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Analysis of Mechanical Face Seals for Design Purpose

Part II : Thermoelastic, Wearing and Vibrational Effects

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설계목적을 위한 기계평면시일의 해석

제 2보 : 열탄성, 마모 및 진동의 영향에 관하여

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요약—기계평면시일의 접촉 운동면에서 유체가 비압축성이고, 점성의 영향을 받는 경우에 대한 체적 누설 유동량과 마찰 토오크를 멱급수의 방법을 이용하여 추정하였다. 본 연구에서 고려되고 있는 설계인자로 시일의 경사도, 접촉 운동면에서의 사인파형, 코우닝, 열탄성 변화량, 마모량, 시일의 스프링 강성도에 따른 축방향의 변화량을 종합적으로 고려하여 해석하였다. 계산된 결과에 의하면 특히 회전속도가 증가되면 열탄성 변화량에 의한 시일의 누설 유동량과 마찰 토오크는 커다란 영향을 받고 있는 것으로 나타나고 있다. (제 6권 제 2호에 이어)

Appendix Integration parameters

$$I_1=2\pi\left[1-a_3\,(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}}\right]^3$$

$$I_2=-3\xi_u\xi_l\,(\xi_u K_{11}+\xi_l K_{12})$$

$$I_3=\frac{1}{(2\bar{h})^3}\left\{\frac{2t^3}{\pi^2}\{|\delta_{wu}|^2(|\delta_{wu}|+3|\delta_{wl}|)+|\delta_{wl}|^2(|\delta_{wl}|+3|\delta_{wu}|)\}+3t\{|\delta_{wu}|+|\delta_{wl}|\}\cdot\left\{\left\{\frac{|\delta_{wu}|\sin(\frac{\Omega_u t}{2})}{\Omega_u}\right\}^2\right.\right.$$
$$\left.+\left\{\frac{|\delta_{wl}|\sin(\frac{\Omega_l t}{2})}{\Omega_l}\right\}^2\right\}+\frac{3|\delta_{wu}||\delta_{wl}|}{\Omega_u\Omega_l}\left\{|\delta_{wu}|\sin(\frac{\Omega_l t}{2})\left\{\frac{2t\sin(\frac{\Omega_u t}{2})}{\pi}K_{03}-\frac{\sin^2(\frac{\Omega_u t}{2})}{\Omega_u}K_{01}\right\}\right.$$
$$\left.\left.+|\delta_{wl}|\sin(\frac{\Omega_u t}{2})\left\{\frac{2t\sin(\frac{\Omega_l t}{2})}{\pi}K_{03}-\frac{\sin^2(\frac{\Omega_l t}{2})}{\Omega_l}K_{02}\right\}\right\}\right\}$$

$$I_4=-a_3^2\{n_u|\delta_{thu}|^3\{-4\pi^2n_u^4-3K_{9u}+3n_uK_{7u}\}+3|\delta_{thu}||\delta_{thl}|\{|\delta_{thu}|\{K_{11}-2n_uK_{1u}+n_u^2K_{7l}-n_l\{K_{9u}-2n_uK_{7u}$$
$$+4\pi^4n_u^2\}\}+|\delta_{thl}|\{K_{12}-2n_lK_{1u}+n_l^2K_{7u}-n_u\{K_{9l}-2n_lK_{7l}+4\pi^4n_l^2\}\}\}+n_l|\delta_{thl}|^3\{-4\pi^2n_l^2-3K_{9l}+3n_lK_{7l}\}\}$$

$$I_5=3\left[1-a_3\{a_4+\delta_{cu}+\delta_{cl}\}+\frac{\delta_d}{2\bar{h}}\right]\left\{\frac{t}{\bar{h}}\{|\delta_{wu}|+|\delta_{wl}|\}+2\pi^2a_3\{n_u|\delta_{thu}|+n_l|\delta_{thl}|\}\right\}$$

$$I_6=3(a_2+a_3a_4)^2\Big\{\pi(1-a_3(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}})-\xi_uK_{22}-\xi_lK_{23}+\frac{1}{2\bar{h}}\Big\{t(|\delta_{wu}|+|\delta_{wl}|)$$

