Saturated - Unsaturated Transient Subsurface Flow Model on a Hillslope

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ABSTRACT/ The governing partial differential equation of flow in porous media is developed on the bases of the continuity equation of fluid for transient flow through a saturated-unsaturated zone, and substitution of Darcy's law. The numerical solution is obtained by the Galerkin finite element method based on the principle of weighted residuals. The analysis is carried out by using the unsteady storm data observed and the functional relationships between the hydraulic conductivities, capillary pressure heads, and volumetric water contents under saturated-unsaturated conditions. As the results the hydraulic conductivities, rates of change of storage and initial moisture conditions are significantly influened on the responses of subsurface flow on a hillslope.

1. Introduction

In hydroscience and water engineering, the subsurface flow analysis is required in contamination control of groundwater and in the design of various hydraulic structures in the upstream forested watershed which is permeable hillslope and hence rarely occured Hortonian overland flow. The flow in porous media is influenced on both saturated and unsaturated zone (Freeze, 1971; Reeves and Duguid, 1975; Fredlund, 1981). In the saturated zone the physical phenomena goverened the change of storage is dominated by change in void volume, on the other hand, in the unsaturated zone is dominated by change in degree of saturation which is function of the pressure head (Narasimhan and Witherspoon, 1977; Narasimhan, 1979; Geo-slope, 1985). Futhermore the subsurface flow is dependent on the rates of infiltration and exfiltration from the land surface. Therefore a saturated-unsaturated

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subsurface flow model would appear to be superior to a saturated only model (Lam and Fredlund, 1984; Lam et al, 1987).

The flow in the soil system is a multiphase flow, air and water-phase, through a porous media. The most practical studies, however, have encountered only water-phase, since the air-phase is continuous and the pore-air pressure is equal to atmospheric pressure and the single water- phase approach will not introduce significant error in results.

The transient analysis is also necessary in the simulation of rainfall -runoff processes with the time rate of change of fluid mass storage. The fundamental feature which distinguishes transient fluid flow from the steady state case is the phenomena of change in storage. Therefore it is as essential to consider the mechanism of change in storage as to fluid motion in saturated-unsaturated flow region.

In this paper, the 2-dimensional saturated-unsaturated transient model is proposed to analyze the rainfall-runoff responses on a steep hillslope. The governing differential equation is derived from the Darcy's law(i. e., mortion of fluid) and the continuity of fluid. The numerical solution is based on the Galerkin finite element method because of the nonlinearity of the governing equation.

2. Governing Equation

The equation of flow in a soil system is developed on the bases of the continuity equation of fluid for transient flow through a saturated- unsaturated porous medium, and substitution of Darcy's law. Assuming the net fluid mass flux through a control volume is equal to zero, the flow will be steady state flow:

$$\frac{\partial(\rho\, q_x)}{\partial x} + \frac{\partial(\rho\, q_z)}{\partial z} = 0 \qquad (1)$$

in which q is the specific discharge in x and z directions and ρ is fluid density.

Assuming ρ is constant because of the incompressivity of fluid in practical problems and substitution of Darcy's law in Eq. (1) yields:

$$\frac{\partial(K_x\, i_x)}{\partial x} + \frac{\partial(K_z\, i_z)}{\partial z} = 0 \qquad (2)$$

in which the K and i are hydraulic conductivities and hydraulic gradients, respectively. And also assuming the net fluid mass flux be equal to the time rate of change of fluid mass storage within the control volume, the Eq. (2) is modified to.

$$\frac{\partial(K_x\, i_x)}{\partial x} + \frac{\partial(K_z\, i_z)}{\partial z} = \frac{\partial n}{\partial t} \qquad (3)$$

and for anisotropic soil, and encountering the pressure field and discharge inward or outward the system,

$$F\frac{\partial h}{\partial t} - \left[\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial H}{\partial x} + K_{xz}\frac{\partial H}{\partial z}\right) + \frac{\partial}{\partial z}\left(K_{zx}\frac{\partial H}{\partial x} + K_{zz}\frac{\partial H}{\partial z}\right)\right] - Q = 0 \tag{4}$$

in which n is porosity, H is total head, h is pressure head and F is the soil hydraulic properties defined by density of water, gravitational constant and pore-water storage. Since F and K are functions of pressure head or volumetric water content, the gorverning equation is nonlinear.

