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CONDITIONAL GENERALIZED WIENER MEASURES

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1. Introduction and preliminaries

For a fixed $T \in (0, \infty)$, let $C_0\,[0, T]$ be the space of all continuous functions x on $[0, T]$ vanishing at the origin. For each partition $0 = \tau_0 < \tau_1 < \ldots < \tau_n = T$, a set of the type

$$W = \{x \in C_0\,[0,T] \mid (x(\tau_1), x(\tau_2), \ldots, x(\tau_n)) \in B\}, \tag{1.1}$$

is called a strict interval of $C_0\,[0,T]$ if $B = \prod_{i=1}^n (a_i, b_i]$. If B is a Borel measurable subset of $\mathbf{R}^n$, then W is called an interval of $C_0\,[0,T]$. The collection $\mathcal{R}$ of all such strict intervals forms a semi-algebra of subsets of $C_0\,[0,T]$. Let $(C_0\,[0,T], \mathcal{W}, m_g)$ denote the generalized Wiener space where m_g is a probability measure on $\mathcal{R}$ defind for W as in (1.1) by

$$m_g(W) = \int_B g(\tau_0, \tau_1, \tau_2, \ldots, \tau_n; u_0, u_1, u_2, \ldots, u_n)\, du_1 du_2 \ldots du_n$$

where

$$g(\tau_0, \tau_1, \tau_2, \ldots, \tau_n; u_0, u_1, u_2, \ldots, u_n) = \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (\beta(\tau_i) - \beta(\tau_{i-1}))}} \exp\left[-\frac{1}{2}\sum_{i=1}^n \frac{\{u_i - \alpha(\tau_i) - u_{i-1} + \alpha(\tau_{i-1})\}^2}{\beta(\tau_i) - \beta(\tau_{i-1})}\right],$$

$u_0 = 0, \alpha(t)$ and $\beta(t)$ are real valued continuous function on $[0,T]$ such that $\alpha(0) = 0, \beta(0) = 0$ and $\beta(t)$ is strictly increasing; $\mathcal{W}$ is the σ-algebra of Caratheodory measurable subsets of $C_0\,[0,T]$ with respect

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to the outer measure derived from the probability measure m_g which contains the σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.

We note that $C_0[0,T]$ is a separable Banach space with the supremum norm and that $\sigma(\mathcal{R}) = \mathcal{B}(C_0[0,T])$, Borel σ-algebra in $C_0[0,T]$.

Let X be a $\mathbf{R}^n$-valued $\mathcal{W}$-measurable function such that $m_g \circ X^{-1} \equiv P_X \ll m_L$, where m_L is a Lebesgue measure on $(\mathbf{R}^n, \mathcal{M})$ and $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets in $\mathbf{R}^n$. Let F be a real valued integrable function on $C_0[0,T]$. The conditional generalized Wiener integral of F given X, written $E[F \mid X]$, is defined by the equivalence class of Lebesgue measurable and P_X integrable functions ψ on $\mathbf{R}^n$ modulo null functions on $(\mathbf{R}^n, \mathcal{M}, P_X)$ such that for all $C \in \mathcal{M}$,

$$\int_{X^{-1}(C)} F(x)\, dm_g(x) = \int_C \psi(\vec{\xi})\, dP_X(\vec{\xi}).$$

By the Radon-Nikodym theorem such a funtction ψ exists and is uniquely determind up to a null function on $(\mathbf{R}^n, \mathcal{M}, P_X)$. We will let $E[F \mid X]$ denote a representation of the equivalence class and so

$$\int_{X^{-1}(C)} F(x)\, dm_g(x) = \int_C E[F \mid X](\vec{\xi})\, dP_X(\vec{\xi}), \qquad C \in \mathcal{M}.$$

J. Yeh [5,6] gave the inversion formula for the conditional Wiener integrals and applied the formula to evaluate the conditional Wiener integrals. C. Park and D. Skoug [4] obtained a simple formula for the conditional Wiener integral when the conditioning function is vector valued, which converts the conditional Wiener integrals to the nonconditional Wiener integrals.

In this paper we define the conditional generalized Wiener measure and then express the conditional generalized Wiener integral over this new measure. In particular we consider a conditional expectation of functionals of the generalized Brownian paths under the condition that the paths pass through the given points $\xi_1, \xi_2, \ldots, \xi_n$ at times $t_1, t_2, \ldots, t_n$, respectively.

