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# FUZZY $P$–LIMIT SPACES

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## 1. Introduction

The notion of convergence in a fuzzy topological space was introduced in various aspects as a local theory of fuzzy topologies [2, 9, 10, 11, 12, 15, 19, 20]. In [12] a notion of fuzzy limitierung on a set is defined in terms of prefilters convergent to a fuzzy point and allows natural function structure in it. In this paper using fuzzy x-filters we introduce a notion of convergence to a point in a set, called a *fuzzy* p-*limitierung*, generalizing that of fuzzy neighborhood system introduced by Warren [18], which characterizes fuzzy topology. The notion of *fuzzy* p-*limitierung* enables us to form a convenient class containing all fuzzy topological spaces and all limit spaces. In section 4 we show that in fuzzy topology there is no natural function space structure. We show that fuzzy p-limit spaces form a quasitopos and give explicitly natural function space structure in it, providing an exponential law $\mathcal{C}(X \times Y, Z) = \mathcal{C}(X, \mathcal{C}(Y, Z))$. Form a categorical point of view the category **FpLim** of fuzzy p-limit spaces andfuzzy continuous maps is shown to be a quasitopos (=topological universe) containing the category **FTop** of fuzzy topological spaces as a bireflective subcategory and the category **Lim** of limit spaces as a bicoreflective subcategory.

Throughout this paper, we use Lowen's notion of fuzzy topology [7]. For categorical background we refer to [4, 6]

## 2. Fuzzy p-limitierung

Warren [18] characterized a fuzzy topology in terms of a fuzzy neighborhood system, which defines a concept of convergence to a point in a fuzzy topological space. A fuzzy set $N$ in $X$ is called a *neighborhood*

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*of a point* $x$ in $(X,\delta)$ iff there exists $U \in \delta$ such that $U \subseteq N$ and $0 < \mu_U(x) = \mu_N(x)$. A fuzzy set $U$ is open in $(X,\delta)$ iff for every $x \in X$ with $\mu_U(x) > 0$, $U$ is a neighborhood of $x$. Thus this neighborhood system determines a fuzzy topology. A function $f : (X,\delta) \to (Y,\delta')$ is fuzzy continuous at $x$ iff for every neighborhood $N$ of $f(x)$ there exists a neighborhood $M$ of $x$ such that $f(M) \subseteq N$ and $\mu_M(x) = \mu_N(f(x))$.

DEFINITION 2.1. Let $X$ be a set and $x \in X$. A *fuzzy* x *-filter* $\mathcal{F}$ on $X$ is a family of fuzzy sets in $X$ subject to the following axioms:

(F1) if $A \in \mathcal{F}$ and $A \subseteq_x B$, then $B \in \mathcal{F}$, ($A \subseteq_x B$ means $A \subseteq B$ and $\mu_A(x) = \mu_B(x)$)
(F2) for all $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$,
(F3) $\underline{0} \notin \mathcal{F}$, where $\underline{0}$ is the constant map with value 0,
(F4) for every subfamily $\{A_i\}_I$ of $\mathcal{F}$, $\cup_{i\in I} A_i \in \mathcal{F}$.

The family of all neighborhoods of $x$ in a fuzzy topological space $X$ is a fuzzy x-filter on $X$, called the *neighborhood fuzzy* x*-filter*.

Let $\mathcal{B}$ be a family of fuzzy sets in $X$. If $\mathcal{B}$ satisfies the conditions, $\underline{0} \notin \mathcal{B}$ and for any $B_1, B_2 \in \mathcal{B}$, there exists $B \in \mathcal{B}$ with $B_1 \cap B_2 \subseteq_x B$, then $B$ generates a fuzzy x-filter $[\mathcal{B}] = \{A \in I^X : \cup_{i\in I} B_i \subseteq_x \text{A for some family } \{B_i\}_I \text{ in } \mathcal{B}\}$. Denote $\mathcal{F}_x(X)$ = the collection of all fuzzy x-filter on $X$. For $\mathcal{F}, \mathcal{G} \in \mathcal{F}_x(X)$, we denote $\mathcal{F} \subseteq \mathcal{G}$ in $\mathcal{F}_x(X)$ iff for each $A \in \mathcal{F}$ there exists $B \in \mathcal{G}$ with $B \subseteq_x A$. For $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X)_x$, the family $\{A \cup B : A \in \mathcal{F}, B \in \mathcal{G}, \mu_A(x) = \mu_B(x)\}$ generates a fuzzy x-filter, denoted by $\mathcal{F} \cap \mathcal{G}$ in $\mathcal{F}_x(X)$. We note that if $\mathcal{F}_i \subseteq \mathcal{G}_i$ in $\mathcal{F}_x(X)$ for $i = 1, 2, \cdots, n$, then $\cap_{i=1}^n \mathcal{F}_i \subseteq \cap_{i=1}^n \mathcal{G}_i$ in $\mathcal{F}_x(X)$.