3. Formulation of Finite Element Equation

Using the Galerkin principle of weighted residuals, the governing equation descrived subsurface flow, Eq. (4), for a flow region is given by,

$$\begin{aligned}
&[\int_\Gamma N_i\ F\ N_j\ ](\ dh_j/dt\ ) + [\ \int_\Gamma(\ \nabla N_i\ )\ K\ (\ \nabla N_j\ )\ d\Gamma\ ]\ h_i \\
&= -\int_\Gamma(\ \nabla N_i\ )\ K\ (\ \nabla z\ )\ d\Gamma + \int_\Gamma n\ K\ (\ \nabla h + \nabla z\ )\ N_i\ d\Gamma \\
&,\ i = 1,2,\ldots,N
\end{aligned} \tag{5}$$

in which N_j is base function, N_i is weighting function, ∇ is del operator and Γ is flow region.

In matrix form,

$$[\ a\ ]\ (\ dh/dt\ ) + [\ b\ ]\ \{\ h\ \} = \{\ D\ \} + \{\ Q\ \} \tag{6}$$

in which [a] is mass matrix , [b] is stiff matrix, { D } is known load vector and { Q } is boundary load vector. If M is the set of elements that have a local side α-β coinciding with the global side i-j, the above matrices are

$$a_{ij} = \int_\Gamma N_i\ F\ N_j\ \ d\Gamma = \Sigma_M \int_{\Gamma_\bullet} N_i\ F\ N_j\ \ d\Gamma = \Sigma_M \int_{\Gamma_\bullet} N^\bullet_\alpha\ F\ N^\bullet_\beta\ \ d\Gamma \tag{7a}$$

$$\begin{aligned}
b_{ij} &= \int_\Gamma(\ \nabla N_i\ )\ K\ (\ \nabla N_j\ )\ d\Gamma = \Sigma_M \int_{\Gamma_\bullet}(\ \nabla N_i\ )\ K\ (\ \nabla N_j\ )\ d\Gamma \\
&= \Sigma_M \int_{\Gamma_\bullet}(\ \nabla N^\bullet_\alpha\ )\ K\ (\ \nabla N^\bullet_\beta\ )\ d\Gamma
\end{aligned} \tag{7b}$$

$$\begin{aligned}
D_i &= -\int_\Gamma(\ \nabla N_i\ )\ K\ (\ \nabla z\ )\ d\Gamma = -\Sigma_M \int_{\Gamma_\bullet}(\ \nabla N_i\ )\ K\ \ \nabla z\ d\Gamma \\
&= -\Sigma_M \int_{\Gamma_\bullet}(\ \nabla N^\bullet_\alpha\ )\ K\ \nabla\ \ d\Gamma
\end{aligned} \tag{7c}$$

$$Q = \int_{\Gamma} n\, K\, (\nabla h + \nabla z)\, N_i\, d\Gamma \tag{7d}$$

in which super- and sub-scrips, e is an element.

For transient analysis, the time derivative, $\partial h/\partial t$, in Eq. (6) can be approximated by a implicit finite difference procedure. The implicit scheme is found to be more effective in damping the numerical oscillations frequently encountered in highly nonlinear flow systems (Neumann, 1973). The relationship between the model heads of an element in two successive time steps can be expressed by the following equations,

$$[M_{ij}](\{h_j\}_{t+\Delta t} - \{h_j\}_t)/\Delta_t + [S_{ij}]\{h_j\}_{t+\Delta t} + \{D_i\} + \{Q_i\} = 0 \tag{8}$$

and be simplified as follows:

$$[C_{ij}]\{h_j\} = \{R_i\} - \{Q_i\} \tag{9}$$

in which $[C_{ij}]$ is element coefficient matrix, $\{h_j\}$ is the unknown vector to be found and $\{R_i\}$ is the element load vector.

