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REMARKS ON SOME VARIATIONAL INEQUALITIES

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1. Introduction and Preliminaries

This is a continuation of the author's previous work [17]. In this paper, we consider mainly variational inequalities for single-valued functions.

We first obtain a generalization of the variational type inequality of Juberg and Karamardian [10] and apply it to obtain strengthened versions of the Hartman-Stampacchia inequality and the Brouwer fixed point theorem. Next, we obtain fairly general versions of Browder's variational inequality [5] and its subsequent generalizations due to Brezis et al. [4], Takahashi [23], Shih and Tan [19], Simons [20], and others. Finally, in this paper, we obtain a variational inequality for non–real locally convex t.v.s. which generalizes a result of Shih and Tan [19].

For terminology and notations, we follow [17]. For a subset X of a vector space E and $x \in E$, the *inward* and *outward* sets of X at x, $I_K(x)$ and $O_K(x)$, are defined as follows:

$$I_X(x) = \{x + r(u - x) \in E : u \in X,\ r > 0\},$$
$$O_X(x) = \{x - r(u - x) \in E : u \in X,\ r > 0\}.$$

We begin with the following form of [17, Theorem 1], which can be deduced from a generalized Fan-Browder fixed point theorem in [15], [16] as in [17].

THEOREM 0. *Let X be a convex space, $p, q : X \times X \to \mathbf{R} \cup \{+\infty\}$ and $h : X \to \mathbf{R} \cup \{+\infty\}$ functions satisfying*

(i) *$q(x,y) \le p(x,y)$ for $(x,y) \in X \times X$ and $p(x,x) \le 0$ for all $x \in X$;*

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(ii) *for each* $y \in X$, $\{x \in X : p(x,y) + h(y) > h(x)\}$ *is convex or empty;*

(iii) *for each* $x \in X$, $\{y \in X : q(x,y) + h(y) > h(x)\}$ *is compactly open; and*

(iv) *there exist a nonempty compact subset* K *of* X *and, for each finite subset* N *of* X, *a compact convex subset* L_N *of* X *containing* N *such that* $y \in L_N \backslash K$ *implies* $q(x,y) + h(y) > h(x)$ *for some* $x \in L_N$.

Then there exists a point $y_0 \in K$ *such that*

$$q(x, y_0) + h(y_0) \leq h(x) \quad \text{for all} \quad x \in X.$$

Moreover, the set of all such solutions y_0 *is a compact subset of* K.

2. Main results

Let E be a real vector space, F a nonempty set, and $\langle \cdot, \cdot \rangle : E \times F \to \mathbf{R}$ a real–valued function which is linear in the first variable in the sense : for each given $y \in F$, $\langle \cdot, y \rangle$ maps E linearly into $\mathbf{R}$.

THEOREM 1. *Let* X *be a convex space in* E, $h : X \to \mathbf{R} \cup \{+\infty\}$ *and* $f, g : X \to F$ *functions satisfying*

(i) $\langle x - y, gy \rangle \leq \langle x - y, fy \rangle$ *for* $(x,y) \in X \times X$;

(ii) *for each* $y \in X$, $\{x \in X : \langle x - y, fy \rangle + h(y) > h(x)\}$ *is convex or empty;*

(iii) *for each* $x \in X$, $\{y \in X : \langle x - y, gy \rangle + h(y) > h(x)\}$ *is compactly open; and*

(iv) *there exist a nonempty compact subset* K *of* X *and, for each finite subset* N *of* X, *a compact convex subset* L_N *of* X *containing* N *such that* $y \in L_N \backslash K$ *implies* $\langle x - y, gy \rangle + h(y) > h(x)$ *for some* $x \in L_N$.

Then there exists a $y_0 \in K$ *such that*

$$\langle x - y_0, gy_0 \rangle + h(y_0) \leq h(x) \quad \text{for all} \quad x \in X.$$

Moreover, if $h : E \to \mathbf{R} \cup \{+\infty\}$ *is convex, then the inequality holds for all* $x \in I_X(y_0)$.

