \sigma^2/c),\ i=1,\cdots,k,\ \text{independently},$$

and the prior of σ^2 is assumed to be

$$p(\sigma^2)=1/\sigma^2.$$

Then the joint posterior distribution of $\theta_1,\cdots,\theta_k$, given $\bar{y}_1,\cdots,\bar{y}_k$ and s^2, is as follows:

$$(\theta_1,\cdots,\theta_k)' \mid \bar{y}_1,\cdots,\bar{y}_k,\ s^2 \sim T_k(f,\ \mathbf{x},\ s_0^2 I_k), \tag{2.1}$$

where $T_k(f,\ \mathbf{x},\ s_0^2 I_k)$ denotes the k-variate t distribution with p.d.f. proportional to

$$\left[1+\frac{1}{f}\ \frac{\Sigma_{i=1}^k(\theta_i-x_i)^2}{s_0^2}\right]^{-(f+k)/2},$$

with $\mathbf{x}=(x_1,\cdots,x_k)'$ and

$$f=r+k,\ x_i=\frac{n\bar{y}_i+c\psi}{n+c},\ i=1,\cdots,k \tag{2.2}$$

$$s_0^2=\frac{1}{f}\left[\frac{rs^2}{n+c}+\frac{nc\ \Sigma_{i=1}^k(\bar{y}_i-\psi)^2}{(n+c)^2}\right].$$

The posterior distribution in (2.1) implies that, for $\mathbf{y}=(\bar{y}_1,\cdots,\bar{y}_k)'$,

$$\max_{1\leq i<j\leq k} \frac{\mid (\theta_i-x_i)-(\theta_j-x_j)\mid}{s_0} \mid \mathbf{y},\ s^2 \sim Q_{k,f},$$

where $Q_{k,f}$ denotes the Studentized range distribution with parameter k and f degrees of freedom. Letting $Q_{k,f}^{(\alpha)}$ be the upper α quantile of $Q_{k,f}$, we have

$$P_{\psi,c}[\theta_i-\theta_j \in x_i-x_j \pm Q_{k,f}^{(\alpha)}\ s_0,\ \forall i\neq j \mid \mathbf{y},\ s^2]=1-\alpha, \tag{2.3}$$

which provides a $100(1-\alpha)\%$ simultaneous credible set for $\theta_i-\theta_j$, $1\leq i<j\leq k$, and it will be called the *T-type Bayesian MCA(all-pairwise multiple comparison) procedure*. Note that, in the case of noninformative prior $p(\theta,\ \sigma^2)=1/\sigma^2$, f becomes r and we can formally plug in 0 for c in (2.2). Thus in such a case the T-type Bayesian MCA procedure coincides with the classical Tukey's T-procedure in (1.1).

Before we present a parametric empirical Bayes procedure, consider an optimal property of T-type Bayesian MCA procedure. Assume the following loss structure for interval estimates $[a_{ij}, b_{ij}]$ of $\theta_i-\theta_j$ for $1\leq i<j\leq k$:

$$L(\mathbf{a}, \mathbf{b}: \theta)=\sum_{i<j} l(a_{ij}, b_{ij}: \theta_i-\theta_j), \tag{2.4}$$

where, for a fixed constant $\beta\in(0, \frac{1}{2})$,

$$l(a, b:\Delta)=\begin{cases}(b-a)+(a-\Delta)/\beta & \text{for } \Delta<a\\ (b-a) & \text{for } a\leq\Delta\leq b\\ (b-a)+(\Delta-b)/\beta & \text{for } \Delta>b.\end{cases} \tag{2.5}$$

Then we have the following optimality result. Here $t_f^{(\beta)}$ denotes the upper β quantile of the t distribution with f degrees of freedom.

Theorem 2.1 *The T-type Bayesian MCA procedure given by (2.3) has minimum Bayes risk under the loss structure (2.4) and (2.5) where* α and β values are related by $\sqrt{2}t_f^{(\beta)}=Q_{k,f}^{(\alpha)}$.

Proof. See the Appendix.