4. Initial and Boundary Conditions

In the simulations presented, the initial conditions are obtained by solving for the nodal head values, h, from the steady state version of Eq. (2) under time invariant boundary conditions.

The initial conditions in flow region, Γ, is

$$h(x, z, 0) = h_i\ (x, z, t) \tag{10}$$

in which $h(x, z, t)$ is the prescrived pressure head and h_i is initial pressure head as a function of time(t) and spatial coordinates(x, z).

The boundary conditions may be specified by some points of view. Fig. 1. is shown the specification of boundary conditions. The Dirichlet boundary condition type is prescrived head, the Neumann

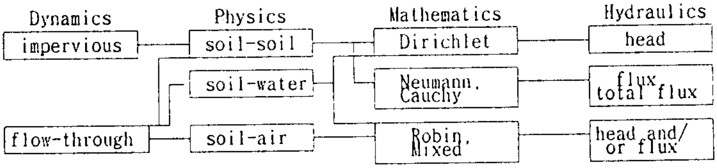

Fig. 1 Specification of boundary conditions for subsurface flow model

type is prescrived derivative flux and the Cauchy type is prescrived total flux. While the variable type is classified into the Robin and mixed ones. If the imposed boundary conditions is a linear combination the Dirichlet and Nuemann types , it is known as the Robin type. And certain portion of the boundary is the Dirichlet and on another portion of the boundary is the Neumann type , it is known as mixed type (Yeh and Ward, 1982; Hoffmann, 1989).

The Dirichlet type for the ponding depth occured as the excess rainfall or depression storage on the ground surface:

$$h = h(x, z, t) \tag{11}$$

The Neumann type for the rainfall-infiltration boundary and flux or impervious boundary prescrived:

$$-\left[\left(K_{xx} \frac{\partial h}{\partial x} + K_{xz} \frac{\partial h}{\partial z} + K_{xz}\right) n_x + \left(K_{zx} \frac{\partial h}{\partial x} + K_{zz} \frac{\partial h}{\partial z} + K_{zz}\right) \right] n_z = q \tag{12}$$

in which n_x and n_z are unit normal vectors.

The variable type for the seepage face can be used Eq. (11) or (12).

5. Hydraulic Characteristics of Soil

To analyze the saturated-unsaturated subsurface flow processes in the unsaturated zone it is required the relationship between the hydraulic conductivity versus capillary pressure head. And for the transient analysis the soil have to be concerned about the rates of change of storage depended on the hydraulic characteristics within the flow region. Below the phreatic line, which is an imaginary surface where the capillary pressure is zero the pressure head, is positive and increases linearly with depth. Above the phreatic line, however, the pressure is negative and decreases with height from the given bottom.

The saturated hydraulic conductivity can be obtained using the standard laboratory or field procedure. But the direct measurement of the unsaturated hydraulic conductivity requires often significant effort and difficult to conduct without suitable equipments (Green and Corey, 1971; Lam et al, 1987; Geo-Slope, 1987).

In this study, the relationships between the hydraulic conductivities and capillary pressure head for the saturated-unsaturated flow analysis and between the volumetric water content and capillary pressure head for the transient analysis procedure are obtained from other researchers' paper and experimental data presented. Fig. 2 shows the relationships with respect to loamy sand (low and medium permeability), loam (low permeability) and silty clay loam (very low permeability).

6. Model Configuration

The 2-dimensional model was formulated with 396 nodes and 350 elements. The soil depth is 2m at the top vertical boundary and 5m at lowest vertical boundary and ground surface slope is 34% as shown on Fig. 3. The elements type is quadlilateral bilinear which may be generally superior to triangular elements. Although the disadvantage of quadlilateral bilinear elements is that the element characteristic matrix formulation requires more computations than for triangular elements, they make mesh layout easier and contribute to better convergence of nonlinear analysis (Pinder and Gray, 1977; Geo-Slope, 1989).