2. Conditional generalized Wiener measures

Let $0 \leq s < t \leq T$ and $\xi, \eta \in \mathbf{R}$. Let $C_{s,\xi;t,\eta}$ denote the space of continuous functions x with $x(s) = \xi$, $x(t) = \eta$. Let $d(x,y) = \max_{s \leq \tau \leq t} |x(\tau) - y(\tau)|$ be the metric on $C_{s,\xi;t,\eta}$.

Let $\mathcal{B}_{s,\xi;t,\eta}$ be the Borel σ-algebra generated by open sets for the metric topology on $C_{s,\xi;t,\eta}$. Let $s = \tau_0 < \tau_1 < \cdots < \tau_n = t$. Then the σ-algebra $\mathcal{B}_{s,\xi;t,\eta}$ is the smallest σ-algebra containing the strict intervals of the form

(2.1)

$$A = \{x \in C_{s,\xi;t,\eta} \mid (x(\tau_1), x(\tau_2), \ldots, x(\tau_{n-1})) \in B\}, \quad B = \prod_{i=1}^{n} (a_i, b_i].$$

Let A be a strict interval given as (2.1). Define

$$(2.2) \quad m_{s,\xi;t,\eta}(A) = \sqrt{2\pi(\beta(t) - \beta(s))} \exp\left\{ \frac{(\eta - \alpha(t) - \xi + \alpha(s))^2}{2(\beta(t) - \beta(s))} \right\}$$
$$\int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} g(\tau_0, \tau_1, \ldots, \tau_n; u_0, u_1, \ldots, u_n)\, du_1 du_2 \cdots du_{n-1}$$

where $u_0 = \xi$, $u_n = \eta$. It can be shown that $m_{s,\xi;t,\eta}$ is countably additive on the semi-algebra $\mathcal{R}_{s,\xi;t,\eta}$ of strict intervals in $C_{s,\xi;t,\eta}$. By using the Caratheodory extention theorem we obtain a probability measure $m_{s,\xi;t,\eta}$ on the complete σ-algebra $\bar{\mathcal{B}}_{s,\xi;t,\eta}$ of subsets in $C_{s,\xi;t,\eta}$, which is called the conditional generalized Wiener measure with parameter ξ and η.

Let $\vec{t} = (t_1, t_2, \ldots, t_n)$ and $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be fixed vectors where $0 = t_0 < t_1 < \cdots < t_n = T$. We consider the product measure space

$$\left(C^* = \prod_{i=1}^{n} C_{t_{i-1},\xi_{i-1};t_i,\xi_i},\ \mathcal{B}^* = \prod_{i=1}^{n} \mathcal{B}_{t_{i-1},\xi_{i-1};t_i,\xi_i},\ m^* = \prod_{i=1}^{n} m_{t_{i-1},\xi_{i-1};t_i,\xi_i} \right)$$

where $\mathcal{B}^*$ is the smallest σ-algebra containing all measurable rectangles $\prod_{i=1}^{n} A_i$, $A_i \in \mathcal{B}_{t_{i-1},\xi_{i-1};t_i,\xi_i}$ and $\xi_0 = 0$.

Let $C_{\vec{t},\vec{\xi}} = \{x \in C_0[0,T] \mid x(t_i) = \xi_i, i = 1,2,\dots,n\}$. Define a map $P : C^* \to C_{\vec{t},\vec{\xi}}$ by $P(x_1, x_2, \dots, x_n)(s) = x_i(s)$, $t_{i-1} \le s \le t_i$, $i = 1,2,\dots,n$. Then P is bijective. Let $\mathcal{R}_{\vec{t},\vec{\xi}} \equiv \{V \subset C_{\vec{t},\vec{\xi}} \mid P^{-1}(V) \in \prod_{j=1}^{n} \mathcal{R}_{t_{j-1},\xi_{j-1};t_j,\xi_j}\}$. Then $\mathcal{R}_{\vec{t},\vec{\xi}}$ is the semi-algebra of sets of the form

$$V = \Big\{ x \in C_{\vec{t},\vec{\xi}} \mid (x(\tau_1), \dots, x(\tau_{m(1)-1}), x(\tau_{m(1)+1}), \dots, x(\tau_{m(2)-1}),$$
$$x(\tau_{m(2)+1}), \dots, x(\tau_{m(n)-1})) \in \prod_{j=1}^{m(n)} (a_j, b_j],$$
$$j \ne m(1), m(2), \dots, m(n) \Big\},$$