For a function $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_x(X)$, we define a fuzzy $f(x)$ -filter $f(\mathcal{F}) = \{B \in I^Y : f(A) \subseteq B \text{ with } 0 < \mu_A(x) = \mu_B(f(x)) \text{ for some } A \in \mathcal{F}\}$ on $Y$. In general, for $A \in \mathcal{F}$, $f(A)$ needs not be a member of $f(\mathcal{F})$. However if $\mu_A(x) = \alpha > 0$, then $f(A \cap \underline{\alpha}) \in f(\mathcal{F})$. Hence we can show that for $f : X \longrightarrow Y, g : Y \longrightarrow Z$ and $\mathcal{F} \in \mathcal{F}_x(X)$, $g \circ f(\mathcal{F}) = g(f(\mathcal{F}))$.

DEFINITION 2.2. A *fuzzy* p*-limit* space (fps for short) is a set $X$ structured with a function $\Delta$, called a *fuzzy* p*-limitierung*, which assigns to every $x \in X$ a set $\Delta(x)$ of fuzzy x-filters on $X$ subject to the following axioms:

(L0) $\mathcal{F} \in \Delta(x) \Rightarrow \underline{\alpha} \in \mathcal{F}$ for all $\alpha > 0$,

(L1) $[x] = \{A \in I^X : \mu_A(x) > 0\} \in \Delta(x)$,
(L2) $\mathcal{F} \in \Delta(x)$ and $\mathcal{F} \subseteq \mathcal{G}$ in $\mathcal{F}_x(X) \Rightarrow \mathcal{G} \in \Delta(x)$,
(L3) $\mathcal{F}, \mathcal{G} \in \Delta(x) \Rightarrow \mathcal{F} \cap \mathcal{G} \in \Delta(x)$.

If $\mathcal{F} \in \Delta(x)$, we say that $\mathcal{F}$ converges to $x$ and sometimes write $\mathcal{F} \longrightarrow x$ instead of $\mathcal{F} \in \Delta(x)$.

REMARK. A subset $A$ of a set $X$ can be identified with the characteristic function $\chi_A$. Thus if we substitute $I$ by $\{0,1\}$, the two point chain, then it is easy to see that the notion of fuzzy p-limitierung is equivalent to that limitierung.

We introduce the notion of initial and final fuzzy p-limitierung. If the identity map $1_X : (X, \Delta_1) \to (X, \Delta_2$ is fuzzy continuous, then we say that $\Delta_1$ is finer than $\Delta_2$ and that $\Delta_2$ is coarser than $\Delta_1$.

THEOREM 2.3. *(Existence of initial structure) Let $X$ be a set, let $\{(Y_i, \Delta_i)\}_I$ be a family of fps's, and for each $i \in I$ let $f_i : X \to Y_i$ be a map. Define a function $\Delta$ on $X$ as follows; for each $x \in X$ a fuzzp x-filter $\mathcal{F} \in \Delta_{(x)}$ iff $f_i\mathcal{F} \in \Delta_i(f_i(x))$ for each $i \in I$. Then $\Delta$ is the initial fuzzy p-limitierung on $X$ w.r.t. the family $\{f_i\}_I$, which is the coarsest fuzzy p-limitierung on $X$ making each map $f_i$ fuzzy continuous.*

*Proof.* It follows from the fact that $f([x]) = [f(x)]$ and $f(\mathcal{F} \cap \mathcal{G}) = f(\mathcal{F}) \cap f(\mathcal{G})$ fuzzy x-filters $\mathcal{F}, \mathcal{G}$.