In order to apply the T-type Bayesian MCA procedure, the values of prior parameters ψ and c should be specified. When this is difficult, empirical estimations of them can be done using the marginals.

Note that for any fixed σ^2,

$$\bar{y}_i\mid\sigma^2\sim N(\psi, \sigma^2(1/n+1/c)),\ i=1,\cdots,k, \text{ independently.}$$

Hence, letting

$$\bar{y}=\frac{1}{k}\sum_{i=1}^{k}\bar{y}_i \text{ and } MS_T=\frac{n}{k-1}\sum_{i=1}^{k}(\bar{y}_i-\bar{y})^2,$$

$\bar{y}$ and MS_T/n may be considered as the ordinary unbiased estimates of ψ and $\sigma^2(1/n+1/c)$ respectively for fixed σ^2. Since s^2 is unbiased for σ^2, c can be estimated by $n/(F-1)$, where $F=MS_T/s^2$, the ordinary F-ratio in analysis of variance.

After making some corrections for negative values of $n/(F-1)$, we propose the estimates of ψ and c as follows:

$$\hat{\psi}=\bar{y}$$

$$\hat{c}=\begin{cases} n/(F-1), & \text{if } F>1\\ \infty, & \text{if } F\leq 1.\end{cases} \tag{2.6}$$

By substituting these estimates in (2.2), we have

$$\hat{x}_i=\begin{cases}(1-\frac{1}{F})\bar{y}_i+\frac{1}{F}\bar{y}, & \text{if } F>1\\ \bar{y}, & \text{if } F\leq 1\end{cases} \tag{2.7}$$

$$\hat{s}_0^2=\begin{cases} 1-\frac{1}{F})(1-\frac{1}{r+k})s^2/\text{n} & \text{if } F>1\\ 0 & \text{if } F\leq 1\end{cases}$$

Applying (2.7) to (2.3), we have the following procedure, which will be called the *T-type parametric empirical Bayes(PEB) MCA procedure*.

$$(\theta_i-\theta_j)\in(1-\frac{1}{F})^+\{(\bar{y}_i-\bar{y}_j)\pm Q_{k,r+k}^{(\alpha)}((1-\frac{1}{r+k})/(1-\frac{1}{F})^+)^{1/2}\frac{s}{\sqrt{n}}\},\ \forall i\neq j, \tag{2.8}$$

where $x^+=max(x,\ 0)$.

It should be noticed that the T-type PEB MCA procedure is very sensitive to the value of F, while the classical T-type procedure is irrelevant to F. For large values of F the former contains less 0's than the latter, and for small values of F the former contains more 0's. The following example with artificial data illustrates this point.

Example 2.1. Suppose that the following artificial data given by Table 2.1 are taken from a two-way layout with one observation per cell.

Table 2.1. Data from Two-way Layout

	B_1	B_2	B_3	B_4	$\bar{y}_{i\cdot}$
A_1	72.4	69.9	72.6	72.7	71.90
A_2	79,1	80.9	85.9	77.7	80.90
A_3	65.4	69.4	67.8	64.6	66.80
A_4	70.2	68.6	67.3	63.3	67.35
A_5	77.7	75.0	81.4	74.1	77.05
$\bar{y}_{\cdot j}$	72.96	72.76	75.00	70.48	$\bar{y}$=72.80

The numerical results of the analysis of variance are presented in Table 2.2 for convenience, which show the high significance of the factor A even at 1%, and no significance of the factor B at 5% ($F_{3,12}^{(0.05)}=3.49$).

Table 2.2. ANOVA Table for the Data of Table 2.1

Source	S.S.	d.f.	M.S.	F
A	600.740	4	150.185	27.510
B	51.248	3	17.083	3.129
Error	65.512	12	5.459	
Total	717.500	19		

Suppose we choose a confidence level $(1-\alpha)=0.95$. Then $Q_{5,17}^{(0.05)}=4.303$, $Q_{5,12}^{(0.05)}=4.508$, $Q_{4,16}^{(0.05)}=4.046$, and $Q_{4,12}^{(0.05)}=4.199$. The proposed T-type PEB MCA and classical T-procedure confidence intervals are thus as given in Table 2.3, and the graphical representations are as follows, where any means underscored by the same line are not significanlty different.