The lower boundary is impermeable because of the distribution of rock. In fact the rock is permeable, through fractured rock, but the hydraulic conductivity is very small than the upper soil matrix and hence the permeability can be ignored in the practical purposes. The lower vertical boundary is also assumed impermeable because of the symmetric configuration and 10 cm of the initial ponding depth, means the normal flow depth in the channel, at the last node is allowed in order to set the initial condition.

For the transient analysis the hydraulic properties of the soil may be used the relationships on the Fig. 2 and assumed homogenous and isotropic soil. Hysteresis is considered only wetting curve in the simulations because of the difficulty to determine the scanning curve between wetting and drying curves, and ain entry value and capillary fringe The seepage face connected with channel is fluctuated from the channel to the upper flux boundary and effected on the position phreatic line, rainfall intensity and hydraulic properties of soil.

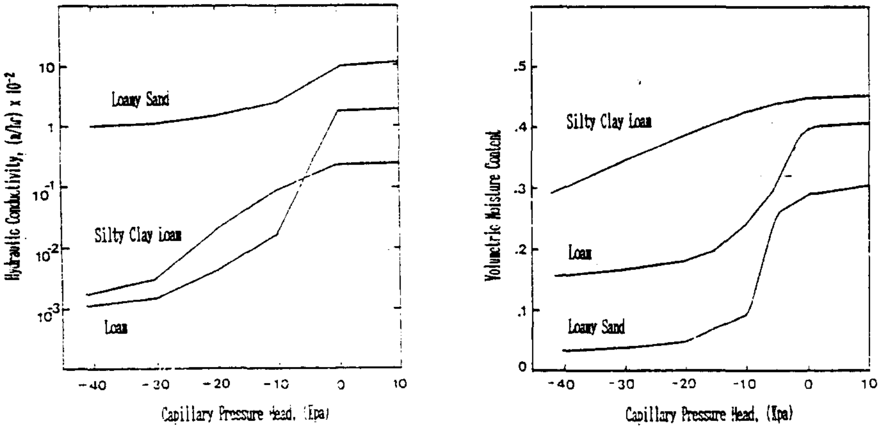

Fig. 2 Hydauric properties of soils

7. Results and Discussions

For the transient analysis the initial nodal conditions are necessary. In this study the initial

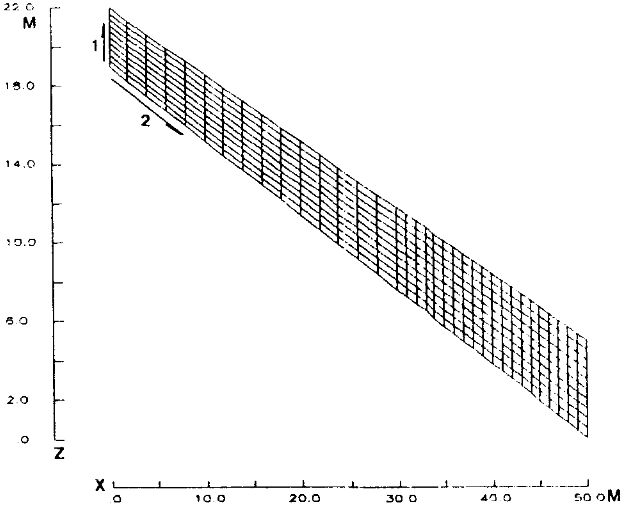

Fig. 3 Finite element mesh for subsurface flow model on a hillslope

nodal heads are obtained from steady state analysis. The convergence tolerance by head and hydraulic conductivity between the time steps sets in 5%.

The analysis is carried out with the actual storm data observed, occured at Oct. 10, 1986 and July 27, 1987 (see Table 1), from Jangpyung watershed which is one of the IHP, from UNESCO, experimental basins in Korea. And rainfall intensities applied in the analysis are considered initial losses by canopy interception, evapotranspiration, ground cover interception and depression storage (Nahm and Choi, 1990). The results are illustrated in Fig. 4 through 8. Fig. 4 shows the transient pressure heads according to the elapsed time in order to dominate the influince of soil textures.