$$-\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}K_{26}-\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}K_{27}\Big\}-a_3(|\delta_{thu}|(K_{22}-\pi^2n_u)+|\delta_{thl}|(K_{23}-\pi^2n_l))\Big)$$

$$I_7=3\Big\{(1-a_3(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}})(\pi(\xi_u^2+\xi_l^2)+2\xi_u\xi_l(a_2+a_3a_4)K_{34}+\frac{1}{2\bar{h}}(t(|\delta_{wu}|+|\delta_{wl}|)$$

$$(\xi_u^2+\xi_l^2+\frac{2}{\pi}\xi_u\xi_lK_{13})-\frac{\xi_l|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}(\xi_lK_{15}+2\xi_uK_{16})-\frac{\xi_u|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(\xi_uK_{14}+2\xi_lK_{17})\Big)$$

$$-a_3(|\delta_{thl}|(\xi_l^2K_{12}+2\xi_u\xi_lK_{11})+|\delta_{thl}|(\xi_u^2K_{11}+2\xi_u\xi_lK_{12})-(n_u|\delta_{thu}|+n_l|\delta_{thl}|)(\xi_u^2K_{9u}+\xi_l^2K_{9l}+2\xi_u\xi_lK_{1u}))\Big\}$$

$$I_8=\frac{3}{4\bar{h}}\Big\{(1-a_3(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}})\Big(\frac{2t^2}{\pi}(|\delta_{wu}|+|\delta_{wl}|)^2+\pi\Big(\Big\{\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}\Big\}^2+\Big\{\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}\Big\}^2\Big)$$

$$+\frac{2|\delta_{wu}||\delta_{wl}|\sin(\frac{\Omega_ut}{2})\sin(\frac{\Omega_lt}{2})}{\Omega_u\Omega_l}K_{03}\Big)-\frac{2t(a_2+a_3a_4)}{\pi}\Big(\frac{|\delta_{wu}|^2\sin(\frac{\Omega_ut}{2})}{\Omega_u}K_{28}+\frac{|\delta_{wl}|^2\sin(\frac{\Omega_lt}{2})}{\Omega_l}K_{29}$$

$$+|\delta_{wu}||\delta_{wl}|\Big(\frac{\sin(\frac{\Omega_ut}{2})}{\Omega_u}K_{28}+\frac{\sin(\frac{\Omega_lt}{2})}{\Omega_l}K_{29}-\frac{\pi}{t}\frac{\sin(\frac{\Omega_ut}{2})\sin(\frac{\Omega_lt}{2})}{\Omega_u\Omega_l}K_{04})\Big)+\frac{2t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)$$

$$\Big(\frac{|\delta_{wu}|\sin(\frac{\Omega_u}{2})}{\Omega_u}(\xi_uK_{21}+\xi_lK_{18})+\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(\xi_uK_{19}+\xi_lK_{20})\Big)+a_3\Big(|\delta_{thu}|\Big\{|\delta_{wu}|^2\Big(\frac{2t\sin(\frac{\Omega_ut}{2})}{\pi\Omega_u}K_{21}$$

$$+n_u\Big(2t^2-\frac{2t\sin(\frac{\Omega_ut}{2})}{\pi\Omega_u}K_{8u}+\frac{\sin^2(\frac{\Omega_ut}{2})}{\Omega_u^2}J_{9u}\Big)\Big)+|\delta_{wl}|^2\Big(\frac{2t\sin(\frac{\Omega_lt}{2})}{\pi\Omega_l}K_{19}+n_u\Big(2t^2-\frac{2t\sin(\frac{\Omega_lt}{2})}{\pi\Omega_l}K_{8l}$$

$$+\frac{\sin^2(\frac{\Omega_lt}{2})}{\Omega_l^2}J_{9l}\Big)\Big)+\frac{2t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)\Big(\frac{\sin(\frac{\Omega_ut}{2})}{\Omega_u}K_{21}+\frac{\sin(\frac{\Omega_lt}{2})}{\Omega_l}K_{19}\Big)+n_u\Big(2\pi t-\frac{\sin(\frac{\Omega_ut}{2})}{\Omega_u}K_{8u}$$