	A_3	A_4	A_1	A_5	A_2	B_4	B_2	B_1	B_3
PEB MCA									
Classical T									

As mentioned previously, the T-type PEB MCA procedure separates more differences of means than the classical T-procedure for the factor A which has a large value of F. On the othe hand, for the factor B which has a small value of $F(<F_{3,12}^{(0.05)})$, the former separates no difference of means while the latter separates B_4 and B_3.

Table 2.3. 95% Confidence Intervals (C.I.) by Two Procedures

differences	C.I. by PEB MCA	C.I. by Classical T
(a_2-a_1)	(3.885, 13.460)*	(3.734, 14.267)*
(a_3-a_1)	(-9.702, -0.127)*	(-10.367, 0.167)
(a_4-a_1)	(-9.172, 0.403)	(-9.817, 0.717)
(a_5-a_1)	(0.175, 9.750)*	(-0.117, 10.417)
(a_3-a_2)	(-18.375, -8.800)*	(-19.367, -8.833)*
(a_4-a_2)	(-17.845, -8.270)*	(-18.817, -8.283)*
(a_5-a_2)	(-8.498, 1.077)	(-9.117, 1.417)
(a_4-a_3)	(-4.257, 5.317)	(-4.717, 5.817)
(a_5-a_3)	(5.090, 14.665)*	(4.983, 15.517)*
(a_5-a_4)	(4.560, 14.135)*	(4.433, 14.967)*
(b_2-b_1)	(-3.513, 3.241)	(-4.588, 4.188)
(b_3-b_1)	(-1.989, 4.765)	(-2.348, 6.428)
(b_4-b_1)	(-5.064, 1.689)	(-6.868, 1.908)
(b_3-b_2)	(-1.852, 4.901)	(-2.148, 6.628)
(b_4-b_2)	(-4.928, 1.825)	(-6.668, 2.108)
(b_4-b_3)	(-6.452, 0.301)	(-8.908, -0.132)*

* denotes the interval not containing zero.

3. Asymptotic Properties

In this section we consider the coverage probability of the T-type PEB MCA procedure. From now on the event *coverage* is the event defined by (2.8).

First we consider the asymptotic coverage probability as $n\to\infty$ from frequentist's point of view.

Theorem 3.1. *Suppose that θ_i's are not all equal, and that $r\to\infty$ as $n\to\infty$. Then*

$$\lim_{n\to\infty} P[coverage \mid \theta_1,\cdots,\theta_k,\ \sigma^2]=1-\alpha.$$

Proof. Since θ_i's are not all equal, we have

$$P[coverage \mid \theta_1, \cdots, \theta_k, \sigma^2]$$
$$=P[F>1,\ (\theta_i-\theta_j)\in(1-\frac{1}{F})\{(\bar{y}_i-\bar{y}_j) \tag{3.1}$$
$$\pm Q_{k,r+k}^{(\alpha)}((1-\frac{1}{r+k})/(1-\frac{1}{F}))^{1/2}\frac{s}{\sqrt{n}}\},\ \forall i\neq j/\theta_1,\cdots\theta_k,\ \sigma^2]$$
$$=P[F>1,\ \max_{i\neq j} \mid A_n\{\frac{(\bar{y}_i-\theta_i)-(\bar{y}_j-\theta_j)}{s/\sqrt{n}}-\frac{\sqrt{n}}{(F-1)}\frac{(\theta_i-\theta_j)}{s}\} \mid$$
$$\leq Q_{k,\infty}^{(\alpha)} \mid \theta_1,\cdots,\theta_k,\ \sigma^2],$$

where

$$A_n=(Q_{k,\infty}^{(\alpha)}/Q_{k,r+k}^{(\alpha)})\ \{(1-\frac{1}{F})/(1-\frac{1}{r+k})\}^{1/2}.$$

Note that, as $n\to\infty$,

$$\frac{1}{n}F=\frac{MS_T}{ns^2}=\frac{1}{k-1}\sum_{i=1}^{k}(\bar{y}_i-\bar{y})^2/s^2\xrightarrow{P}\frac{\Sigma_1^k(\theta_i-\bar{\theta})^2}{(k-1)\sigma^2}.$$