Proof. Putting $p(x,y) \equiv \langle x - y, fy \rangle$ and $q(x,y) \equiv \langle x - y, gy \rangle$ in Theorem 0, we have a $y_0 \in K$ satisfying

$$\langle x - y_0, gy_0 \rangle + h(y_0) \le h(x) \quad \text{for all} \quad x \in X.$$

Moreover, suppose that $h : E \to \mathbf{R} \cup \{+\infty\}$ is convex. If $x \in I_X(y_0) \backslash X$, then there exist $u \in X$ and $r > 1$ such that $x = y_0 + r(u - y_0)$. Hence

$$u - y_0 = \frac{1}{r}(x - y_0) \text{ and } u = \frac{1}{r}x + (1 - \frac{1}{r})y_0 \in X.$$

Since $\langle u - y_0, gy_0 \rangle + h(y_0) \le h(u)$, we have

$$\frac{1}{r}\langle x - y_0, gy_0 \rangle + h(y_0) \le h(u) \le \frac{1}{r}h(x) + (1 - \frac{1}{r})h(y_0)$$

or

$$\langle x - y_0, gy_0 \rangle + h(y_0) \le h(x) \quad \text{for all} \quad x \in I_X(y_0).$$

This completes our proof.

COROLLARY 1.1. *Let X be a convex space in E, $h : X \to \mathbf{R} \cup \{+\infty\}$ a l.s.c. convex function, and $f : X \to F$ a function such that*

(a) *for each $x \in X$, $y \mapsto \langle x - y, fy \rangle$ is l.s.c. on compact subsets of X, and*
(b) *the condition* (iv) *of Theorem 1 holds with $f \equiv g$.*

Then there exists a $y_0 \in K$ such that

$$\langle x - y_0, fy_0 \rangle + h(y_0) \le h(x) \quad \textit{for all} \quad x \in X.$$

Moreover, if $h : E \to \mathbf{R} \cup \{+\infty\}$ is a convex function which is l.s.c. on X, then the inequality holds for all $x \in I_X(y_0)$

Proof. We use Theorem 1 with $f \equiv g$. Since, for each $y \in X$, $x \mapsto \langle x - y, fy \rangle$ is linear and $x \mapsto h(x)$ is convex, the set $\{x \in X : \langle x - y, fy \rangle + h(y) > h(x)\}$ is convex or empty. This shows that the condition (ii) in Theorem 1 holds. Since h is l.s.c., the condition (a) implies (iii). Therefore, by Theorem 1, the conclusion follows.

For $h \equiv 0$, we have the following:

COROLLARY 1.2. *Let X be a convex space in E, and $f : X \to F$ a function.*

(1) *If, for each $x \in X$, $y \mapsto \langle x - y, fy\rangle$ is l.s.c. on compact subsets of X, and if there exist K and L_N as in* (iv) *of Theorem 0 such that $y \in L_N \backslash K$ implies $\langle x - y, fy\rangle > 0$ for some $x \in L_N$, then there exists a $y_0 \in K$ such that*

$$\langle x - y_0, fy_0\rangle \leq 0 \quad \textit{for all} \quad x \in I_X(y_0).$$

(2) *If, for each $x \in X$, $y \mapsto \langle y - x, fy\rangle$ is l.s.c. on compact subsets of X, and if there exist K and L_N as in* (iv) *of Theorem 0 wuch that $y \in L_N \backslash K$ implies $\langle y - x, fy\rangle > 0$ for some $x \in L_N$, then there exists a $y_0 \in K$ such that*

$$\langle x - y_0, fy_0\rangle \leq 0 \quad \textit{for all} \quad x \in O_X(y_0).$$

Proof. The case (1) is a direct consequence of Corollary 1.1 with $h \equiv 0$.

