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THE WEAK F–REGULARITY OF COHEN–MACAULAY LOCAL RINGS

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1. Introduction

In [3],[4] and [5],Hochster and Huneke introduced the notions of the tight closure of an ideal and of the weak F-regularity of a ring.

This notion enabled us to give new proofs of many results in commutative algebra.

A regular ring is known to be F-regular, and a Gorenstein local ring is proved to be F-regular provided that one ideal generated by a system of parameters (briefly s.o.p.) is tightly closed. In fact, a Gorenstein local ring is weakly F-regular if and only if there exists a system of parameters ideal which is tightly closed [3]. But we do not know whether this fact is true or not if a ring is not Gorenstein, in particular, a ring is a Cohen Macaulay (briefly C-M) local ring.

In this paper, we will prove this in the case of an 1-dimensional C-M local ring. For this, we study the F-rationality and the normality of the ring. And we will also prove that a C-M local ring is to be Gorenstein under some additional condition about the tight closure.

2. Main Theorem

From now on, all rings are commutative, with identity, and Noetherian of positive prime characteristic p, unless otherwise specified.

DEFINITION 2.1 [HOCHSTER-HUNEKE]. Let I be an ideal of R and set $R^0 = R - U\{P : P$ is a minimal prime ideal of $R\}$. We say that $x \in I^*$, the tight closure of I, if there exists $c \in R^0$ such that for all $e \gg 0, cx^q \in I_{[q]}$, where $I^{[q]} = (i^q : i \in I)$ when $q = p^e$. If $I = I^*$, then we say that I is tightly closed. If any ideal of a ring R is tightly

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closed, then we say that R is weakly F-regular. If every localization of R at a multiplicative sustem is weakly F-regular, then we say that R is F-regular.

PROPOSITION 2.2. *(i) If R is regular, then R is F-regular [2].*
(ii) R is weakly F-regular if and only if $R_{\underline{m}}$ is weakly F-regular for every maximal ideal $\underline{m}$ of R [5].
(iii) A Noetherian ring of characteristic p is weakly F-regular if and only if every ideal primary to a maximal ideal is tightly closed [5].
(iv) Let R be a Noetherian ring of characteristic p such that no prime is both minimal and maximal. If every principal ideal of height one is tightly closed, then R is normal. In particular, a weakly F-regular ring is normal [5].

THEOREM 2.3. *Let R be a Gorenstein local ring. Then the followings are equivalent.*

(i) *R is weakly F-regular.*
(ii) *There exists an s.o.p. $x_1, \cdots, x_d$ such that the ideal generated by this s.o.p. is tightly closed.*

Proof. [3, Proposition 5.1]

To study the equivalence of Theorem 2.3 in the case of C-M local ring, first we define the weakening notion of the weak F-regularity.

DEFINITION 2.4. *[1]. If every ideal generated by an s.o.p. in a local ring R is tightly closed, then we say that R is F-rational.*

In a C-M local ring, if some s.o.p. ideal is tightly closed, then R is F-rational [1].

LEMMA 2.5. *If $(R, \underline{m})$ is a F-rational local ring, then R is normal.*

Proof. (i) $\dim R = 0$; Then(0) is tightly closed and thus R is reduced. Since R is a finite direct product of fields, R is normal.
(ii) $\dim R = d \geq 1$; Since R is local, there exists no prime ideal both minimal and maximal. Thus it is enough to show that any principal ideal of height one is tightly closed [5]. Let (x) be such an ideal. Then $x \notin Z(R)$ and x can be extended to a s.o.p. $x = x_1, x_2, \cdots, x_d$ for R. And $x, x_2^n, \cdots, x_d^n$ is also an s.o.p. for R, for every $n \in \mathbb{N}$. Then,

by hypothesis, this s.o.p. ideal $(x, x_2^n, \cdots, x_d^n)$ is tightly closed. Thus $(x)^* \subset (x, x_2^n, \cdots, x_d^n)$ for every n. But $(x_2^n, \cdots, x_d^n) \subset \underline{m}^n$. We have

$$(x)^* \subset \bigcap_{n \in \mathbf{N}} (x, x_2^n, \cdots, x_d^n) \subset \bigcap_{n \in \mathbf{N}} [(x) + (x_2, \cdots, x_d)^n] = (x),$$

by Krull's Intersection Theorem. Thus (x) is tightly closed, and R is normal.