Hence in the expression (3.1), as $n\to\infty$, we have

$$A_n\xrightarrow{P}1 \text{ and } \frac{\sqrt{n}}{(F-1)}\frac{(\theta_i-\theta_j)}{s}\xrightarrow{P}0.$$

Moreover it is clear that in (3.1),

$$\frac{(\bar{y}_i-\theta_i)-(\bar{y}_j-\theta_j)}{s/\sqrt{n}}\xrightarrow{d}Z_i-Z_j \text{ as } n\to\infty,$$

where $Z_1,\cdots,Z_k$ are independently distributed as $N(0,\ 1)$.

Summarizing these, we have for given $\theta_1,\cdots,\theta_k$ and σ^2,

$$\max_{i\neq j} \mid A_n\{\frac{(\bar{y}_i-\theta_i)-(\bar{y}_j-\theta_j)}{s/\sqrt{n}}-\frac{\sqrt{n}}{(F-1)}\frac{(\theta_i-\theta_j)}{s}\} \mid \xrightarrow{d}Q_{k,\infty} \text{ as } n\to\infty,$$
$$\text{and} \quad \lim_{n\to\infty}P[F>1 \mid \theta_1,\cdots,\theta_k,\sigma^2]=1.$$

Thus the probability in (3.1) converges to $1-\alpha$ as $n\to\infty$.

It should be noted that, by applying the dominated covergence theorem to Theorem 3.1, we have

$$\lim_{n\to\infty}P_{\psi,c}[coverage]=1-\alpha.$$

for any fixed ψ and $c>0$. Namely, the marginal coverage probability also converges to $1-\alpha$ as $n\to\infty$.

Now, as typical in empirical Bayesian approach, we consider the asymptotic posterior coverage probability as the number of treatments k gets large. For this purpose we need the following Lemmas, whose proofs are given in the Appendix.

Lemma 3.1. *Let $V\sim Q_{k,r}$ and $\frac{1}{r}=O(\frac{1}{k})$. Then $(V-2a_k)/b_k$ has limiting distribution* as $k\to\infty$ *with density*

$$g(z)=2e^{-z}K_0(2e^{-z/2}), \tag{3.2}$$

where $K_0(z)$ is a modified Bessel function of the second kind, and a_k and b_k are given as follows :

$$a_k=(2\log k)^{1/2}-\frac{1}{2}(\log\log k+\log 4\pi)(2\log k)^{-1/2}, \tag{3.3}$$
$$b_k=(2\log k)^{-1/2}.$$

Lemma 3.2 For any fixed $\sigma^2>0$ *and under the assumption of* $\frac{1}{r}=O(\frac{1}{k})$, *the following identities hold as* $k\to\infty$:

$$(i)\ \frac{1}{F}=\frac{c}{n+c}[1+O((\frac{\log\log k}{k})^{1/2})]\ a.s.,$$

$$(ii)\ \frac{\hat{s}_0}{s_0}=1+O((\frac{\log\log k}{k})^{1/2})\ a.s.,$$

$$(iii)\ \max_{1\le i<j\le k}|(\hat{x}_i-\hat{x}_j)-(x_i-x_j)|=O((\frac{\log\log k}{k})^{1/2})[2(2\log k)^{1/2}+o(1)]\ a.s.,$$

$$(iv)\ Q_{k,r}^{(\alpha)}=2a_k+b_kK^{(\alpha)}+o(b_k),$$

where $K^{(\alpha)}$ denotes the upper α quantile of the density g given by (3.2) and a_k, b_k *are given by (3.3).*

It should be remarked that, in Lemma 3.2, s^2 is assumed to be represented as the average of r *i.i.d.* $\sigma^2\chi_1^2$ random variables. Now we are ready to present the asymptotic result for large k as follows :