For (2), considering $\langle y - x, fy\rangle$ instead of $\langle x - y, fy\rangle$ in (1), we obtain a $y_0 \in K$ such that

$$\langle y_0 - x', fy_0\rangle \leq 0 \quad \text{for all } x' \in I_X(y_0).$$

For any $x \in O_X(y_0)$, let $x' = 2y_0 - x \in I_X(y_0)$. Then

$$\langle x - y_0, fy_0\rangle \leq 0 \quad \text{for all } x \in O_X(y_0).$$

REMARKS.

1. If E is a t.v.s. and if $x \mapsto \langle x, y\rangle$ is continuous on E for each fixed $y \in F$, then the inward [resp. outward] set in Corollary 3.2 can be replaced by its closure.
2. The coercivity assumption in (1) is implied by the following:

   (*) there exists a nonempty compact convex subset L of X such that, for each $y \in X \backslash L$, there is an $x \in L$ satisfying $\langle x - y, fy\rangle > 0$.

Corollary 1.2(1) with the assumption $(*)$ improves the "variational type" inequality of Juberg and Karamardian [10, Theorem]. In fact, they assumed closedness of X and local convexity of E, and obtained weaker conclusion.

3. For a compact X, the condition $(*)$ holds automatically. Therefore, from Corollary 1.2, we have the following:

COROLLARY 1.3. *Let X be a compact convex subset in a t.v.s. E, F a topological space, and $f : X \to F$ a function such that $(x, y) \mapsto \langle x, fy\rangle$ is continuous on $E \times X$. Then there exists a $y_0 \in X$ such that*

$$\langle x - y_0, fy_0\rangle \leq 0 \quad \textit{for all} \quad x \in W(y_0).$$

REMARK. Here $W(y_0)$ denotes any of $\overline{I}_X(y_0)$ or $\overline{O}_X(y_0)$. Corollary 1.3 strengthens Juberg and Karamardian [10, Lemma]. They showed that Corollary 1.2 follows from Corollary 1.3 in a particular case.

Let $\langle\cdot,\cdot\rangle$ denote the inner product of a real inner product space. Then Corollary 1.3 reduces to the following:

COROLLARY 1.4. *Let X be a compact convex subset in an inner product space E and $f : X \to E$ a continuous map. Then there exists an $x_0 \in X$ satisfying*

$$\langle fx_0, y - x_0\rangle \leq 0 \quad \textit{for all} \quad y \in W(x_0)$$

REMARK. The origin of Corollary 1.4 goes back to Hartman and Stampacchia [9] in 1966 for $\mathbf{R}^n$. See also Stampacchia [22, Theorem 2.2] and Moré [13, Theorem 2.1].

We now show that Corollary 1.4 implies the following well-known generalization of the Brouwer fixed point theorem.

COROLLARY 1.5. *Let X be a compact convex subset in an inner product space E and $g : X \to E$ a continuous map such that $gx \in W(x)$ for all $x \in Bd\,X$. Then g has a fixed point.*

Proof. For any $x \in X$ we have $gx \in W(x)$. In fact, for any $x \in \operatorname{Int} X$, we have $gx \in E = I_X(x) = O_X(x)$. Define $f \equiv g - 1_X : X \to E$. Then by Corollary 1.4, there exists an $x_0 \in X$ such that

$$\langle gx_0 - x_0, y - x_0\rangle \leq 0 \quad \text{for all} \quad y \in W(x_0).$$

Since $gx_0 \equiv y$ lies in $W(x_0)$, we must have $x_0 = gx_0$ as desired.

Let E be a real t.v.s., E^* its topological dual (i.e., the vector space of all continuous linear functionals $E \to \mathbf{R}$), and $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbf{R}$ denote the natural pairing.