Table 1 Storm data observed and infiltrated

Time(hr)	July 10, 1985		July 27, 1987	
	Observed	Infiltrated	Observed	Infiltrated
1	3.6	0.1	1.8	0.0
2	1.2	0.0	4.2	0.1
3	6.4	0.2	4.6	0.1
4	9.6	0.3	7.0	0.2
5	6.1	2.6	2.8	0.1
6	28.1	12.0	5.8	1.9
7	18.4	11.6	15.7	11.6
8	18.1	11.2	17.1	11.2
9	4.4	3.9	4.8	4.3
10	2.1	1.8	5.4	4.8
11			4.6	4.1
12			5.6	6.0
13			2.0	1.7
14			2.0	1.7
15			1.6	1.4
Σ	88.0	43.7	85.0	49.3

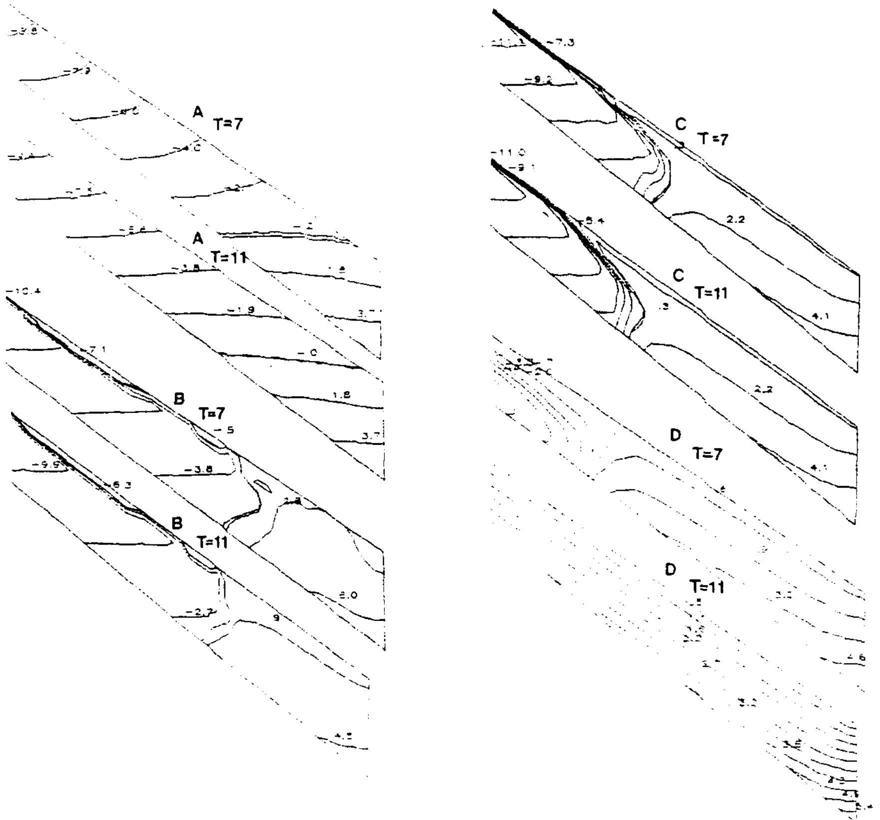

Fig. 4 Distribution of pressure heads for various soil textures: (a) loamy sand, (b) loam, (c) silty clay loam, and (d) loamy sand considering antecedent storm

Fig. 4 shows that the distribution of pressure head for each soil texture are varied along the elapsed time. As the results the hydraulic conductivity and the rates of change of storage in soil system effect on the distribution of pressure head and the position of phreatic line. The case of (d) on Fig. 4 can come out when the ground surface is covered by canopy and litter, storm occured in short period after antecedent precipitation, and appeared infiltration from snow melt. In the cases of (a), (b) and (c) on Fig. 4 the distributions of pressure head are similar to steady state at the first time step, i. e. elapsed time is 1hr, on the other hand, in the case of (d) is rapidly increased and after 7 hrs elapsed the change is small since the initial soil moisture deficits are less than other cases.