Theorem 3.2 *For any finite* ψ *and c and under the assumption of* $\frac{1}{r}=O(\frac{1}{k})$, *we have*

$$\lim_{k\to\infty}P_{\psi,c}[coverage\mid \mathbf{y},\ s^2]=1-\alpha\ a.s.$$

Proof. Note that

$$P_{\psi,c}[coverage\mid \mathbf{y},\ s^2]$$
$$=P_{\psi,c}[\frac{\max_{1\le i<j\le k}|(\theta_i-\hat{x}_i)-(\theta_j-\hat{x}_j)|}{s_0}\le Q_{k,r+k}^{(\alpha)}\mid \mathbf{y},\ s^2]. \tag{3.4}$$

Since

$$\max_{1\le i<j\le k}|(\theta_i-\hat{x}_i)-(\theta_j-\hat{x}_j)|$$
$$\in\max_{1\le i<j\le k}|(\theta_i-x_i)-(\theta_j-x_j)|\pm\max_{1\le i<j\le k}|(\hat{x}_i-\hat{x}_j)-(x_i-x_j)|,$$

the probability in (3.4) is bounded by the following two probabilities :

$$P_{\psi,c}[\frac{\max_{1\le i<j\le k}|(\theta_i-x_i)-(\theta_j-x_j)|}{s_0}$$

$$\leq \frac{\hat{s}_0}{s_0} Q_{k,r+k}^{(\alpha)} \pm \frac{\max_{1\leq i<j\leq k} |(\hat{x}_i-\hat{x}_j)-(x_i-x_j)|}{s_0} \quad | \mathbf{y},\ s^2],$$

which are equal to

$$P_{\psi,c}\left[\frac{\max_{1\leq i<j\leq k} |(\theta_i-x_i)-(\theta_j-x_j)|/s_0-2a_k}{b_k}\right. \tag{3.5}$$

$$\left.\leq\left\{\frac{\hat{s}_0}{s_0} Q_{k,r+k}^{(\alpha)} \pm \max_{1\leq i<j\leq k} |(\hat{x}_i-\hat{x}_j)-(x_i-x_j)|/s_0-2a_k\right\}/b_k \,|\, \mathbf{y},\ s^2\right],$$

respectively.

Furthermore it can be shown that the right hand side of the inside of the probability in (3.5) converges a.s. to $K^{(\alpha)}$ by Lemma 3.2 and the dominated covergence theorem. Since the left hand side of the inside of the probability in (3.5) has the limiting distribution with density g in (3.2) by Lemma 3.1, the result follows.

As a final remark of this section, it should be pointed out that for large k, the T-type PEB MCA procedure also provides and approximate minimum Bayes risk solution for the problem given by Dixon and Duncan(1975) with the K value in their paper determined so that $\sqrt{2}t(K) = Q_{k,r+k}^{(\alpha)}(1-1/(k+r))$.

4. Extensions to Unbalanced Designs

In the case of unbalanced designs, we have $\bar{y}_i \sim N(\theta_i, \sigma^2/n_i)$, $i=1,\cdots,k$, independently, and $rs^2/\sigma^2 \sim \chi_r^2$, independently of $\bar{y}_1,\cdots,\bar{y}_k$. The prior distributions of $\theta_1,\cdots,\theta_k$ and σ^2 are just the same as in Section 2.

Then simple calculations lead to the following joint posterior distribution of $\theta_1,\cdots,\theta_k$ *given* $\bar{y}_1,\cdots,\bar{y}_k$ and s^2,

$$\left(\frac{\theta_1-x_1}{s_1},\cdots,\frac{\theta_k-x_k}{s_k}\right)' \,|\, \mathbf{y},\ s^2 \sim T_k(f,O,I_k), \tag{4.1}$$

where f, x_i and s_i are given as follows:

$$\begin{cases} f=r+k, \quad x_i=\dfrac{n_i\bar{y}_i+c\psi}{n_i+c}, \\ s_i^2=\dfrac{1}{f\cdot(n_i+c)}\left[\sum_{j=1}^{k}\dfrac{cn_j(\bar{y}_j-\psi)^2}{n_j+c}+rs^2\right],\ i=1,\cdots,k. \end{cases} \tag{4.2}$$

Applying the result of Hayter(1984) to (4.1), we have the following inequality.