THEOREM 2. *Let X be a convex space in E and let*

$$p(x,y) \equiv \langle fx, y-x \rangle + h(x) - h(y)$$

where $h : X \to \mathbf{R}$ is a l.s.c. convex function and $f : X \to E^*$ is a function such that

(a) *for each $y \in X$, $x \mapsto \langle fx, y-x \rangle$ is l.s.c. on compact subsets of X, and*
(b) *there exist a nonempty compact subset K of X and, for any finite subset N of X, a compact convex subset L_N of X containing N such that $x \in L_N \backslash K$ implies $p(x,y) > 0$ for some $y \in L_N$.*

Then there exists an $x_0 \in K$ such that

$$p(x_0, y) \leq 0 \quad \textit{for all} \quad y \in X.$$

Moreover, if $h : E \to \mathbf{R}$ is a convex function which is l.s.c. on X, then the conclusion holds for all $y \in I_X(x_0)$.

Proof. In Corollary 1.1, interchange x and y and put $F = E^*$.

REMARKS.

1. Note that Brézis, Nirenberg, and Stampacchia [4, Application 3] obtained Theorem 2 under the stronger assumption that f is pseudo-monotone and continuous with a much stronger condition than (b). Theorem 2 improves Brézis [3, Corollary 29] and Hartman and Stampacchia [9, Theorems 1.1 and 5.1].
2. Theorem 2 also improves Allen [1, Corollary 1]. In fact, he assumed the following particular form of (b):

(b)′ let L be a nonempty compact convex subset of X and suppose that for each $x \in X \backslash L$ there exists $y \in L$ such that $p(x,y) > 0$.

From now on, let E^* have any topology such that a continuous function $f : X \to E^*$ satisfies the requirement (a) of Theorem 2. For example, we equip E^* with the topology of uniform convergence on bounded subsets of E.

COROLLARY 2.1. *Let X be a convex subset of E, and $f : X \to E^*$ continuous.*

(1) *If there exist K and L_N as in* (b) *of Theorem 2 such that $x \in L_N \backslash K$ implies $\langle fx, y - x\rangle > 0$ for some $y \in L_N$, then there exists an $x_0 \in K$ such that*

$$\langle fx_0, y - x_0\rangle \le 0 \quad \text{for all} \quad y \in \overline{I}_X(x_0).$$

(2) *If there exist K and L_N as in* (b) *of Theorem 2 such that $x \in L_N \backslash K$ implies $\langle fx, x - y\rangle > 0$ for some $y \in L_N$, then there exists an $x_0 \in K$ such that*

$$\langle fx_0, y - x_0\rangle \le 0 \quad \text{for all} \quad y \in \overline{O}_X(x_0).$$

Proof. (1) By putting $h \equiv 0$ in Theorem 2, we know that there exists an $x_0 \in K$ such that

$$\langle fx_0, y - x_0\rangle \le 0 \quad \text{for all} \quad y \in I_X(x_0).$$

Since $fx_0 \in E^*$, this implies the conclusion.

(2) By the case for $\langle fx, y - x\rangle$ in Theorem 2, we know that there exists a point $x_0 \in K$ such that $\langle fx_0, x_0 - y'\rangle \le 0$ for all $y' \in \overline{I}_X(x_0)$ as in (1). For any $y \in O_X(x_0)$, let $y' = 2x_0 - y \in I_X(x_0)$. Then

$$\langle fx_0, y - x_0\rangle = \langle fx_0, x_0 - y'\rangle \le 0$$

for all $y \in O_X(x_0)$. Hence, $\langle fx_0, y - x_0\rangle \le 0$ holds for all $y \in \overline{O}_X(x_0)$.

REMARKS.

1. In case X is compact, Corollary 2.1 reduces to Park [14, Theorem 2], which strengthens Browder [5, Theorem 2].

2. In case X is a closed convex subset of a t.v.s. E, if there exists a compact convex subset L of X such that

$$K \equiv \{x \in X : \langle fx, y - x \rangle \leq 0 \text{ for all } y \in L\} \subset L,$$

is compact, then the same conclusion holds. This improves Takahashi [23, Theorem 3].

3. Instead of the continuity of f, it suffices to assume the condition (a) of Theorem 2. Hence, Corollary 2.1 improves Allen [1, Corollary 2].