$$-\frac{\sin(\frac{\Omega_lt}{2})}{\Omega_l}K_{8l}\Big)\Big\}\Big)+|\delta_{thl}|\Big\{|\delta_{wu}|^2\Big(\frac{2t\sin(\frac{\Omega_ut}{2})}{\pi\Omega_u}K_{18}+n_l\Big(2t^2-\frac{2t\sin(\frac{\Omega_ut}{2})}{\pi\Omega_u}K_{8u}+\frac{\sin^2(\frac{\Omega_ut}{2})}{\Omega_u^2}J_{9u}\Big)\Big)$$

$$+|\delta_{wl}|^2\Big(\frac{2t\sin(\frac{\Omega_lt}{2})}{\pi\Omega_l}K_{20}+n_l(2t^2-\frac{2t\sin(\frac{\Omega_lt}{2})}{\pi\Omega_l}K_{8l}+\frac{\sin^2(\frac{\Omega_lt}{2})}{\Omega_l^2}J_{9l})\Big)+\frac{2t}{\pi}(|\delta_{wu}|+\delta_{wl}|)$$

$$\Big(\frac{\sin(\frac{\Omega_ut}{2})}{\Omega_u}K_{18}+\frac{\sin(\frac{\Omega_lt}{2})}{\Omega_l}K_{20}\Big)+n_l\Big(2\pi t-\frac{\sin(\Omega_ut)}{\Omega_u}K_{8u}-\frac{\sin(\frac{\Omega_lt}{2})}{\Omega_l}K_{8l}\Big)\Big\}$$

$$I_9=3a_3^2\{(1-a_3(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}})\{|\delta_{thu}|^2(\pi-2n_uK_{4u}+\frac{8\pi^3n_u^2}{3})+|\delta_{thl}|^2(\pi-2n_lK_{4l}+\frac{8\pi^3n_l^2}{3})$$

$$+2|\delta_{thu}||\delta_{thl}|(K_{13}-n_uK_{4l}-n_lK_{4u}+n_un_l\frac{8\pi^3}{3})\}+2(a_2+a_3a_4)\{n_u|\delta_{thu}|^2(2\pi n_u-K_{2u})$$

$$+n_l|\delta_{thl}|^2(2\pi n_l-K_{2l})+|\delta_{thu}||\delta_{thl}|(K_{34}-n_uK_{2l}n_lK_{2u}+4\pi n_un_l)\Big\}-\xi_u\{n_u|\delta_{thu}|^2(-2K_{9u}+n_uK_{7u})$$

$$+|\delta_{thl}|^2(K_{12}-2n_lK_{1u}+n_l^2K_{7u})2|\delta_{thu}||\delta_{thl}|(K_{11}-n_uK_{1u}-n_lK_{9u}+n_un_lK_{7u})\}-\xi_l\{|\delta_{thu}|^2(K_{11}-2n_uK_{1u}$$

$$+n_u^2K_{7l})+n_l|\delta_{thl}|^2(-2K_{9l}+n_lK_{7l})+2|\delta_{thu}||\delta_{thl}|(K_{12}-n_uK_{9l}-n_lK_{1u}+n_un_lK_{7l})\}+\frac{1}{2\bar{h}}\{\frac{t}{\pi}(|\delta_{wu}|$$

$$+|\delta_{wl}|)\{|\delta_{thu}|^2(\pi-2n_uK_{4u}+\frac{8\pi^3n_u^2}{3})+|\delta_{thl}|^2(\pi-2n_lK_{4l}+\frac{8\pi^3n_l^2}{3})+2|\delta_{thu}||\delta_{thl}|(K_{13}-n_uK_{4l}-n_lK_{4u}$$

$$+\frac{8\pi^3n_un_l}{3})\}-\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}(n_u|\delta_{thu}|^2(-2K_{5u}+n_uJ_{7u})+|\delta_{thl}|^2(K_{15}-2n_lK_{6l}+n_l^2J_{7u})+2|\delta_{thu}||\delta_{thl}|$$

$$(K_{16}-n_uK_{6l}-n_lK_{5u}+n_un_lJ_{7u})\}-\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(|\delta_{thu}|^2(K_{14}-2n_uK_{6u}+n_u^2J_{7l})+n_l|\delta_{thl}|^2(-2K_{5l}+n_lJ_{7l})$$