$$P_{\psi,c}[\theta_i-\theta_j \in x_i-x_j \pm Q_{k,f}^{(\alpha)}\sqrt{(s_i^2+s_j^2)/2},\ \forall i\neq j \,|\, \mathbf{y},\ s^2] \geq 1-\alpha. \tag{4.3}$$

This provides a conservative $100(1-\alpha)\%$ simultaneous credible set for $\theta_i-\theta_j$, $1\leq i<j\leq k$. Note that, in the case of noninformative prior for θ, f becomes r and the resulting confidence intervals coincide with the classical Tukey-Kramer intervals proposed by Tukey(1953) and Kramer (1956).

In fact, the well-known TK-procedure has been recommended most frequently over other MCA procedures in the unbalanced designs(See Hochberg and Tamhane(1987), §3.4.). However the conservative nature of the TK-procedure was proven much later by Hayter(1984).

In this sense the Bayesian confidence intervals given by (4.3) will be called the *TK-type Bayesian MCA procedure*.

Now we consider an empirical version of the TK-type Bayesian MCA procedure. Through the analogous methods of Section 2, the estimates of ψ and c are proposed as follows:

$$\begin{cases} \hat{\psi}=\bar{y} \\ \hat{c}=\begin{cases} k_0/(F-1) & \text{for } F>1 \\ \infty & \text{for } F\leq 1, \end{cases} \end{cases}$$

where

$$\bar{y}=(\sum_{i=1}^{k} n_i\bar{y}_i)/(\sum_{i=1}^{k} n_i),\quad k_0=\{(\sum_{i=1}^{k} n_i)^2-\sum_{i=1}^{k} n_i^2\}/\{(k-1)\sum_{i=1}^{k} n_i\},$$

and F denotes the ordinary F-ratio in analysis of variance. Substituting these into (4.2), we have

$$\hat{x}_i=\frac{n\bar{y}_i+\hat{c}\hat{\psi}}{n_i+\hat{c}},\quad i=1,\cdots,k \tag{4.4}$$
$$\hat{s}_i^2=\frac{1}{(n_i+\hat{c})(r+k)}\left[\sum_{j=1}^{k}\frac{\hat{c}n_j(\bar{y}_j-\hat{\psi})^2}{n_j+\hat{c}}+rs^2\right],\quad i=1,\cdots,k.$$

Applying (4.4) to (4.3), we have the following procedure which will be called the *TK-type PEB MCA procedure*.

$$(\theta_i-\theta_j)\in\hat{x}_i-\hat{x}_j\pm Q_{k,r+k}^{(\alpha)}\sqrt{(\hat{s}_i^2+\hat{s}_j^2)/2},\ \forall i\neq j. \tag{4.5}$$

Of course if the n_i's are all equal, then the PEB MCA procedures of TK-type and T-type give same intervals for $\theta_i-\theta_j$, $1\leq i<j\leq k$.

The numerical results of Unsipaikka(1985) and Spurrier and Isham(1985), and simulation results of Dunnett(1980) show that the extent of conservatism in the classical TK-procedure is small even for cases of rather severe imbalance. For our Bayesian or PEB MCA procedures of TK-type, this property is preserved since the conservatisms of these three procedures are based on the same result of Hayter(1984).

The following example with data taken from McClave and Dietrich II(1979) illustrates the performance of the TK-type PEB MCA procedure in comparison with that of the classical TK-procedure.

Example 4.1. Some varieties of nematodes(round worms that live in the soil and frequently are so small they are invisible to the naked eye) feed upon the roots of lawn grasses and other plants. This pest, which is particularly troublesome in warm climates, can be treated by the application of nematicides. Data collected on the percentage of kill of nematodes for four particular rates of application(letting A_1, A_2, A_3 *and* A_4 for convenience) are given in Table 4.1.

