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# A GENERALIZATION OF DIFFERENTIAL FORMS AND ITS APPLICATION

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## Introduction

Our final purpose may be to introduce generalized differential forms on the space $\mathrm{Map}(S, M)$ of mappings from a manifold $S$ into a manifold $M$ and discuss the differential geometry of the space $\mathrm{Map}(S, M)$ from the point of the generalized forms.

Here we take a subspace $X$ of the space $\mathrm{Map}(S, M)$ and we introduce the generalized differential forms on $X$, taking the dual to the chain space with the flat norm. This method of construction allows us to discuss a sufficient condition for a subspace $Y$ of $X$ to admit the generalized differential forms and the natural integration as the dual operation.

## Flat norm and flat forms

We review the flat norm of the polyhedral chains, introduced by Whitney ([W] chap.5, §3). First denote by $|s|$ the volume of the Euclidean simplex $s$ and define the mass $|c|$ of the chain $c = \sum a(i)s(i)$ for finitely many disjoint simplexes $s(i)$ of dimension $r$ by

$$|c| = \sum |a(i)||s(i)|.$$

Then the flat norm $\|c\|$ of a polyhedral chain $c$ in Euclidean space is defined to be

$$\|c\| = \inf\{|c - bd(A)| + |A|\},$$

using all the $(r + 1)$ dimensional polyhedral chain $A$, for the boundary operator $bd$.

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The function $\|c\|$ of $c$ is proved to be a norm([W] Chap. 5 ,§3-12), using the sharp norm obtained by small parallel displacement of chain $c$.

If we take $c = bd(e)$ and $A = e$ then above definition yields that

$$\|bd(e)\| \leq \|e\| \leq |e| \tag{1}$$

which implies the continuity of the operator $bd$ on the space $C(E)$ of the polyhedral chain with the flat norm on the Euclidean space, hence we see that the dual operator $bd*$ of $bd$ in the dual space of the chain is again a continuous operator.

We may consider the dual space $C^*(E)$ of $C(E)$ as the space of cochain and the dual operator $bd*$ to be the coboundary operator, because of the following equality:

$$(bd*)(bd*) = 0.$$

A generalization of the Fubini's theorem shows that the cochains can be expressed as forms with measurable coefficients instead of the differntiable functions and the coboundary is also known to follow the same law as in the differentiable case so as to call them the flat forms and the (flat) differential operator.

Making use of the flat norm, we can consider the limit in the space of the polyhedral chains.

If we assume a continuous mapping satisfies Lipschitz condition in addition, we see that the linear mapping on the space of the polyhedral chains induced by the mapping is continous and functorial.

We can extend the notion of the flat norm to the Lipschitz singular chains $S(M)$ on a Riemannian manifold $M$ by a sequence of subdivisions and the notion of limit under the flat norm.

For this space of chains, we can consider the dual space $S^*(M)$ as the space of cochain and the dual operator $bd*$ to be the coboundary operator.

As in the Euclidean case, we can show that the cochains are expressed as forms on the manifold with measurable coefficients which may be called the flat forms with differential operator.

If we assume a continuous mapping between manifolds satisfies Lipschitz condition with respect to the Riemannian metrics, we see again the linear mapping on the space of the Lipschitz singular chains induced by the mapping is continuous and functorial.

## Generalization of differential forms

We may consider that the space of differential forms on the Euclidean space is a linear subspace of the cochain space which can be expressed by the integration together with its differential and that the space of differential forms on a (finite dimensional) manifold is a linear subspace of the cochain space of the manifold which can be locally expressed by that on the Euclidean space with respct to the local coordinate of the manifold so that the coordinate transformation gives the transformation of the forms.

If we take the flat norm on the space of the Euclidean chain then the continuous function on the space turns out to have the expression by the integration together with its differential and if we take the Lipshitz mapping then there corresponds a chain homomorphism so as to satisfy the chain rule, thus we may claim that the dual space $S^*(M)$ is the space of generalized differential forms.

In what follows, unless otherwise stated, we simply use the mapping to imply Lipschitz mapping, also singular to Lipschitz singular and we define generalized differential forms as a linear subspace of the dual space of the chain space with respect to the flat norm for a (generalized) manifold of Lipschitz coordinate covering by (generalized) Euclidean space.

Let $X$ be the mapping space of an Euclidean $n$-simplex $e$ into a Riemannian manifold $M$ which reduces to a point on the boundary $bd(e)$ of $e$.

If we take $e$ to be the line segment [0,1], then $X$ turns out to be the path space of $M$ so that our construction of the flat forms covers that space of curves.

An $n$-simplex $\sigma$ in $X$ is defined to be a mapping of an Euclidean $n$-simplex $s$ into $X$, and therefore represented as a mapping of $f$ the product space $s \times e$ into $M$, the space of the chains of these simplexes is denoted by $\Sigma(M)$.

After a suitable subdivision, we may assume that $f$ is a chain on $M$ which have the flat (pseudo-)norm $\|f\|$, because $M$ is a manifold, admitting the parallel displacement of $f$, which is necessary for the sharp norm.

The boundary operator $bd$ on $s \times e$ splits into that on $s$ and on $e$, which we denote by $bd'$ and by $bd''$ respectively.

We then have

$$bd = bd' + bd''$$

and that the boundary operator $Bd$ in $X$ corresponds to the operator $bd'$;

$$Bd \ \ f = f \mid bd'(s \times e)$$

Since $f$ is in $X, f|s \times bd(e)$ degenerates to a point, having 0 mass provided that $\dim f > 1$, hence we see that

$$bd\ f = bd' f \quad \text{for} \ \ f \ \ \text{of} \ \ \dim f > 1.$$

Thus we have that

$$\|bd \ \ f\| = \|bd' \ \ f\|$$

Therefore from (1), we deduce that

$$\|Bd \ \ f\| = \|bd \ \ f\| \leqq \|f\| \tag{2}$$

Since $M$ is a manifold, we can perform the parallel displacement of $f$, which is necessary for the sharp norm and to make the flat pseudo norm into a norm.

We define the flat norm of a singular chain $f$ in $X$, which we denote it again by $\|f\|$, to be the flat norm $\|f\|$ of $f$ considered as a chain of $s \times e$ into $M$, then we have from (2) that the operator $Bd$ is continuous with respect to flat norm thus defined.

Consider a tubular neighbourhood of a curve in $M$, which we may assume diffeomorphic to the Euclidean space, then the restriction map and the inclusion map give the restriction and the inclusion homomorphisms between $\Sigma(E)$ and $\Sigma(M)$, respectively.

Thus the functorial property of the homomorphisms gives the coordinate tranformtion on the space of the cochains $\Sigma*(E)$.

We can generalize above process as follows:

THEOREM 1. *Let $X$ be the mapping space $X$ of a Riemannian manifold $S$ into a Riemannian manifold $M$, satisfying that $f(bd(S))$ reduces to a point, then the function $\|f\|$ defined on the chain of $X$ by*

$$\|f\| = \inf\{|f - bd(A)| + |A|\},$$

*using all the $(r+1)$ dimensional chain $A$ in $X$, turns out to be a norm and the generalized defferetial form is defined on the space $X$.*

Take a closed linear subspace $Y$ in $X$, then the flat norm on $X$ induces a norm on $Y$, thus we can consider the dual space to $Y$ and we have that

THEOREM 2. *Let $Y$ be a closed linear subspace of the mapping space $X$ of $S$ into a Riemannian manifold $M$, then the generalized defferential form is defined on the subspace $Y$.*

## References

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