The existence of initial structures guarantees that of final structures, However, we present here an explicit form of final fuzzy p-limitierung: Let $Y$ be a set, let $\{(X_i, \Delta_i)\}_I$ be a family of fps's and for each $i \in I$ let $f_i : X_i \longrightarrow Y$ be a map. Define a function $\Delta$ on $Y$ as follows, for each $y \in Y$ a fuzzy y-filter $\mathcal{F} \in \Delta(y)$ iff $\mathcal{F} \supseteq [y]$ or $\mathcal{F} \supseteq \cap_{k=1}^{n} f_{ik}(\mathcal{F}_k)$, where $\mathcal{F}_k \in \Delta_{ik}(x_k)$ for some $x_k \in f_{ik}^{-1}(y), i_k \in I$ and $k = 1, \cdots, n$. Then $\Delta$ is the final fuzzy p-limitierung on $Y$ w.r.t. the family $\{f_i\}_I$, which is the finest fuzzy p-limitierung on $Y$ making each $f_i$ fuzzy continuous. In a usual way, the notions of subspace, product, quotient space and coproduct can be defined.

## 3. Relationship with fuzzy topology and limitierung

Let $(X, \delta)$ be a fuzzy topological space (fts for short). For each $x \in X$, let $\Delta_\delta(x) = \{\mathcal{F} \in \mathcal{F}_x(X) : \mathcal{N}_x \subseteq \mathcal{F}\}$, where $\mathcal{N}_x$ is the neighborhood fuzzy x-filter. Then $\Delta_\delta$ is a fuzzy p-limitierung on $X$. We call a fuzzy p-limitierung *fuzzy topological* if the convergent fuzzy x-filters are precisely those of a fuzzy topology. We show that a fuzzy p-limitierung is not fuzzy topological in general. Let $(X, \Delta)$ be a fps. A fuzzy set $U$ in $X$ is said to be *open* iff for every $x \in X$ with $\mu_U(x) > 0, \mathcal{F} \to x$ implies $U \in \mathcal{F}$. Then the collection $\delta_\Delta$ of all open fuzzy sets in $(X, \Delta)$ forms a fuzzy topology on $X$. Moreover, by routine work, we have the following.

PROPOSITION 3.1. *(1) For any fuzzy topology $\delta$ on $X, \delta = \delta_{\Delta_\delta}$.*
*(2) A fuzzy p-limitierung $\Delta$ on $x$ is fuzzy topological iff $\Delta = \Delta_{\delta_\Delta}$.*
*(3) For every $x \in X, \Delta(x) \subseteq \Delta_{\delta_\Delta}(x)$.*

Let $X$ and $Y$ be fps's. A function $f : X \to Y$ is said to be *fuzzy continuous at* $x \in X$ iff $f(\mathcal{F}) \to f(x)$ in $Y$, whenever $\mathcal{F} \to x$ in $X$. A function $f : X \longrightarrow Y$ is fuzzy continuous iff it is *fuzzy continuous* at every $x \in X$. Note that the identity map $1_X : X \longrightarrow X$ is fuzzy continuous since $1_X(\mathcal{F}) = \mathcal{F} \cap [x]$ for a fuzzy x-filter $\mathcal{F}$. Recall that for fts's $X, Y$, a map $f : X \longrightarrow Y$ is fuzzy continuous at $x \in X$ iff $\mathcal{N}_{f(x)} \subseteq f(\mathcal{N}_x)$ in $\mathcal{F}_{f(x)}(Y)$. Thus by definitions we have

PROPOSITION 3.2. *(1) A function $f : (X, \delta) \longrightarrow (Y, \delta')$ is fuzzy continuous iff $f : (X, \Delta_\delta) \longrightarrow (Y, \Delta_{\delta'})$ is fuzzy continuous.*
*(2) If $f : (X, \Delta) \longrightarrow (Y, \Delta')$ is fuzzy continuous, then $f : (X, \delta_\Delta) \longrightarrow (Y, \delta_{\Delta'})$ is fuzzy continuous.*