where $\tau_{m(i)} = t_i$, $i = 1,2,\dots,n$. Furthermore it can be shown that $\mathcal{R}_{\vec{t},\vec{\xi}} = \mathcal{R} \cap C_{\vec{t},\vec{\xi}}$ and $\mathcal{B}_{\vec{t},\vec{\xi}} \equiv \sigma(\mathcal{R}_{\vec{t},\vec{\xi}}) = \mathcal{B}(C_0[0,T]) \cap C_{\vec{t},\vec{\xi}}$. Thus we can define a measure $m_{\vec{t},\vec{\xi}}$ on $\mathcal{R}_{\vec{t},\vec{\xi}}$ by

$$(2.3) \qquad m_{\vec{t},\vec{\xi}}(V) \equiv m^*(P^{-1}(V)) = \prod_{j=1}^{n} m_{t_{j-1},\xi_{j-1};t_j,\xi_j}(A_j)$$

where $P^{-1}(V) = \prod_{j=1}^{n} A_i$ and $A_j = \{x \in C_{t_{j-1},\xi_{j-1};t_j,\xi_j} \mid (x(\tau_{m(j-1)+1}), \dots, x(\tau_{m(j)-1})) \in \prod_{i=m(j-1)+1}^{m(j)-1} (a_i, b_i]\}$.

The measure $m_{\vec{t},\vec{\xi}}$ on $\mathcal{R}_{\vec{t},\vec{\xi}}$ has a unique countably additive extension, called the conditional Wiener measure with parameter $\vec{\xi}$, which is yet denoted as it is, to the complete σ-algebra $\bar{\mathcal{B}}_{\vec{t},\vec{\xi}}$ of $C_{\vec{t},\vec{\xi}}$.

THEOREM 2.1. *Let X be a measurable function on $(C_0[0,T], \mathcal{W})$ defined by*

$$X(x) = (x(t_1), x(t_2), \dots, x(t_n)).$$

(1) *Let $F \in L^1(C_0[0,T], \mathcal{W}, m_g)$. Then the restriction F_R of F on $C_{\vec{t},\vec{\xi}}$ is $\bar{\mathcal{B}}_{\vec{t},\vec{\xi}}$-measurable for a.e. $\vec{\xi}$ and*

$$(2.4) \qquad \int_{C_0[0,T]} F(x)\, dm_g(x) = \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} F(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}).$$

Thus $\int_{C_{\vec{t},\vec{\xi}}} F(x)\, dm_{\vec{t},\vec{\xi}}(x)$ *exists a.e.* $\vec{\xi}$. *In particular, if* F *is* $\mathcal{B}(C_0\,[0,T])$*-measurable, then* F_R *on* $C_{\vec{t},\vec{\xi}}$ *is* $\mathcal{B}_{\vec{t},\vec{\xi}}$*-measurable.*

(2) *If* $F \in L^1(C_0\,[0,T]\,, \mathcal{W}, m_g)$, *then there exists a version* $E\,[F \mid X]$ *such that*

$$E\,[F \mid X]\,(\vec{\xi}) = \int_{C_{\vec{t},\vec{\xi}}} F(x)\, dm_{\vec{t},\vec{\xi}}(x). \tag{2.5}$$

Proof. Let $\mathcal{A} = \{E \in \mathcal{B}(C_0\,[0,T]) \mid$ the equality (2.4) hold for $F = \chi_E\}$. Then it is clear that $\mathcal{A}$ is a monotone class containing the algebra of intervals in $C_0\,[0,T]$. Hence $\mathcal{A} = \mathcal{B}(C_0\,[0,T])$ by the monotone class theorem.

Let S be an m_g-null set. Then there exists a set $N \in \mathcal{B}(C_0\,[0,T])$ such that $S \subset N$ and $m_g(N) = 0$. Now observe that

$$\begin{aligned} 0 = m_g(N) &= \int_{C_0[0,T]} \chi_N(x)\, dm_g(x) \\ &= \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} \chi_N(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}) \end{aligned}$$

since (2.4) holds for all χ_E, $E \in \mathcal{B}(C_0\,[0,T])$, and so

$$\int_{C_{\vec{t},\vec{\xi}}} \chi_N(x)\, dm_{\vec{t},\vec{\xi}}(x) = m_{\vec{t},\vec{\xi}}(N \cap C_{\vec{t},\vec{\xi}}) = 0 \qquad \text{a.e. } \vec{\xi}.$$

