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SOME PROPERTIES OF HYPOELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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I. Introduction

Let m, ρ and δ be real numbers; $0 \leq \delta \leq 1$, $0 \leq \rho \leq 1$. The class $S^m_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n)$ consists of functions $\sigma(x,\zeta) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ such that for any multi-indices α, β and any compact set $K \subset \mathbf{R}^n$ a constant $C_{\alpha,\beta,K}$ exists for which

$$(1.1) \qquad \left|\partial_\zeta^\alpha \partial_x^\beta \sigma(x,\zeta)\right| \leq C_{\alpha,\beta,K}|\zeta|^{m-\rho|\alpha|+\delta|\beta|}$$

where $x \in K$ and $\zeta \in \mathbf{R}^n$. Instead of $S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ we simply write $S^m(\mathbf{R}^n \times \mathbf{R}^n)$. We also put $S^{-\infty} = \cap_m S^m$.

A function $\sigma(x,\zeta) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ is called a hypoelliptic symbol if the following conditions are fulfilled:

(i) there exist real numbers m_0 and m, such that for an arbitrary compact set $K \subset \mathbf{R}^n$ one can find positive constants R, C_1 and C_2 such that

$$(1.2) \qquad C_1|\zeta|^{m_0} \leq |\sigma(x,\zeta)| \leq C_2|\zeta|^m$$

where $|\zeta| \geq R$ and $x \in K$.

(ii) there exist numbers ρ and δ, with $0 \leq \delta < \rho \leq 1$, and for each compact set $K \subset \mathbf{R}^n$ a positive constant R such that for any multi-indices α and β

$$(1.3) \qquad \left|[\partial_\zeta^\alpha \partial_x^\beta \sigma(x,\zeta)]\sigma^{-1}(x,\zeta)\right| \leq C_{\alpha,\beta,K}|\zeta|^{-\rho|\alpha|+\delta|\beta|}, \quad |\zeta| \geq R,\ x \in K$$

with some constant $C_{\alpha,\beta,K}$.

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Denote by $HS^{m,m_0}_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n)$ the class of symbols satisfying (1.1) and (1.2) for fixed m, m_0, ρ and δ. From (1.1) and (1.2), it obiously follows that $HS^{m,m_0}_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n) \subset S^m(\mathbf{R}^n \times \mathbf{R}^n)$ (see p. 38 of Shubin [6]). We will denote by $HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$ the class of properly supported pseudodifferential operator T for which $\sigma_T(x,\zeta) \in HS^{m,m_0}_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n)$.

DEFINITION 1.1. A pseudodifferential operator T is called to be hypoelliptic if there exists a properly supported pseudodifferential operator $T_1 \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$ such that $T = T_1 + R_1$, where $R_1 \in L^{-\infty}(\mathbf{R}^n)$, i.e., R_1 is an operator with infinitely differentiable kernel.

REMARK 1.2. If $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$ and if $m = m_0$, then it follows from proposition 5.1 in Shubin [1] that T is elliptic.

The aim of this paper is to study some properties of hypoelliptic pseudodifferential operators on $L^p(\mathbf{R}^n)$, $1 < p < \infty$. In section 2, we prove that if $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$ and $D(T) = S$, then T is closable. Here S is the Schwartz class. This result is proposition 3.1 in Wong [5] if $\rho = 1$, $\delta = 0$, and $m = m_0$. Also, we prove that if $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$ and if there exist two positive constants C, C' such that (2.2) holds, then $D(T_{\min}) = H^{m,p}$. For $\rho = 1$, $\delta = 0$, and $m = m_0$, it follows from Theorem 3.5 in Wong [5] that $D(T_{\min}) = H^{m,p}$. If $\sigma_T \in S^m(\mathbf{R}^n \times \mathbf{R}^n)$ is any symbol independent of x, then it follows from theorem 2.4 in Wong [3] that the minimal and the maximal operators associated with T coincide in $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$. See Chapter 4 of Schecter [2] for the minimal and maximal operators.

REMARK 1.3. If $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$, then it follows from proposition 5.3 in Schubin [1] that $T^* \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$, where T^* is the adjoint of T.

II. Main results

PROPOSITION 2.1. *If $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$ and the domain $D(T)$ of T is S, then T is closable.*

Proof. Let $\{\Phi_k\}$ be a sequence of functions in S such that $\Phi_k \to 0$, $T\Phi_k \to f$ in $L^p(\mathbf{R}^n)$ as $k \to \infty$. Then for any function ψ in S, we have $(T\Phi_k, \psi) = (\Phi_k, T^*\psi)$. Let $k \to \infty$. Then we have $(f, \psi) = 0$ for all function ψ in S. Since S is dense in $L^p(\mathbf{R}^n)$, it follows that $f = 0$.