In the study of the tight closure, we find an important fact: if R is the homomorphic image of a C-M F-regular ring, then R is C-M [3]. Moreover, if we apply the Lemma 2.5, then we can prove the following theorem.

THEOREM 2.6. *Let R be a local ring which can be represented as the quotient of a C-M ring. If R is F-rational, then R is normal and C-M.*

Proof. Since R is F-rational local ring, R is normal, and hence a domain. Thus R is equidimensional. To apply [3, Theorem 3.3], we let $x_1, \cdots, x_d$ generate an s.o.p. ideal denoted by I_d, and let $I_i = (x_1, \cdots, x_i)R$ for $i = 1, \cdots, d$. Then $(I_i :_R x_{i+1}) \subset I_i^*$. But the F-rationality of R implies that every part of an s.o.p. ideal is also tightly closed. For, if the ideal (x) in the proof of Lemma 2.5 is replaced by regular sequence. Hence R is C-M.

PROPOSITION 2.7. *If R is an 1-diminsional homomorphic image of a C-M local ring, and if there exists an s.o.p. ideal which is tightly closed, then R is Cohen-Macaulay.*

For the proof, refer to [8].

THEOREM 2.8. *Let $(R, \underline{m})$ be a reduced local ring of dimension 1. If there exists a single s.o.p. ideal which is tightly closed, then R is weakly F-regular.*

Proof. The hypothesis implies that R is Cohen-Macaulay, hence R is F-rational and normal by Lemma 2.5. Since R is of dimension one, R is regular and thus R is weakly F-regular.

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PROPOSITION 2.7. *If R is an 1-diminsional homomorphic image of a C-M local ring, and if there exists an s.o.p. ideal which is tightly closed, then R is Cohen-Macaulay.*

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Proof. The hypothesis implies that R is Cohen-Macaulay, hence R is F-rational and normal by Lemma 2.5. Since R is of dimension one, R is regular and thus R is weakly F-regular.

3. The conditions of C-M local ring to be Gorenstein

LEMMA 3.1 [LIPMAN-SATHAYE]. *Let R be a commutative Noetherian ring. If $x_1, \cdots, x_r$ is a regular sequence in R with $R/(x_1, \cdots, x_r)R$ is normal, then for any $y \in R, (x_1, \cdots, x_r)R+ yR$ is integrally closed in R.*

For the proof, refer to [6].

THEOREM 3.2. *Let R be a C-M local ring of dimension d. If there exists a single s.o.p. $x_1, \cdots, x_d$ such that the image of an ideal (x_d) in $R/(x_1, \cdots, x_{d-1})R$ is tightly closed, then R is Gorenstein and R is F-regular.*

To prove the Theorem 3.2, we need the following Lemma.

LEMMA 3.3. *If all of the hypothesis of Theorem 3.2 are satisfied, then the ideal $I = (x_1, \cdots, x_d)$ is integrally closed and R is F-rational.*

proof. Let $\overline{R} = R/(x_1, \cdots, s_{d-1})R$ and $(\overline{x_d})$ be the image of (x_d) in $\overline{R}$. Then $\overline{R}$ is a 1-dimensional C-M local ring, for $x_1, \cdots, x_{d-1}$ form a regular sequence in R. Since an s.o.p. ideal $(\overline{x_d})$ in $\overline{R}$ is tightly closed, $\overline{R}$ is weakly F-regular by Theorem 2.8. Thus $\overline{R}$ is normal. Since $I = (x_1. \cdots, x_{d-1})R + x_dR, I$ is integrally closed by Lemma 3.1. Hence $I = I^*$ and R is F-rational by Cohen-Macaulayness of R.

Proof of Theorem 3.2. Since $\overline{R} = R/(x_1, \cdots, x_{d-1})R$ is a 1-dimensional normal ring, $\bar{R}$ is regular. Thus $\overline{R}$ is a Gorenstein local ring. Since $x_1, \cdots, x_{d-1}$ form a regular sequence in $R, \overline{R}$ is Gorenstein if and only if R is Gorenstein. But by Lemma 3.3, R is F-rational. Hence by Theorem 2.3, R is weakly F-regular, and R is known to be F-regular in this case.

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