Then

$$\int_{C_{\vec{t},\vec{\xi}}} \chi_S(x)\, dm_{\vec{t},\vec{\xi}}(x) = m_{\vec{t},\vec{\xi}}(S \cap C_{\vec{t},\vec{\xi}}) = 0 \qquad \text{a.e. } \vec{\xi}.$$

Thus we have

(2.6)

$$\begin{aligned} \int_{C_0[0,T]} \chi_S(x)\, dm_g(x) &= m_g(S) = 0 \\ &= \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} \chi_S(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}). \end{aligned}$$

We note that $\mathcal{W} \cap C_{\vec{t},\vec{\xi}} \subset \mathcal{B}_{\vec{t},\vec{\xi}}$ for a.e. $\vec{\xi}$. Thus for a Borel set B in $\mathbf{R}$,

$$F_{\mathbf{R}}^{-1}(B) = F^{-1}(B) \cap C_{\vec{t},\vec{\xi}} \in \mathcal{B}_{\vec{t},\vec{\xi}} \qquad \text{a.e. } \vec{\xi}.$$

since $F^{-1}(B) \in \mathcal{W}$ and $\mathcal{W} \cap C_{\vec{t},\vec{\xi}} \subset \mathcal{B}_{\vec{t},\vec{\xi}}$ for a.e. $\vec{\xi}$. We also note that if F is Borel-measurable in $C_0\,[0,T]$, then F_R on $C_{\vec{t},\vec{\xi}}$ is $\mathcal{B}_{\vec{t},\vec{\xi}}$-measurable on $C_{\vec{t},\vec{\xi}}$.

Let $E \in \mathcal{W}$. Then there exist G and N in $\mathcal{B}(C_0\,[0,T])$ such that $E = G \cup S$, $S \subset N$, $m_g(N) = 0$ and $G \cap S = \phi$. Now we obtain

$$\begin{aligned}
\int_{C_0[0,T]} \chi_E(x)\, dm_g(x) &= m_g(G \cup S) \\
&= m_g(G) + m_g(S) \\
&= \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} \chi_G(x) + \chi_S(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}) \\
&= \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} \chi_{G\cup S}(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}) \\
&= \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} \chi_E(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}).
\end{aligned}$$

Now that (2.4) holds when F is any characteristic function of a set of $\mathcal{W}$ we can follow the usual procedure in integration theory to show that (2.4) holds for $\mathcal{W}$ measurable functions on $C_0\,[0,T]$. This completes the proof of (1).

For each Lebesque measurable set C in $\mathbf{R}^n$, we obtain, using (2.3),

$$\begin{aligned}
\int_{X^{-1}(C)} F(x)\, dm_g(x) &= \int_{C_0[0,T]} \chi_{X^{-1}(C)}(x) F(x)\, dm_g(x) \\
&= \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} \chi_C(X(x)) F(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}) \\
&= \int_{\mathbf{R}^n} \chi_C(\vec{\xi}) \int_{C_{\vec{t},\vec{\xi}}} F(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}) \\
&= \int_C \int_{C_{\vec{t},\vec{\xi}}} F(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi}).
\end{aligned}$$

But since F is integrable, $E[F \mid X]$ exists and for any Lebesque measurable set C in $\mathbf{R}^n$,

$$\int_{X^{-1}(C)} F(x)\, dm_g(x) = \int_C E[F \mid X](\vec{\xi})\, dP_X(\vec{\xi}), \qquad C \in \mathcal{M}.$$

Hence we obtain a version of $E[F \mid X]$ as in (2.5).

COROLLARY 2.1. $\bar{\mathcal{B}}_{\vec{t},\vec{\xi}} = \mathcal{W} \cap C_{\vec{t},\vec{\xi}}$ for a.e. $\vec{\xi}$.

Proof. It has been shown in the proof of Theorem 2.1 that $\bar{\mathcal{B}}_{\vec{t},\vec{\xi}} \subset \mathcal{W} \cap C_{\vec{t},\vec{\xi}}$ for a.e. $\vec{\xi}$. Let N be a $m_{\vec{t},\vec{\xi}}$-null set. Then ther exists $M \in \bar{\mathcal{B}}_{\vec{t},\vec{\xi}}$ with $N \subset M$ and $m_{\vec{t},\vec{\xi}}(M) = 0$. Since $\mathcal{B}_{\vec{t},\vec{\xi}} = \mathcal{B}(C_0\,[0,T]) \cap C_{\vec{t},\vec{\xi}}$, $M = B \cap C_{\vec{t},\vec{\xi}}$ and $B \in \mathcal{B}(C_0\,[0,T])$. Thus $N \subset B \cap C_{\vec{t},\vec{\xi}}$ and since (2.4) holds for $B \in \mathcal{B}(C_0\,[0,T])$, we have