Table 4.1. Percentage of Kill

	Rate of application			
	A_1	A_2	A_3	A_4
	86	87	94	90
	82	93	99	85
	76	89	97	86
			91	
$\bar{y}_i$	81.333	89.667	95.250	87.000

From the given data we have $F=8.634$, which shows that the homogeneity hypothesis is rejected even at 1% assuming the usual one-way model. Choosing $\alpha=0.05$, we find that $Q_{4,13}^{(0.05)}=4.151$ and $Q_{4,9}^{(0.05)}=4.415$. Table 4.2 gives the 95% confidence intervals for the mean differences by the TK-type PEB MCA procedure and the classical TK-procedure. In this case, TK-type PEB MCA procedure separates $\theta_3-\theta_1$ and $\theta_4-\theta_3$, while the classical TK-procedure does only $\theta_3-\theta_1$.

However, to see the performances of the procedures under small F values, let us add 5 to each element of first and fourth columns(i.e. 91, 87, 81 for A_1 and 95, 90, 91 for A_4) so that $F=3.638$ which shows no significance of the mean differences at 5% by the usual F-test. Then after simple calculations it can be shown that our 95% TK-type PEB MCA procedure declares no significant mean differences, while the 95% classical TK-procedure still separates $\theta_3-\theta_1$. In fact, the former provides (-0.148, 13.266) and the latter does(0.207, 17.626) for $\theta_3-\theta_1$, corresponding to the largest difference of sample means. These indicate the point that the TK-type PEB MCA procedure provides sharp intervals for large F values, and conservative intervals for small F values, in comparison with the classical TK-procedure.

Table 4.2. 95% Confidence Intervals by Two Procedures

difference	TK-type PEB MCA	Classical TK-procedure
$(\theta_2-\theta_1)$	(-0.524, 15.130)	(-0.978, 17.644)
$(\theta_3-\theta_1)$	(5.001, 19.747)*	(5.207, 22.626)*
$(\theta_4-\theta_1)$	(-2.861, 12.793)	(-3.644, 14.978)
$(\theta_3-\theta_2)$	(-2.299, 12.444)	(-3.126, 14.293)
$(\theta_4-\theta_2)$	(-10.164, 5.490)	(-11.978, 6.644)
$(\theta_4-\theta_3)$	(-14.781, -0.038)*	(-16.960, 0.460)

* denotes the interval not containing zero.

Finally it should be remarked that the asymptotic confidence level of the TK-type PEB MCA procedure can be handled in a way similar to Section 3.

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Appendix

Proof of Theorem 2.1 :

From the additivity of loss, it follows that the Bayes risk for the decision problem is minimized by minimizing the subcomponent risks. It is easy to see that the posterior distribution of $\theta_i-\theta_j$ is given by

$$\frac{(\theta_i-\theta_j)-(x_i-x_j)}{\sqrt{2}s_0} \mid \mathbf{y},\ s^2 \sim t_f. \tag{A.1}$$

Let the posterior p.d.f. of $\theta_i-\theta_j$ be $p_{ij}(\Delta \mid \mathbf{y},\ s^2)$. Then for the interval estimate $[a_{ij},\ b_{ij}]$ of $\theta_i-\theta_j$, the expected subcomponent loss with respect to the posterior of $\theta_i-\theta_j$ can be obtained as follows :

$$\int l(a_{ij},\ b_{ij} :\ \Delta)p_{ij}(\Delta \mid \mathbf{y},\ s^2)d\Delta$$
$$=(b_{ij}-a_{ij})+\frac{1}{\beta}\int_{-\infty}^{a_{ij}}(a_{ij}-\Delta)p_{ij}(\Delta \mid \mathbf{y},\ s^2)d\Delta+\frac{1}{\beta}\int_{b_{ij}}^{\infty}(\Delta-b_{ij})p_{ij}(\Delta \mid \mathbf{y},\ s^2)d\Delta$$

By differentiating this expression with respect to a_{ij}, we have

$$\frac{\partial}{\partial a_{ij}}\int l(a_{ij},\ b_{ij} :\ \Delta)p_{ij}(\Delta \mid \mathbf{y},\ s^2)d\Delta=-1+\frac{1}{\beta}\int_{-\infty}^{a_{ij}}p_{ij}(\Delta \mid \mathbf{y},\ s^2)d\Delta,$$

which becomes zero by (A.1) when

$$a_{ij}=x_i-x_j-\sqrt{2}t_f^{(\beta)}\ s_0.$$

A similar work for b_{ij} completes the proof.