REMARK 2.2. A consequence of Proposition 2.1 is that $T : S \to L^p(\mathbf{R}^n)$ has a closed extension in $L^p(\mathbf{R}^n)$. We denote the smallest such by $T_{\min}$ and call it the minimal operator of T. It can be shown easily that the domain $D(T_{\min})$ of $T_{\min}$ consists of all functions u in $L^p(\mathbf{R}^n)$ for which a sequence $\{\Phi_k\}$ in S can be found such that $\Phi_k \to u$ in $L^p(\mathbf{R}^n)$ and $T\Phi_k \to f$ for some f in $L^p(\mathbf{R}^n)$. Moreover, $T_{\min} u = f$, see again Wong[5].

REMARK 2.3. Now, it follows from Schechter [2, pp. 60–61] that there exist the maximal extension of $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$. Indeed, we can define another closed extension of T_1 of T on S as follows. We say that $u \in D(T_1)$ and $T_1 u = f$ if u and f are in $L^p(\mathbf{R}^n)$ and $(u, T^*\psi) = (f, \psi)$ for all $\psi \in S$. It is clear from the definition that T_1 is a closed extension of T on S. It is called the maximal or weak extension of T. It is the maximal in the sense that it is the largest closed extension having S in the domain of its adjoint. We denote T_1 by $T_{\max}$. It is clear that $D(T_{\max})$ consists of all function u in $L^p(\mathbf{R}^n)$ for which T_u is in $L^p(\mathbf{R}^n)$ (see Wong [5, Remark 3.3]).

The main result in this paper is that if $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$ and if there exist two positive constants C, C' such that (2.2) holds, then $D(T_{\min}) = H^{m,p}$. To this end, we recall a very important result in the theory of pseudodifferential operators.

THEOREM 2.4. *Let $T \in HL^{m,m_0}_{\rho,\delta}(M)$, with either $1 - \rho \leq \delta < \rho$ or $\rho < \delta$ and M a domain in $\mathbf{R}^n$. Then there exists an operator $Q \in HL^{-m,-m_0}_{\rho,\delta}(M)$, such that*

$$QT = I + R_1, \ TQ = I + R_2 \tag{2.1}$$

where $R_j \in L^{-\infty}(M)$, $j = 1, 2$, and I is the identity operator.

For a proof of Theorem 2.4, see Theorem 5.1 of Schubin [1]. We recall that $H^{s,p}$ is the L^p Sobolev space of order s. See Ch. 2, section 4 of Schechter [2] for a discussion of these spaces.

THEOREM 2.5. *Let ρ, δ be as in Theorem 2.4 and let $T \in HL^{m,m_0}_{\rho,\delta}(\mathbf{R}^n)$. If there exist two constants C, C' such that*

$$\|\Phi\|_{m,p} \le C(\|T\Phi\|_{0,p} + \|\Phi\|_{0,p}) \le C'\|\Phi\|_{m,p}, \quad \Phi \in S \tag{2.2}$$

then $D(T_{\min}) = H^{m,p}$.

Proof. If $u \in H^{m,p}$, then we can take a sequence $\{\Phi_k\}$ of functions in S such that $\Phi_k \to u$ in $H^{m,p}$. Hence, by (2.2), $\{T\Phi_k\}$ and $\{\Phi_k\}$ are Cauchy sequences in $L^p(\mathbf{R}^n)$. So $\Phi_k \to u$ and $T\Phi_k \to f$ for some u and f in $L^p(\mathbf{R}^n)$. Hence $u \in D(T_{\min})$, and $T_{\min}u = f$. On the other hand, if $u \in D(T_{\min})$, then we can find a sequence $\{\Phi_k\}$ in S such that $\Phi_k \to u$ in $L^p(\mathbf{R}^n)$ and $T\Phi_k \to f$ for some f in $L^p(\mathbf{R}^n)$. Hence, $\{\Phi_k\}$ and $\{T\Phi_k\}$ are Cauchy sequences in $L^p(\mathbf{R}^n)$, so, by (2.2), $\{\Phi_k\}$ is a Cauchy sequence in $H^{m,p}$. Since $H^{m,p}$ is complete, $\Phi_k \to v$ for some v in $H^{m,p}$. Suppose $m \ge 0$. Then the inclusion map $H^{m,p} \to L^p(\mathbf{R}^n)$ is continuous. Thus, $\Phi_k \to v$ in $L^p(\mathbf{R}^n)$. Hence $u = v$, and consequently lies in $H^{m,p}$. If $m < 0$, then the inclusion map $L^p(\mathbf{R}^n) \to H^{m,p}$ is continous. Thus, $\Phi_k \to u$ in $H^{m,p}$. Hence $u = v$, and consequently lies in $H^{m,p}$.

REMARK 2.6. Since $T_{\max}$ is the maximal closed extension of T, $H^{m,p}$ is contained in the domain of $T_{\max}$.

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