$$(2.7) \qquad m_g(B) = \int_{\mathbf{R}^n} \int_{C_{\vec{t},\vec{\xi}}} \chi_B(x)\, dm_{\vec{t},\vec{\xi}}(x) dP_X(\vec{\xi})$$
$$= \int_{\mathbf{R}^n} m_{\vec{t},\vec{\xi}}(B \cap C_{\vec{t},\vec{\xi}})\, dP_X(\vec{\xi}) = 0.$$

Hence this completes the proof.

REMARK. Note that for a.e. $\vec{\xi}$, we have

$$(2.8) \quad E[F \mid X](\vec{\xi}) = \int_{C_{\vec{t},\vec{\xi}}} F(x)\, dm_{\vec{t},\vec{\xi}}(x)$$
$$= \int_{\prod_{j=1}^n C_{t_{j-1},\xi_{j-1};t_j,\xi_j}} F(P(x_1, x_2, \dots, x_n)) \prod_{j=1}^n dm_{t_{j-1},\xi_{j-1};t_j,\xi_j}(x_j).$$

THEOREM 2.2. *Let $F \in L^1(C_{s,\xi;t,\eta}, \bar{\mathcal{B}}_{s,\xi;t,\eta}, m_{s,\xi;t,\eta})$. Then*

(2.9)
$$\int_{C_{s,\xi;t,\eta}} F(x(\tau))\, dm_{s,\xi;t,\eta}(x) = \int_{C_{t-s,\eta-\xi}} F(x(\tau - s) + \xi)\, dm_{t-s,\eta-\xi}(x),$$

where $(x+\xi)(\tau)=x(\tau)+\xi$.

Proof. Let $\phi : C_{t-s,\eta-\xi} \to C_{s,\xi;t,\eta}$ be the map defined by $\phi(x)(\tau) = x(\tau - s)+\xi$, $s \le \tau \le t$. Then ϕ is continuous under the metric topology and so it is $\bar{\mathcal{B}}_{t-s,\eta-\xi} - \bar{\mathcal{B}}_{s,\xi;t,\eta}$ measurable. It is obvious from the definitions of measure $m_{t-s,\eta-\xi}$ and mapping ϕ that $m_{s,\xi;t,\eta}(V) = m_{t-s,\eta-\xi}(\phi^{-1}(V))$ for each interval V in $C_{s,\xi;t,\eta}$ and so for all Borel sets in $C_{s,\xi;t,\eta}$.

We next note that for every $m_{s,\xi;t,\eta}$-null set N in $C_{s,\xi;t,\eta}$, $m_{s,\xi;t,\eta}(N) = 0$ iff $m_{t-s,\eta-\xi}(\phi^{-1}(N)) = 0$. We can see that $m_{s,\xi;t,\eta} = m_{t-s,\eta-\xi}\circ\phi^{-1}$ on $\bar{\mathcal{B}}_{s,\xi;t,\eta}$. Hence (2.9) is proved by using the change of variable formula.

We consider the map $T_{\vec{\xi}} : (C_0\,[0,T]\,, \mathcal{W}, m_g) \to (C_{\vec{t},\vec{\xi}}, \bar{\mathcal{B}}_{\vec{t},\vec{\xi}}, m_{\vec{t},\vec{\xi}})$ defined by

$$T_{\vec{\xi}}(x)(s) = x(s) - [x]\,(s) + [\vec{\xi}](s), \qquad 0 \le s \le T \tag{2.10}$$

where $[x]$ and $[\vec{\xi}]$ are the polygonal functions on $[0,T]$ by

$$[x](s) = x(t_{j-1}) + \frac{s-t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1}))$$

and

$$[\vec{\xi}](s) = \xi_{j-1} + \frac{s-t_{j-1}}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}), \quad t_{j-1} \le s \le t_j,$$
$$j = 1, 2, \dots, n, \ \xi_0 = 0.$$

Then $T_{\vec{\xi}}$ is a continuous and surjective function. Thus $T_{\vec{\xi}}$ is $\mathcal{B}(C_0\,[0,T])$ $- \mathcal{B}_{\vec{t},\vec{\xi}}$ measurable, and moreover $\mathcal{W} - \bar{\mathcal{B}}_{\vec{t},\vec{\xi}}$ measurable by Theorem 2.3.