$$+2|\delta_{thu}||\delta_{thl}|(K_{17}-n_uK_{5l}-n_lK_{6u}+n_un_lJ_{7l})\}\}$$

$$I_{10}=6(1-a_3(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}})\{-a_2+a_3a_4\}(\xi_uK_{24}+\xi_lK_{25})+\frac{1}{2\bar{h}}(\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}(\xi_uK_{21}+\xi_lK_{16})$$

$$+\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(\xi_uK_{19}+\xi_lK_{20})\}+\frac{a_3}{2\bar{h}}\Big\{|\delta_{thu}|(\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}(K_{34}-n_uK_{8u})+2\pi tn_u(|\delta_{wu}|+|\delta_{wl}|)$$

$$+\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(K_{19}-n_uK_{8l})\}+|\delta_{thl}|(\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}(K_{16}-n_lK_{8u})+2\pi tn_l(|\delta_{wu}|+|\delta_{wl}|)$$

$$+\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(K_{35}-n_lK_{8l})\}\Big\}+a_3\{\xi_u(|\delta_{thu}|(\pi-n_uK_{4u})+|\delta_{thu}|(K_{13}-n_lK_{4u})\}+\xi_u(|\delta_{thu}|K_{13}-n_uK_{4l})$$

$$+|\delta_{thl}|(\pi-n_lK_{4l})\}\Big\}$$

$$I_{11}=-6(a_2+a_3a_4)\Big\{(1-a_3(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}})a_3(|\delta_{thu}|K_{24}+|\delta_{thl}|K_{25})+\frac{t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)(\xi_uK_{24}+\xi_lK_{25})$$

$$-\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}(\xi_uK_{30}+\xi_lK_{33})-\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(\xi_uK_{32}+\xi_lK_{31})+a_3\Big\{\frac{t}{\pi}(|\delta_{wu}|+|\delta_{wl}|)(|\delta_{thu}|K_{24}+$$

$$|\delta_{thl}|K_{25})+|\delta_{thu}|(\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}(n_uK_{3u}-K_{30})+\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(n_uK_{3l}-K_{32})\Big)+|\delta_{thl}|\Big(\frac{|\delta_{wu}|\sin(\frac{\Omega_ut}{2})}{\Omega_u}$$

$$(n_lK_{3u}-K_{33})+\frac{|\delta_{wl}|\sin(\frac{\Omega_lt}{2})}{\Omega_l}(n_uK_{3l}-K_{31})\Big)\Big\}\Big\}$$

$$I_{12}=6a_3\left\{(a_2+a_3a_4)\{\xi_u(|\delta_{thl}|(K_{34}-n_lK_{2u})-n_u|\delta_{thu}|K_{2u})+\xi_l(|\delta_{thu}|(K_{25}-n_uK_{2l})-n_l|\delta_{thl}|K_{2l}\right\}$$

$$+\frac{1}{2\bar{h}}\left\{t(|\delta_{wu}|+|\delta_{wl}|)(\xi_u|\delta_{thu}|+\xi_l|\delta_{thl}|)+\xi_u\left\{|\delta_{thu}|\left(\frac{|\delta_{wl}|\sin(\frac{\Omega_l t}{2})}{\Omega_l}(n_uK_{6u}-K_{14})-n_u(\frac{tK_{4u}}{\pi}(|\delta_{wu}|\right.\right.\right.$$

$$\left.\left.+|\delta_{wl}|)-\frac{|\delta_{wu}|\sin(\frac{\Omega_u t}{2})}{\Omega_u}K_{5u}\right)\right)-n_l|\delta_{thl}|\left(\frac{t}{\pi}K_{4u}(|\delta_{wu}|+|\delta_{wl}|)-\frac{|\delta_{wu}|\sin(\frac{\Omega_u t}{2})}{\Omega_u}K_{5u}\right.$$