4. If $x_0 \in \operatorname{Int} X$ or $X = E$ in Corollary 2.1, it is obvious that there exists $x^* \in E$ such that $fx^* = 0$. In fact, $\langle fx_0, y - x_0 \rangle \leq 0$ for all $y \in E = I_X(x_0)$ implies $fx_0 = 0$.

5. Corollary 2.1 has a very interesting interpretation when X is a cone in E as follows :

A nonempty closed subset X is a *cone* in E if $\alpha x + \beta y \in X$ for all $\alpha, \beta \geq 0$ and $x, y \in X$. The *polar* X^* of a cone X is the cone defined by

$$X^* \equiv \{p \in E^* : \langle p, x \rangle \geq 0 \quad \text{for all} \quad x \in X\}.$$

COROLLARY 2.2. *Let X be a cone in E and $f : X \to E^*$ continuous. If there exist K and L_N as in* (b) *of Theorem 2 such that $x \in L_N \backslash K$ implies $\langle fx, x - y \rangle > 0$ for some $y \in L_N$, then there exists an $x_0 \in X$ such that*

$$fx_0 \in X^* \text{ and } \langle fx_0, x_0 \rangle = 0.$$

Proof. By Corollary 2.1(2), there exists $x_0 \in K$ such that $\langle fx_0, y - x_0 \rangle \geq 0$ for all $y \in X$. Since $\langle fx_0, \alpha y \rangle \geq \langle fx_0, x_0 \rangle$ for all $\alpha > 0$ and $y \in X$, we obtain $\langle fx_0, y \rangle \geq 0$ for all $y \in X$, i.e., $fx_0 \in X^*$. Since $\langle fx_0, 0 - x_0 \rangle \geq 0$, we have $\langle fx_0, x_0 \rangle = 0$.

REMARKS.

1. The problem of finding a vector $x_0 \in X$ satisfying the conclusion is known as the complementarity problem ; several problems in mathematical programming, game theory, economics, operations research, and mechanics can be presented in this form.

2. Corollary 2.2 generalizes Takahashi [23, Theorem 4]. Also Takahashi [24, Corollary 2.1] proved Karamardian's complementarity problem [11] by using a particular form of Theorem 0.

COROLLARY 2.3. *Let X be a convex subset of E, and $T : X \to 2^{E^*}$ a multifunction having a continuous selection $f : X \to E^*$.*

(1) If there exist K and L_N as in (b) of Theorem 2 such that $x \in L_N \backslash K$ implies $\langle fx, y-x\rangle > 0$ for some $y \in L_N$, then there exist $x_0 \in K$ and $x_0^* \in E^*$ such that

$$x_0^* \in Tx_0 \quad \text{and} \quad \langle x_0^*, y - x_0\rangle \leq 0 \text{ for all } y \in \overline{I}_X(x_0).$$

(2) If there exist K and L_N as in (b) of Theorem 2 such that $x \in L_N \backslash K$ implies $\langle fx, x-y\rangle > 0$ for some $y \in L_N$, then the same conclusion holds for all $y \in \overline{O}_X(x_0)$.

Proof. Put $x_0^* = fx_0$ in Corollary 2.1.

REMARK. Corollary 2.3 is a particular form of the generalized quasi-variational inequalities. For related results, see, e.g., Shih and Tan [18].

The following is a simple consequence of Corollary 2.3.

COROLLARY 2.4. *Let X be a compact convex subset of E and $T : X \to 2^{E^*}$ a multifunction satisfying*

(i) Tx is nonempty and convex for each $x \in X$; and

(ii) $T^{-1}y$ is open for each $y \in Y$.

Then there exist $x_0 \in X$ and $x_0^* \in E^*$ such that

$$x_0^* \in Tx_0 \quad \text{and} \quad \langle x_0^*, y - x_0\rangle \leq 0 \quad \text{for all} \quad y \in W(x_0).$$

Proof. T has a continuous selection by a result in [2].