THEOREM 2.3. *$m_g \circ T_{\vec{\xi}}^{-1} = m_{\vec{t},\vec{\xi}}$ on $\bar{\mathcal{B}}_{\vec{t},\vec{\xi}}$.*

REMARK. The following theorem shows that the conditional generalized Wiener measure is nothing more than the pinned generalized Wiener measure(i.e. probability measure induced by the tied down generalized Brownian Motion).

Proof. Let V be a strict interval in $C_{\vec{t},\vec{\xi}}$ as in (2.2), i.e.

$$V = \{x \in C_{\vec{t},\vec{\xi}} \mid (x(\tau_1), x(\tau_2), \dots, x(\tau_{m(1)-1}), x(\tau_{m(1)+1}, \dots, x(\tau_{m(n)-1}))$$
$$\in \prod_{j=1}^{m(n)} (a_j, b_j], \quad j \neq m(1), m(2), \dots, m(n)\}.$$

Then $V \in \bar{\mathcal{R}}_{\vec{t},\vec{\xi}}$ and we note that

$$\begin{aligned} T_{\vec{\xi}}^{-1}(V) &= \{x \in C_0\,[0,T] \mid T_{\vec{\xi}}(x)(\tau_j) \in (a_j, b_j], \\ &\qquad j = 1, \dots, m(1)-1, m(1)+1, \dots, m(n)-1\} \\ &= \{x \in C_0\,[0,T] \mid x(\tau_j) - [x](\tau_j) + [\vec{\xi}](\tau_j) \in (a_j, b_j], \\ &\qquad j = 1, 2, \dots, m(1)-1, m(1)+1, \dots, m(n)-1\} \end{aligned}$$

and $\{x(\tau) - [x](\tau) \mid t_{j-1} \leq \tau \leq t_j\}$, $j = 1, 2, \dots, n$ are independent stochastic process[4]. Thus we write

(2.11)

$$\begin{aligned} m_g \circ T_{\vec{\xi}}^{-1}(V) = \prod_{i=1}^{n} m_g\{x \in C_0\,[0,T] \mid x(\tau_j) - [x](\tau_j) + [\vec{\xi}](\tau_j) \in (a_j, b_j], \\ j = m(i-1)+1, m(i-1)+2, \dots, m(i)-1\} \end{aligned}$$

where $m(0) = 0$. We now show that the right side of (2.11) is equal to $m_{\vec{t},\vec{\xi}}(V)$. To show this, let $W_i = \{x \in C_0\,[0,T] \mid x(\tau_j) - [x](\tau_j) + [\vec{\xi}](\tau_j) \in (a_j, b_j],\ j = m(i-1)+1, m(i-1)+2, \dots, m(i)-1\}$. We need to prove that $m_g(W_i) = m_{t_{i-1},\xi_{i-1};t_i,\xi_i}(A_i)$. Then we have

$$m_\omega \circ T_{\vec{\xi}}^{-1}(V) = \prod_{j=1}^{n} m_{t_{j-1},\xi_{j-1};t_j,\xi_j}(A_j) = m^* \circ P^{-1}(V) = m_{\vec{t},\vec{\xi}}(V)$$

for $V \in \mathcal{R}_{\vec{t},\vec{\xi}}$, since $P^{-1}(V) = \prod_{j=1}^{n} A_j$. Hence it follows that $m_g \circ T_{\vec{\xi}}^{-1} = m_{\vec{t},\vec{\xi}}$ on $\mathcal{B}_{\vec{t},\vec{\xi}}$. Finally by nothing the fact that for any $m_{\vec{t},\vec{\xi}}$-null set N

$$m_g \circ T_{\vec{\xi}}^{-1}(N) = m_{\vec{t},\vec{\xi}}(N) = 0,$$

we obtain the desired result, i.e. $m_g \circ T_{\vec{\xi}}^{-1} = m_{\vec{t},\vec{\xi}}$ on $\bar{\mathcal{B}}_{\vec{t},\vec{\xi}}$.