In order to confirm the influences of initial moisture depended on the antecedent precipitation and ground surface environments, the velocity vectors are plotted on Fig. 5 for the cases (a) and (d) in Fig. 4. The results show that the initail moisture condition is very sensitive to the subsurface storm flow. The phreatic lines are quickly rising in comparision of the cases (a) and (d) in Fig. 5.

Fig. 6 illustrates the transient positions of the phreatic line as it advances from its initial steady state condition to a final state with respect to three soil textures mentioned above.

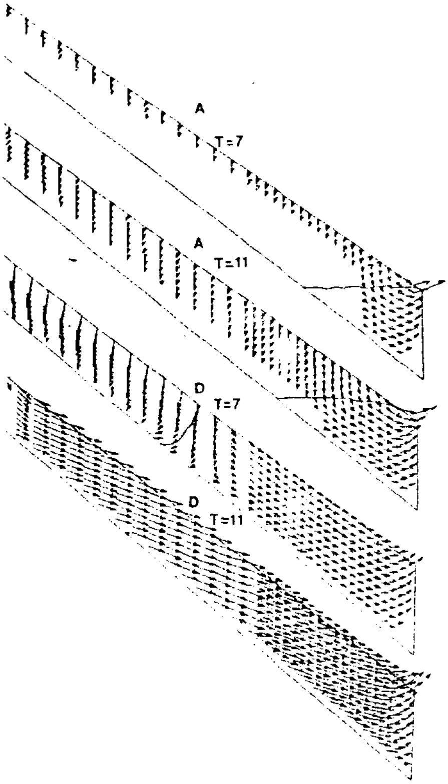

Fig. 5 Comparision of the velocity vectors to confirm the influences of antecedent moisture condition.

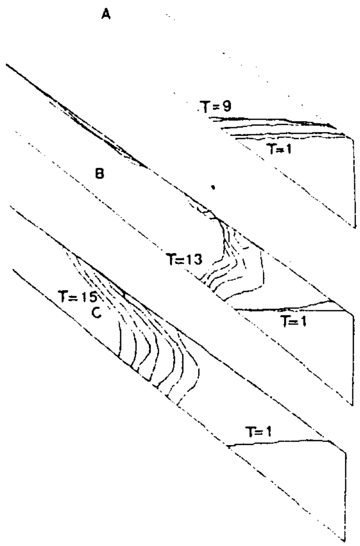

Fig. 6 Transient positions of phreatic line from steady state condition (a) loamy sand, (b) loam and (c) silty clay loam.

Fig. 6 shows that the higher hydraulic conductivity may be strongly effective to the transient position of phreatic line, however, the lower one acts on the time lag, time of concentration in saturated zone, to be considered as a part of the subsurface storm flow response. Beven(1982) stated that for the soil depths greater than 2 m the results suggested that the time lag, not flow rates, in unsaturated zone is going to be of the order of several days unless rainfall intensities are very high, the soil is very wet or saturated conductivities are high.

In general the changes in slope of ground surface, slope length and soil thickness do not affect to the hydrograph significantly (Freeze, 1972; Beven, 1982). Therefore, in present study, the effects of the soil hydraulic properties are analyzed.

The analysis are adopted to verify the effects of antecedent storm and hydraulic properties of each soil texture on the subsurface storm flow rates. The results are shown on Fig. 7 and 8. To demonstrate the effects of antecedent storm event the inflow hydrographs to channel through seepage face located at the near the channel wetlands are plotted on Fig. 7, and the hydrographs derived from unsteady rainfall events for each soil texture and layered with loamy sand and loam are shown on Fig. 8.