Let $(X, \wedge)$ be a limit space and $\mathcal{F} \to x$ in $X$. Then the set $\mathcal{F}_\mathcal{F}$ of all fuzzy sets $A$ in $X$ such that $\mu_A(\mathcal{F}) \to \mu_A(x)$ in $I_r$ (=I with the right topology) and $\mu_A(x) > 0$ form a fuzzy x-filter on $X$. Let $\Delta_\wedge(x) = \{\mathcal{F} \in \mathcal{F}_x(X) : \mathcal{F} \supseteq \mathcal{F}_\mathcal{F}$ for some $\mathcal{F} \in \wedge(x)\}$ Then $\Delta_\wedge$ is a fuzzy p-limitierung on $X$. Note that $\mathcal{F}_{\mathcal{F} \cap \mathcal{G}} = \mathcal{F}_\mathcal{F} \cap \mathcal{F}_\mathcal{G}$ for $\mathcal{F}, \mathcal{G} \in \wedge(x)$. Let $(X, \Delta)$ be a fps. For each $\mathcal{F} \in \Delta(x)$ let $\Omega_\mathcal{F}$ be the set of all filters $\mathcal{F}$ on $X$ such that $\mu_A(\mathcal{F}) \to \mu_A(x)$ in $I_r$ for each $A \in \mathcal{F}$ with $\mu_A(x) > 0$. Then $\wedge_\Delta(x) = \cap_{\mathcal{F} \in \Delta(x)} \Omega_\mathcal{F}$ defines a limitierung on $X$, since $\mathcal{F} \cap \mathcal{G}$ in $\Omega_{\mathcal{F} \cap \mathcal{G}}$ for $\mathcal{F} \in \Omega_\mathcal{F}$, $\mathcal{G} \in \Omega_\mathcal{G}$.

PROPOSITION 3.3. *(1) For any limitierung $\wedge$ on $X, \wedge = \wedge_{\Delta_\wedge}$.*
*(2) For any fuzzy p-limitierung $\Delta$ on $X, \Delta_{\wedge_\Delta}(x) \subseteq \Delta(x)$.*

*Proof.* (1) By defintions, $\wedge(x) \subseteq \wedge_{\Delta_\wedge}(x)$. Let $\mathcal{F} \in \wedge_{\Delta_\wedge}(X)$. Then $\mathcal{F} \in \Omega_\mathcal{F}$ for some $\mathcal{F} \in \Delta_\wedge(x)$ and $\mathcal{F} \supseteq \mathcal{F}_\mathcal{G}$ for some $\mathcal{G} \in \wedge(x)$. We

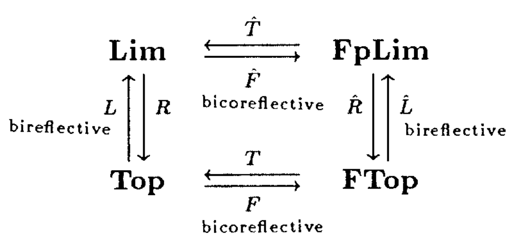
show that $\mathcal{F} \supseteq \mathcal{G} \cap <x>$ : Let $V \in \mathcal{G} \cap < x >$ . Then the characteristic function $\chi_V \in \mathcal{F}_{\mathcal{G}}$ and hence $\chi_V(\mathcal{F}) \to \chi_V(x)$ in $I_r$. Thus $V \supseteq F$ for some $F \in \mathcal{F}$.
(2) It is immediate from definitions.

PROPRSITION 3.4. *(1) If a map $f : (X, \Delta) \to (Y, \Delta')$ is fuzzy continuous, then $f : (X, \wedge_\Delta) \to (Y, \wedge_{\Delta'})$ is continuous.*
*(2) A map $f : (X, \wedge) \to (Y, \wedge)$ is continuous iff $f : (X, \Delta_\wedge) \to (Y, \Delta_\wedge)$ is fuzzy continuous.*

*Proof.* It is easy to show by definitions and Proposition 5.1.