$$\left.-\frac{|\delta_{wl}|\sin(\frac{\Omega_l t}{2})}{\Omega_l}K_{6u}\right)\right\}+\left(\frac{t}{\pi}K_{13}(|\delta_{wu}|+|\delta_{wl}|)-\frac{|\delta_{wu}|\sin(\frac{\Omega_u t}{2})}{\Omega_u}K_{16}-\frac{|\delta_{wl}|\sin(\frac{\Omega_l t}{2})}{\Omega_l}K_{17}\right)(\xi_u|\delta_{wu}|$$

$$+\xi_u|\delta_{wl}|)+\xi_l\left\{-n_u|\delta_{thu}|\left(\frac{t}{\pi}K_{4l}(|\delta_{wu}|+|\delta_{wl}|)-\frac{|\delta_{wu}|\sin(\frac{\Omega_u t}{2})}{\Omega_u}K_{6l}-\frac{|\delta_{wl}|\sin(\frac{\Omega_l t}{2})}{\Omega_l}K_{5l}\right)\right.$$

$$\left.+|\delta_{thu}|\left(\frac{|\delta_{wu}|\sin(\frac{\Omega_u t}{2})}{\Omega_u}(n_lK_{6l}-K_{15})-n_l\left(\frac{t}{\pi}K_{4l}(|\delta_{wu}|+|\delta_{wl}|)-\frac{|\delta_{wl}|\sin(\frac{\Omega_l t}{2})}{\Omega_l}K_{6l}\right)\right)\right\}\right\}$$

$$I_{13}=-\frac{3}{h}(a_2+a_3a_4)\left(1-a_3(a_4+\delta_{cu}+\delta_{cl})+\frac{\delta_d}{2\bar{h}}\right)\left(\frac{|\delta_{wu}|\sin(\frac{\Omega_u t}{2})}{\Omega_u}K_{28}+\frac{|\delta_{wl}|\sin(\frac{\Omega_l t}{2})}{\Omega_l}K_{29}\right)$$

where

$$|\delta_{thu}|=\frac{\alpha_u\xi_u}{K_un_u^2},\ |\delta_{thl}|=\frac{\alpha_l\xi_l}{K_ln_l^2},\ |\delta_{wu}|=\frac{w|q_u|}{\mu_a},\ |\delta_{wl}|=\frac{w|q_l|}{\mu_a}$$

$$K_{14}=-\delta_{(2n_u)(n_l)}\frac{\pi}{2}\sin\left(\phi_l-\left(2\Omega_u-\frac{\Omega_l}{2}\right)t\right)$$

$$K_{15}=\delta_{(2n_l)(n_u)}\frac{\pi}{2}\sin\left(\phi_u+\left(\frac{\Omega_u}{2}-2\Omega_l\right)t\right)$$

$$K_{16}=-\delta_{(2n_u)(n_l)}\frac{\pi}{2}\sin\left(\phi_u+\left(\frac{3\Omega_u}{2}-\Omega_l\right)t\right)$$

$$K_{17}=\delta_{(n_u)(2n_l)}\frac{\pi}{2}\sin\left(\phi_l+\left(\Omega_u-\frac{3\Omega_l}{2}\right)t\right)$$

$$K_{18}=\delta_{(n_u)(n_l)}\pi\cos\left(\left(\frac{\Omega_u}{2}-\Omega_l\right)t+\phi_u\right)$$

$$K_{19}=\delta_{(n_u)(n_l)}\pi\cos\left(\left(\frac{\Omega_l}{2}-\Omega_u\right)t-\phi_l\right)$$

$$K_{20}=\pi\cos\left(\frac{\Omega_l t}{2}+\phi_l\right)$$

$$K_{21}=\pi\cos\left(\frac{\Omega_u t}{2}-\phi_u\right)$$

$$K_{26}=-\delta_{(n_u)(2)}\frac{\pi}{2}\sin\left(\frac{\Omega_u t}{2}+\phi_u\right)$$

$$K_{27}=-\delta_{(n_l)(2)}\frac{\pi}{2}\sin\left(\frac{\Omega_l t}{2}-\phi_l\right)$$