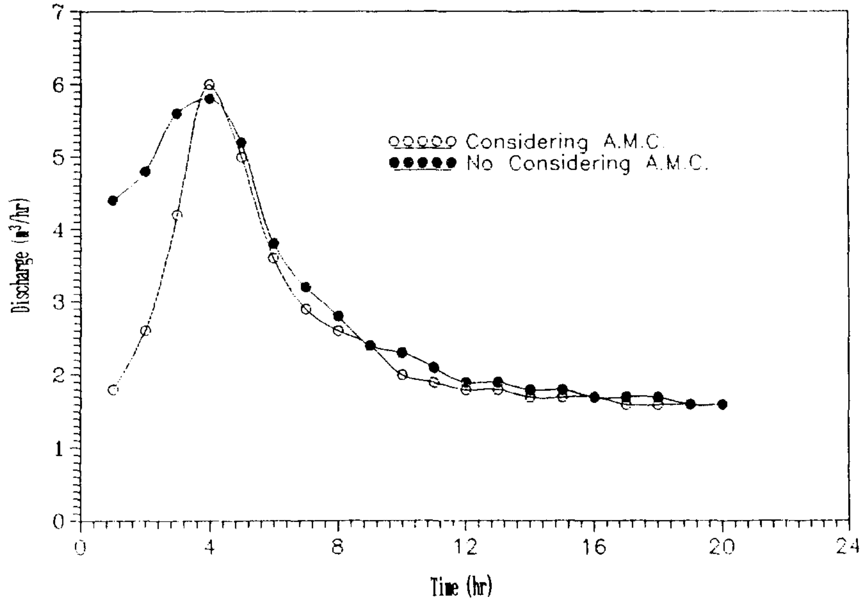

Fig. 7 Comparision of the subsurface storm flow hydrographs considering antecedent storm and no considering for the loamy sand

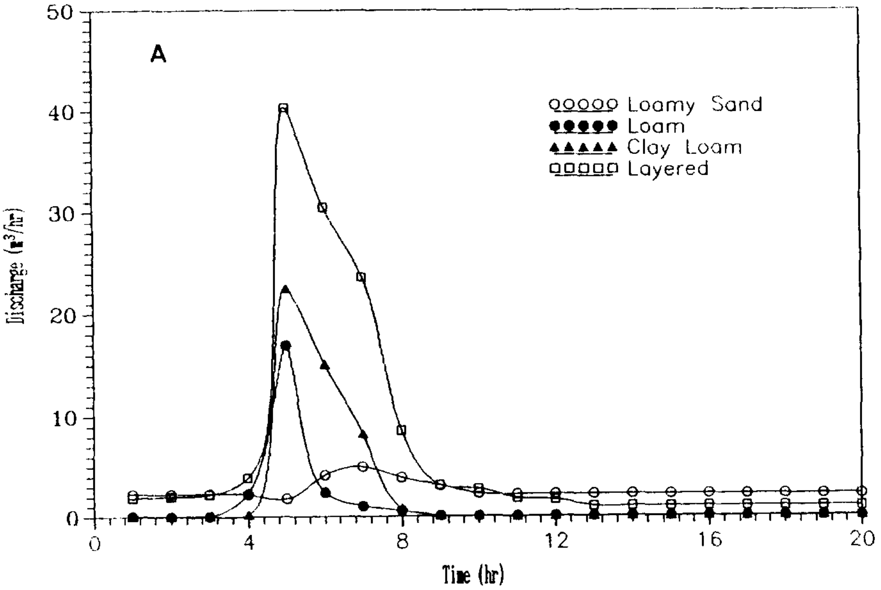

Fig. 8 Comparision of the runoff hydrographs for the storm of July 10, 1985.

8. Conclusions

The proposed 2 dimensional saturated-unsaturated transient finite element model based on Galerkin finite element scheme is well suited to analyze the subsurface storm flow responses on a hillslope. The analysis is carried out by using the observed storm data and the functional relationships between the hydraulic conductivties, capillary pressure head, and volumetric water content under unsaturated- saturated conditions. The results demonstrated that the hydraulic conductivity of the various hydraulic properties of soils is the most affected to the source area runoff generation and subsurface flow responses such as velocity vector, transient phreatic line, pressure head, time lag and channel inflow hydrograph shape. And the distribution of velocity vector, positions of phreatic line, rising and falling states, and distribution of pressure head with elapsed time are significantly influenced on the given soil properties which are hydraulic conductivities, storage rates and initial soil moisture conditions.

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