For simplicity, instead of proving for sets W_i and A_i we shall prove $m_g(W_0) = m_{t_{i-1},\xi_{i-1};t_i,\xi_i}(A_0)$ for sets W_0 and A_0 of the following form

$$W_0 = \{x \in C_0\,[0,T] \mid x(s_j) - [x](s_j) + [\vec{\xi}](s_j) \in (a_j, b_j], \\ t_{i-1} < s_j < t_i,\ j = 1,2,\ldots,k-1\}$$

and

$$A_0 = \{x \in C_{t_{i-1},\xi_{i-1};t_i,\xi_i} \mid x(s_j) \in (a_j, b_j], \\ t_{i-1} < s_j < t_i,\ j = 1,2,\ldots,k-1\}.$$

Now observe that

(2.12)

$$\begin{aligned} m_g(W_0) &= \int_{C_0[0,T]} \chi_{W_0}(x)\,dm_g(x) \\ &= \int_{\mathbf{R}^n}\int_{C_{\vec{t},\vec{\eta}}} \chi_{W_0}(x)\,dm_{\vec{t},\vec{\eta}}(x) dP_X(\vec{\eta}) \\ &= \int_{\mathbf{R}^n} m_{\vec{t},\vec{\eta}}(W_0 \cap C_{\vec{t},\vec{\eta}})\,dP_X(\vec{\eta}) \\ &= \int_{\mathbf{R}^n} m^* \circ (P^{-1}(W_0 \cap C_{\vec{t},\vec{\eta}}))\,dP_X(\vec{\eta}) \\ &= \int_{\mathbf{R}^n} m_{t_{i-1},\eta_{i-1};t_i,\eta_i}(U_0)\,dP_X(\vec{\eta}) \\ &= \int_{\mathbf{R}^n} \Bigg[g(t_{i-1},t_i;\eta_{i-1},\eta_i)^{-1} \int_{p_1}^{q_1}\cdots\int_{p_{k-1}}^{q_{k-1}} g(s_0,s_1,\ldots,s_k;u_0, \\ &\quad u_1,\ldots,u_k) du_1 du_2 \cdots du_{k-1}\Bigg] g(t_0,t_1,t_2,\ldots,t_n;\eta_0,\eta_1,\eta_2,\ldots, \\ &\quad \eta_n) d\eta_1,\cdots,d\eta_n \end{aligned}$$

where the second equality is obtained by (2.4) and the fifth equality follows from

$$P^{-1}(W_0 \cap C_{\vec{t},\vec{\eta}}) = C_{t_1,\eta_1} \times C_{t_1,\eta_1;t_2,\eta_2} \times \cdots \\ \times C_{t_{i-2},\eta_{i-2};t_{i-1}\eta_{i-1}} \times U_0 \cdots \times C_{t_{n-1},\eta_{n-1};t_n,\eta_n},$$

where

$$U_0 = \{x \in C_{t_{i-1},\eta_{i-1};t_i,\eta_i} \mid x(s_j) - [x](s_j) + [\vec{\xi}](s_j) \in (a_j, b_j], \\ j = 1, 2, \dots, k-1,\ t_{i-1} < s_j < t_i\}.$$

Also the upper and lower limits p_i, q_i of the last integral in (2.12) are equal to

$$p_j = a_j + \{(s_j - t_{i-1})/(t_i - t_{i-1})\}(\eta_i - \eta_{i-1} - \xi_i + \xi_{i-1}) + \eta_{i-1} - \xi_{i-1},$$
$$q_j = b_j + \{(s_j - t_{i-1})/(t_i - t_{i-1})\}(\eta_i - \eta_{i-1} - \xi_i + \xi_{i-1}) + \eta_{i-1} - \xi_{i-1},$$
$$u_0 = \eta_{i-1},\ u_k = \eta_i,\ s_0 = t_{j-1},\ s_k = t_j.$$

Integrating the last integral of (2.12) over $\eta_n, \eta_{n-1}, \dots, \eta_{i+1}$ and then $\eta_1, \eta_2, \dots, \eta_{i-2}$, respectively, we have

$$(2.13)\quad m_g(W_0) = \int_{\mathbf{R}^2} g(t_{i-1}, \eta_{i-1}; t_i, \eta_i) \int_{p_1}^{q_1} \cdots \int_{p_{k-1}}^{q^{k-1}} g(s_0, \dots, \\ s_k; u_0, \dots, u_k)\, du_1 du_2 \cdots du_{k-1} d\eta_{i-1} d\eta_i.$$