## Categorical comments

The category **FTop** is known to be a topological category [8]. A category **FpLim** is formed by all fuzzy p-limit spaces and fuzzy continuous maps. By Propositions 3.1 and 3.2, **FTop** is shown to be a bireflective subcategry of **FpLim** in a natural way. By Theorem 2.3 and the definition of fuzzy p-limitierung, **FpLim** is a topological category. By Propositions 3.3 and 3.4 the category **Lim** is shown to be a bicoreflective subcategory of **FpLim**. We recall that the category **Top** of topological spaces is a bicoreflective subcategory of **FTop**[7] and **Top** is a bireflective subcategory of **Lim**[3]. Let $L(\hat{L})$ be the embedding functor from **Top**(**FTop**) into **Lim**(**FpLim**) with reflectors $R(\hat{R})$, respectively. Let $F(\hat{F})$ be the embedding functor form **Top**(**Lim**) into **FTop**(**pLim**) with coreflector $T(\hat{T})$, respectively. We note that $F(X) = (X, \mathcal{C}(X, I_r))$ and $T(X, \delta) = (X, \mathcal{T}_\delta)$, where $\mathcal{T}_\delta$ is initial topology w.r.t. the family $\{\mu : X \to I_r : \mu \in \delta\}$. Then we have the following diagram:



THEOREM 3.5. *(1)* $\hat{T}\circ\hat{L} = L\circ T$, *(2)* $\hat{R}\circ\hat{F} = F\circ R$, *(3)* $\hat{L}\circ F = \hat{F}\circ L$.

*Proof.* (1) and (2) are equivalent by the uniqueness of adjoint functor. We show (1). $\mathcal{F} \to x$ in $\hat{T}\circ\hat{L}(X,\delta)$ iff $\mathcal{F} \in \Omega_{\mathcal{N}_x}$, where $\mathcal{N}_x$ is the neighborhood fuzzy x-filter in $(X,\delta)$, iff $\mathcal{F} \supseteq \mathcal{N}_x$ in $T(X,\delta)$, where $\mathcal{N}_x$ is the neighborhood filter of $x$ in $T(X,\delta)$ iff $\mathcal{F} \to x$ in $L\circ T(X,\delta)$. (3) $\mathcal{F} \to x$ in $\hat{L}\circ\mathcal{F}(X,\mathcal{T})$ iff $\mathcal{F} \supseteq \mathcal{N}_x$, where $\mathcal{N}_x$ is the neighborhood fuzzy x-filter in $F(X,\mathcal{T}) = (X,\delta_{\mathcal{T}})$. Note that if $\mu_A : X \to I_r$ is continuous at $x$, then there exists a continuous map $\mu_B : X \to I$ such that $B \subseteq A$ and $\mu_B(x) = \mu_A(x)$. Hence $\mathcal{N}_x = \mathcal{F}_{\mathcal{N}_x}$, where $\mathcal{N}_x$ is the neighborhood filter of $x$ in $(X,\mathcal{T})$. Therefore $\mathcal{F} \to x$ in $\hat{L}\circ F(X,\mathcal{T})$ iff $\mathcal{F} \to x$ in $\hat{F}\circ L(X,\mathcal{T})$.

REMARK. This Theorem shows that $\hat{T}$ and $\hat{\mathcal{F}}$ are extensions of $T$ and $F$ by (1) and (3), repectively, and $\hat{L}$ and $\hat{R}$ are extensions of $L$ and $R$ by (3) and (2), respectively.

COROLLARY. *If a fps $(X,\Delta)$ is fuzzy topological, then $\hat{T}(X,\Delta)$ is topological.*

*Proof.* By Proposition 3.1 and Theorem 3.5, $\hat{T}(X,\Delta) = \hat{T}\hat{L}\hat{R}(X,\Delta)$ $= LT\hat{R}(X,\Delta)$.

REMARK. It is well known [16] that there exists a limit space $(X,\wedge)$, which is not equal to $L(X,\mathcal{T})$ for any topology $\mathcal{T}$ on $X$. Hence $\hat{F}(X,\wedge)$ $= \hat{L}(X,\delta)$ for any fuzzy topology $\delta$ on $X$, i.e., $\hat{F}(X,\wedge)$ is not fuzzy topological, by Proposition 3.3(1) and Theorem 3.5(1). Moreover there exists a fts $(X,\delta)$ which is not equal to $F(X,\mathcal{T})$ for any topology $\mathcal{T}$ on $X$ [7]. Hence $\hat{L}(X,\delta) = \hat{F}(X,\wedge)$ for any limitierung $\wedge$ on $X$ by Proposition 3.1(1) and Theorem 3.5(2).