$$K_{28}=-\delta_{(n_u)(1)}\pi\sin\left(\frac{\Omega_u t}{2}+\phi_u\right)$$

$$K_{29}=-\delta_{(n_l)(1)}\pi\sin\left(\frac{\Omega_l t}{2}-\phi_l\right)$$

$$K_{30}=-\delta_{(2n_u)(1)}\frac{\pi}{2}\cos\left(\frac{3\Omega_u t}{2}+\phi_u\right)$$

$$K_{31}=-\delta_{(2n_l)(1)}\frac{\pi}{2}\cos\left(\frac{3\Omega_l t}{2}-\phi_l\right)$$

$$K_{32}=\left(\delta_{(n_u)(n_l+1)}+\delta_{(n_u+1)(n_l)}\right)\frac{\pi}{2}\cos\left(\phi_l+\left(\Omega_u-\frac{\Omega_l}{2}\right)t\right)$$

$$K_{33}=\left(\delta_{(n_u)(n_l+1)}+\delta_{(n_u+1)(n_l)}\right)\frac{\pi}{2}\cos\left(\phi_u+\left(\frac{\Omega_u}{2}-\Omega_l\right)t\right)$$

$$K_{01}=-\delta_{(2n_u)(n_l)}\frac{\pi}{2}\sin\left(\left(\Omega_u-\frac{\Omega_l}{2}\right)t+2\phi_u+\phi_l\right)$$

$$K_{02}=\delta_{(2n_l)(n_u)}\frac{\pi}{2}\sin\left(\left(\frac{\Omega_u}{2}-\Omega_l\right)t+\phi_u+2\phi_l\right)$$

$$K_{03}=\delta_{(n_u)(n_l)}\pi\cos\left(\left(\Omega_u-\Omega_l\right)\frac{t}{2}+\phi_u+\phi_l\right)$$

$$K_{04}=\left(\delta_{(n_u)(n_l+1)}+\delta_{(n_l)(n_u+1)}\right)\frac{\pi}{2}\cos\left(\left(\Omega_u-\Omega_l\right)\frac{t}{2}+\phi_u+\phi_l\right)$$

$$K_{1u}=\begin{cases}\pi\left\{\pi\cos[(\Omega_u-\Omega_l)t]+\dfrac{\sin[(\Omega_u+\Omega_l)t]}{2n_u}\right\} & \text{when } n_u=n_l\\ \pi\left\{-\dfrac{\sin[(\Omega_u-\Omega_l)t]}{n_u-n_l}+\dfrac{\sin[(\Omega_u+\Omega_l)t]}{n_u+n_l}\right\} & \text{when } n_u\neq n_l\end{cases}$$

$$K_{2l}=\begin{cases}-\pi\left(\pi\sin[\Omega_l t]+\dfrac{\cos[\Omega_l t]}{n_l+1}\right) & \text{when } n_l=1\\ \dfrac{2\pi n_l}{1-n_l^2}\cos[\Omega_l t] & \text{when } n_l\neq 1\end{cases}$$

$$K_{2u}=\begin{cases} -\pi\left(\pi\sin(\Omega_u t)+\dfrac{\cos(\Omega_u t)}{n_u+1}\right) & \text{when } n_u=1 \\ \dfrac{2\pi n_u}{1-n_u^2}\cos(\Omega_u t) & \text{when } n_u\neq 1 \end{cases}$$

$$K_{3l}=\begin{cases} -\pi\left(\pi\sin\left(\dfrac{\Omega_l t}{2}-\phi_l\right)+\dfrac{\cos\left(\dfrac{\Omega_l t}{2}-\phi_l\right)}{n_l+1}\right) & \text{when } n_l=1 \\ \dfrac{2\pi n_l}{1-n_l^2}\cos\left(\dfrac{\Omega_l t}{2}-\phi_l\right) & \text{when } n_l\neq 1 \end{cases}$$

$$K_{3u}=\begin{cases} -\pi\left(\pi\sin\left(\dfrac{\Omega_u t}{2}+\phi_u\right)+\dfrac{\cos\left(\dfrac{\Omega_u t}{2}+\phi_u\right)}{n_u+1}\right) & \text{when } n_u=1 \\ \dfrac{2\pi n_u}{1-n_u^2}\cos\left(\dfrac{\Omega_u t}{2}+\phi_u\right) & \text{when } n_u\neq 1 \end{cases}$$