Proof of Lemma 3.1 :

Without loss of generality, V may be assumed to be $\max_{1\leq i<j\leq k}|Z_i-Z_j|/S$ where Z_i's are $i.i.d.$ $N(0,1)$ and S^2 is the average of r $i.i.d.$ χ_1^2 random variables independent of Z_i's. Note that

$$\frac{\max_{1\leq i<j\leq k}|Z_i-Z_j|/S-2a_k}{b_k}$$

$$=\{\frac{\max_{1\leq i<j\leq k}|Z_i-Z_j|-2a_k}{b_k}-\frac{2(S-1)a_k}{b_k}\}/S. \qquad (A.2)$$

It follows from Gumbel(1947) that in (A.2),

$$(\max_{1\leq i<j\leq k}|Z_i-Z_j|-2a_k)/b_k$$

has limiting distribution as $k\to\infty$ with density g given by (3.2). Since $S\xrightarrow{P}1$ *as* $k\to\infty$ clearly, it suffices to show that $2(S-1)a_k/b_k\xrightarrow{P}0$ as $k\to\infty$.

It follows from the law of the iterated logarithm and the assumption $\frac{1}{r}=O(\frac{1}{k})$ that

$$S^2=1+O((\frac{\log\log k}{k})^{1/2}) \text{ as } k\to\infty \text{ a.s.},$$

which implies

$$S=1+O((\frac{\log\log k}{k})^{1/2}) \text{ as } k\to\infty \text{ a.s.}.$$

Thus it can be shown that as $k\to\infty$,

$$2(S-1)a_k/b_k=o(1) \text{ a.s.},$$

which completes the proof.

Proof of Lemmma 3.2 :

(i) It is easy to see from the law of the iterated logarithm that as $k\to\infty$,

$$MS_T=\frac{(n+c)}{c}\sigma^2[1+O((\frac{\log\log k}{k})^{1/2})] \text{ a.s.},$$

$$s^2=\sigma^2[1+O((\frac{\log\log r}{r})^{1/2})] \text{ a.s.}. \qquad (A.3)$$

Thus we have, as $k\to\infty$,

$$\frac{1}{F}=\frac{c}{(n+c)}\{\frac{1+O((r^{-1}\log\log r)^{1/2})}{1+O((k^{-1}\log\log k)^{1/2})}\}$$

$$=\frac{c}{(n+c)}[1+O((\frac{\log\log k}{k})^{1/2})] \text{ a.s.}. \qquad (A.4)$$

(ii) Since $F>1$ a.s. as $k\to\infty$ by (i), it follows from (A.3) and (A.4) that as $k\to\infty$,

$$\hat{s}_0^2=\frac{\sigma^2}{n+c}\quad[1+O((\frac{\log\log k}{k})^{1/2})]\text{ a.s.},$$

where $\hat{s}_0^2$ is given by (2.7).

Also simple calculations show that as $k\to\infty$,

$$s_0^2=\frac{\sigma^2}{n+c}\quad[1+O((\frac{\log\log k}{k})^{1/2})]\text{ a.s.}.$$

Hence the result follows.

(iii) Since $F>1$ a.s. as $k\to\infty$ by (i), we have as $k\to\infty$,

$$(\hat{x}_i-\hat{x}_j)-(x_i-x_j)=(1-\frac{1}{F}-\frac{n}{n+c})(\bar{y}_i-\bar{y}_j)$$

$$=O((\frac{\log\log k}{k})^{1/2})(\bar{y}_i-\bar{y}_j)\text{ a.s.}.$$

Then the result follows by noting that

$$\max_{1\leq i<j\leq k}|\bar{y}_i-\bar{y}_j|=[2(2\log k)^{1/2}+o(1)](\frac{1}{n}+\frac{1}{c})^{1/2}\sigma\text{ a.s.},$$

as $k\to\infty$ (see Serfling (1980), p.91).

(iv) The result follows from Lemma 3.1 with a little algebra.