## 4. Function spaces

Since the category **Top** is a bicoreflective subcategory of **FTop** and the full embedding functor from **Top** into **FTop** preserves initial sources, by Proposition 2.4 in [13] **FTop** is not cartesian closed. This means that there is no natural function space structure in **FTop**. However we can show that **FpLim** has a natural function space structure.

In fact, **FpLim** is shown to be a quasitopos(= topological universe [14]) in the sense of Herrlich [5], equivalently cartesian colsed and hereditary.

THEOREM 4.1. *Final epi-sinks in* **FpLim** *are preserved by pullbacks.*

*Proof.* Let $\{f_i : X_i \to Y\}_I$ be a final epi-sink in **FpLim**, $g : Z \to Y$ be a fuzzy continuous map and for each $i \in I$ the diagram

$$\begin{array}{ccc} W_i & \xrightarrow{k_i} & X_i \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle f_i} \\ Z & \xrightarrow{g} & Y \end{array}$$

a pullback. We will show that $\{h_i : W_i \to Z\}_I$ is a final epi-sink in **FpLim**.

Note that for each $i \in I$, $W_i = \{(z, x_i) : f_i(x_i) = g(z)\}$ is a subspace of the product $Z \times X$ and $h$ and $k$ are projections. Clearly $\{h_i\}_I$ is epi-sink. Let $\mathcal{H} \to b$ in $Z$. Then $g(\mathcal{H} \cap [b]) \to g(b)$ in $Y$ and hence there exists $\mathcal{F}_i \to a_i$ in $X_i (i = 1, \cdots, n)$ such that $g(\mathcal{H} \cap [b]) \supseteq \cap_{i=1}^{n} f_i(\mathcal{F}_i \cap [a_i]$ in $\mathcal{F}_{g(b)}(Y)$ and $f_i(a_i) = g(b)$. Let $\mathcal{G}_i$ be the fuzzy $(b, a_i)$-filter on $W_i$ generated by $\{[H, F_i] : H \in \mathcal{H} \cap [b], F_i \in \mathcal{F}_i \cap [a_i]\}$, where $\mu_{[H,F_i]}(z, x_i) = \mu_H(z) \wedge \mu_{F_i}(x_i)$. Then $h_i(\mathcal{G}_i) \to b$ in $Z$, since $h_i(\mathcal{G}_i) \supseteq \mathcal{H} \cap [b]$ (Note that $h_i([H, F_i] \cap t) \subseteq H$, where $t = \mu_{[H,F_i]}(a_i)), k_i(\mathcal{G}_i) \to a_i$ in $X$ and hence $\mathcal{G}_i \to (b, a_i)$ in $W_i$. We claim that $\mathcal{H} \cap [b] \supseteq \cap_{i=1}^{n} h_i(\mathcal{G}_i)$. Take $A \in \cap_{i=1}^{n} h_i(\mathcal{G}_i)$. Then there exist $H_i \in \mathcal{H} \cap [b]$ and $F_i \in \mathcal{F}_i \cap [a_i]$ such that $\cup_{i=1}^{n} h_i([H_i, F_i]) \subseteq A$ with $\mu_{H_i}(b) \cap \mu_{F_i}(a_i) = \mu_A(b) > 0$. for each $i = 1, \cdots, n$. Denote $\alpha^* = \mu_A(b)$. Note that $\mu_{f_i(\mathcal{F}_i \cap \underline{\alpha}^*)}(f_i(a_i)) = \alpha^*$ for each $i = 1, \cdots, n$ and hence $\cup_{i=1}^{n} f_i(F_i \cap \underline{\alpha}^*) \in \cap_{i=1}^{n} f_i(\mathcal{F}_i[a_i])$. Thus there exists $H \in \mathcal{H} \cap [b]$ such that $g(H) \subseteq U_{i=1}^{n} f_i(F_i \cap \underline{\alpha}^*)$ with $0 < \mu_H(b) = \alpha^*$. Now we show that $K = (\cap_{i=1}^{n} H_i) \cap H \subseteq_b A$. Take any $z \in Z$. Then $\mu_H(z) \leq \mu_{g(H)}(g(z)) \leq \vee_{i=1}^{n} \sup_{x_i' \in f_i^{-1}(g(z))} \mu_{F_i}(x_i') \wedge \alpha^*$ and hence