$$K_{4l}=-\frac{2\pi}{n_l}\cos(\Omega_l t)$$

$$K_{4u}=-\frac{2\pi}{n_u}\cos(\Omega_u t)$$

$$K_{5l}=\pi\left(\pi\cos\left(\frac{\Omega_l t}{2}+\phi_l\right)+\frac{1}{2n_u}\sin\left(\frac{3\Omega_l t}{2}-\phi_l\right)\right)$$

$$K_{5u}=\pi\left(\pi\cos\left(\frac{\Omega_u t}{2}-\phi_u\right)+\frac{1}{2n_u}\sin\left(\frac{3\Omega_u t}{2}+\phi_u\right)\right)$$

$$K_{6l}=\begin{cases} \pi\left\{\dfrac{\sin\left(\left(\Omega_u+\dfrac{\Omega_l}{2}\right)t+\phi_u\right)}{n_u+n_l}+\dfrac{\sin\left(\left(\dfrac{\Omega_u}{2}-\Omega_l\right)t+\phi_u\right)}{n_u-n_l}\right\} & \text{when } n_u\neq n_l \\ \pi\left\{\pi\cos\left(\left(\dfrac{\Omega_u}{2}-\Omega_l\right)t+\phi_u\right)+\dfrac{\sin\left(\left(\Omega_u+\dfrac{\Omega_l}{2}\right)t+\phi_u\right)}{2n_u}\right\} & \text{when } n_u=n_l \end{cases}$$

$$K_{6u}=\begin{cases} \pi\left\{\dfrac{\sin\left(\left(\dfrac{\Omega_l}{2}-\Omega_u\right)t-\phi_l\right)}{n_u-n_l}+\dfrac{\sin\left(\left(\Omega_u+\dfrac{\Omega_l}{2}\right)t-\phi_l\right)}{n_u+n_l}\right\} & \text{when } n_u\neq n_l \\ \pi\left\{\pi\cos\left(\left(\Omega_u-\dfrac{\Omega_l}{2}\right)t+\phi_l\right)+\dfrac{\sin\left(\left(\Omega_u+\dfrac{\Omega_l}{2}\right)t-\phi_l\right)}{2n_u}\right\} & \text{when } n_u=n_l \end{cases}$$

$$K_{7l} = -\frac{4\pi}{n_l}\left\{\pi\cos[\Omega_l t] + \frac{\sin[\Omega_l t]}{n_l}\right\}$$

$$K_{7u} = -\frac{4\pi}{n_u}\left\{\pi\cos[\Omega_u t] + \frac{\sin[\Omega_u t]}{n_u}\right\}$$

$$J_{7l} = -\frac{4\pi}{n_l}\left\{\pi\cos\left\{\frac{\Omega_l t}{2} - \phi_l\right\} + \frac{\sin\left\{\frac{\Omega_l t}{2} - \phi_l\right\}}{n_l}\right\}$$

$$J_{7u} = -\frac{4\pi}{n_u}\left\{\pi\cos\left\{\frac{\Omega_u t}{2} + \phi_u\right\} + \frac{\sin\left\{\frac{\Omega_u t}{2} + \phi_u\right\}}{n_u}\right\}$$

$$K_{8l} = -\frac{2\pi}{n_l}\cos\left\{\frac{\Omega_l t}{2} - \phi_l\right\}$$

$$K_{8u} = -\frac{2\pi}{n_u}\cos\left\{\frac{\Omega_u t}{2} + \phi_u\right\}$$

$$K_{9l} = \pi\left\{\pi + \frac{\sin[2\Omega_l t]}{2\Omega_l}\right\}$$

$$K_{9u} = \pi\left\{\pi + \frac{\sin[2\Omega_u t]}{2\Omega_u}\right\}$$

$$J_{9l} = \pi\left\{\pi + \frac{1}{2n_l}\sin(\Omega_l t - 2\phi_l)\right\}$$

$$J_{9u} = \pi\left\{\pi + \frac{1}{2n_u}\sin(\Omega_u t + 2\phi_u)\right\}$$

where a symbol, $\delta_{(m)(n)}$ is defined by

$$\delta_{(m)(n)} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$