REMARK. Corollary 4.4 strengthens Simons [21, Theorem 4.5]. For another proof, see Komiya [12]. This generalizes and unifies fixed point theorems for multifunctions due to Browder [5], Fan [8], Takahashi [23], [25] and Cellina [7]. Simons [21] gave several comments on related results to Corollary 2.4 and deduced some fixed point theorems from Corollary 2.4.

For reflexive Banach spaces, Theorem 2 reduces to the following:

COROLLARY 2.5. *Let X be a convex subset of a real reflexive Banach space E, $f : X \to E^*$ is a weakly continuous function, and $h : X \to \mathbf{R}$ a weakly l.s.c. convex function. If*

(∗) *there exist a bounded subset K of X and, for each finite subset N of X, a closed bounded convex subset L_N of X containing N such that $x \in L_N \backslash K$ implies $\langle fx, x-y \rangle + h(x) > h(y)$ for some $y \in L_N$,*

then there exists an $x_0 \in K$ such that

$$\langle fx_0, x_0 - y \rangle + h(x_0) \leq h(y) \text{ for all } y \in X.$$

Moreover, if h is defined on E, then the conclusion holds for all $y \in I_X(x_0)$.

Proof. Switch to the weak topology.

REMARK. Browder [6, Theorem 6] obtained Corollary 2.4 under stronger assumptions, i.e.,

(1) f is pseudo–monotone in the sense in [6] (which implies f is continuous from any finite topology of X to the weak topology of X^*), and

(2) for some $y_0 \in X$, there exists an $R_0 \in \mathbf{R}$ such that

$$\langle fx, x - y_0 \rangle + h(x) > h(y_0)$$

for all $x \in X$ with $\|x\| > R_0$.

Note that (2) implies (∗). In fact,

$$K \equiv \{x \in X : \langle fx, x - y_0 \rangle + h(x) \leq h(y_0)\} \subset \{x \in X : \|x\| \leq R_0\}$$

is bounded.

Finally in this paper, we add a variational inequality for a non–real Hausdorff locally convex space (simply, l.c.s.).

THEOREM 3. *Let X be a nonempty bounded convex subset of a l.c.s. E, and $f : X \to E^*$ continuous from X to the strong topology of E^* such that*

(∗) *there exist a nonempty compact subset K of X and, for each finite subset N of X, a compact convex subset L_N of X containing N such that $y \in L_N \backslash K$ implies $\mathrm{Re}\,\langle fy, y - x \rangle > 0$ for some $x \in L_N$.*

Then there exists a point $y_0 \in K$ such that

$$\mathrm{Re}\ \langle y_0, y_0 - x\rangle \leq 0 \quad \textit{for all } x \in \overline{I}_X(y_0).$$

Proof. Define $p : X \times X \to \mathbf{R}$ by

$$p(x,y) \equiv \mathrm{Re}\ \langle fy, y\rangle \quad \text{for all } x, y \in X.$$

Then, for each $x \in X$, $p(x,\cdot)$ is continuous by [19, Lemma 1]. By applying Theorem 0 with $h \equiv 0$, the conclusion follows.

REMARKS.

1. Theorem 3 generalizes Shih and Tan [19, Theorem 10] since they assumed the following stronger condition than $(*)$:

$(**)$ there exists a compact convex subset L of X such that, for each $y \in X\backslash L$, there is an $x \in L$ with $\mathrm{Re}\ \langle fy, y - x\rangle > 0$.

2. If X is closed in Theorem 3, $(*)$ is implied by the following:

$(***)$ for some nonempty compact subset C of E and $x_0 \in X \cap C$,

$$\mathrm{Re}\ \langle fy, y - x_0\rangle > 0 \text{ for all } y \in X\backslash C.$$

Therefore, Theorem 3 generalizes Shih and Tan [19, Theorem 11].

3. For compact X, Theorem 5 improves Browder's variational inequality in [5], [6].

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