$$\begin{aligned} \mu_K(z) &= (\wedge_{i=1}^{n} \mu_{H_i}(z)) \wedge \mu_H(z) \\ &\leq (\wedge_{i=1}^{n} \mu_{H_i}(z)) \wedge (\vee_{i=1}^{n} \sup_{(z,x_i') \in W_i} (\mu_{F_i}(x_i') \wedge \alpha^*)) \end{aligned}$$

$$\leq \vee_{i=1}^{n}(\mu_{H_i}(z) \wedge \sup_{(z,x_i')\in W_i} \mu_{F_i}(x_i'))$$
$$= \vee_{i=1}^{n} \sup_{(z,x_i')\in W_i} \mu_{[H_i,F_i]}(z,x_i')$$
$$= \vee_{i=1}^{n} \mu_{h_i([H_i,F_i])}(z) \leq \mu_A(z).$$

Obviously $\mu_K(b) = \mu_A(b)$. Hence the result follows.

This Theorem means that **FpLim** is a quasitopos and hence cartesian closed and hereditary. (So, for any fps $X$ the comma category **FpLim**$/X$ is cartesian closed. [5]) However there is no subobject classifier in **FpLim**, since a bimorphism is not an isomorphism in general. Cartesian closedness implies the existence of a natural function space structure, which is now introduced explicitly.

For fps's $X, Y$, let $\mathcal{C}(X,Y)$ be the set of all fuzzy continuous maps from $X$ into $Y$. For a fuzzy set $L$ in $\mathcal{C}(X,Y)$ and a fuzzy set $A$ in $X$, we define a fuzzy set $L(A)$ in $Y$ by $\mu_{L(A)}(y) = \sup_{(x,g)\in ev^{-1}(y)} \mu_L(g) \wedge \mu_A(x)$, if $ev^{-1}(y) = \emptyset$, and 0, otherwise, where $ev$ is the evaluation map. Let $\mathcal{H}$ be a fuzzy f-filter on $\mathcal{C}(X,Y)$ and $\mathcal{A}$ be a fuzzy x-filter on $X$. The family $\{B \in I^Y : L(A) \subseteq B, 0 < \mu_L(f) = \mu_A(x) = \mu_B(f(x))$ for some $L \in \mathcal{H}, A \in \mathcal{A}\}$ generates a fuzzy $f(x)$-filter $\mathcal{H}(\mathcal{A})$ on $Y$. Define a function $\Delta$ on $\mathcal{C}(X,Y)$ as follows; for each $f \in \mathcal{C}(X,Y)$ a fuzzy f-filter $\mathcal{H} \in \Delta(f)$ iff for each $x \in X, \mathcal{H}(\mathcal{A}) \to f(x)$ in $Y$, whenever $\mathcal{A} \to x$ in $X$. Then $\Delta$ is the natural fuzzy p-limitierung on $\mathcal{C}(X,Y)$, i.e., with respect to this function space structure, $ev : X \times \mathcal{C}(X,Y) \to Y$ is fuzzy continuous and for a fuzzy continuous map $h : X \times Z \to Y$ there exists a unique fuzzy continuous map $h^* : Z \to \mathcal{C}(X,Y)$ such that $ev\circ(1_X \times h^*)$. (Cf. [12]) Thus we have the following convenient properties [4]:

(1) $\mathcal{C}(X \times Y, Z) = \mathcal{C}(X, \mathcal{C}(Y,Z))$ (first exponential law)
(2) $\mathcal{C}(X, \prod Y_i) = \prod \mathcal{C}(X, Y_i)$ (second exponential law)
(3) $\mathcal{C}(\coprod X_i, Y) = \prod \mathcal{C}(X_i, Y)$ (third exponential law)
(4) $X \times \coprod X_i = \coprod (X \times X_i)$ (distributive law)
(5) Finite products of quotient maps are quotient maps.

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