In order to calculate the right side of (2.13), let

$$v_j = u_j - \frac{s_j - t_{i-1}}{t_i - t_{i-1}}(u_k - u_0 - \xi_i + \xi_{i-1}) - u_0 + \xi_{i-1},\ j = 1, 2, \dots, k-1,$$
$$v_0 = \eta_{i-1} = u_0, \quad v_k = \eta_i = u_k$$

and then by applying the change of variable formula and integrating with respect to v_k, v_0, we have

$$m_g(W_0) = g(t_{i-1}, t_i; \xi_{i-1}, \xi_i) \\ \int_{a_1}^{b_1} \cdots \int_{a_{k-1}}^{b_{k-1}} g(s_0, s_1, \dots, s_k; v_0, v_1, \dots, v_k)\, dv_1 dv_2 \cdots dv_{k-1} \\ = m_{t_{i-1},\xi_{i-1};t_i,\xi_i}(A_0)$$

where $v_0 = \xi_{i-1}$, $v_k = \xi_i$.

COROLLARY 2.2. *Let $F \in L^1(C_0[0,T], \mathcal{W}, m_g)$. Then*

$$\int_{C_0[0,T]} F(T_{\vec{\xi}}(x))\, dm_g = \int_{C_{\vec{t},\vec{\xi}}} F(x)\, dm_{\vec{t},\vec{\xi}}(x).$$

Proof. The proof follows from Theorem 2.3 and the change of variable formula.

3. Example

In this section we will give examples for evaluation.

(1) Let $F(x) = \int_0^T x(t)\, dt$, $x \in C_0[0,T]$

$$\begin{aligned}
&E\left(\int_0^T x(t)\, dt \mid X(x) = \vec{\xi}\right) \\
&= \int_{C_{\vec{t},\vec{\xi}}} \int_0^T x(t)\, dt dm_{\vec{t},\vec{\xi}}(x) \\
&= \sum_{i=1}^n \int_{C_{\vec{t},\vec{\xi}}} \left[\int_{t_{i-1}}^{t_i} x(t)\, dt\right] dm_{\vec{t},\vec{\xi}}(x) \\
&= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[\int_{C_{\vec{t},\vec{\xi}}} x(t)\, dm_{\vec{t},\vec{\xi}}(x)\right] dt \\
&= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[\int_{C_{t_{i-1},\xi_{i-1};t_i,\xi_i}} x_i(t)\, dm_{t_{i-1},\xi_{i-1};t_i,\xi_i}(x)\right] dt \\
&= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{\beta(t_i) - \beta(t)}{\beta(t_i) - \beta(t_{i-1})}\{\xi_{i-1} - \alpha(t_{i-1})\} \\
&\qquad + \frac{\beta(t) - \beta(t_{i-1})}{\beta(t_i) - \beta(t_{i-1})}\{\xi_i - \alpha(t_i)\} + \alpha(t)\, dt
\end{aligned}$$

where the fourth equality is justified by (2.8). In particular, if $\alpha(t) = 0$ and $\beta(t) = t$, then

$$E\left(\int_0^T x(t)\,dt \mid X(x)=\vec{\xi}\right)=\frac{1}{2}\sum_{i=1}^{n}(\xi_i+\xi_{i-1})(t_i-t_{i-1}).$$

Hence our result is the one in [4].

(2) Let $F(x)=\int_0^T (x(t))^2\,dt$, $x\in C_0\,[0,T]$.

$$\begin{aligned}
&E\left(\int_0^T x(t)^2\,dt \mid X(x)=\vec{\xi}\right)\\
&=\int_{C_{\vec{t},\vec{\xi}}}\int_0^T x(t)^2\,dt dm_{\vec{t},\vec{\xi}}(x)\\
&=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\int_{C_{\vec{t},\vec{\xi}}} x(t)^2\,dm_{\vec{t},\vec{\xi}}(x)\,dt\\
&=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\int_{C_{t_{j-1},\xi_{j-1};t_j,\xi_j}} x_j(t)^2\,dm_{t_{j-1},\xi_{j-1};t_j,\xi_j}(x)\,dt\\
&=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} a^2+\frac{(\beta(t)-\beta(t_{j-1}))(\beta(t_j)-\beta(t)}{\beta(t_j)-\beta(t_{j-1})}\,dt
\end{aligned}$$

since

$$\begin{aligned}
&\int_{C_{t_{j-1},\xi_{j-1};t_j,\xi_j}} x_j(t)^2\,dm_{t_{j-1},\xi_{j-1};t_j,\xi_j}(x_j)\\
&=\sqrt{2\pi(\beta(t_j)-\beta(t_{j-1}))}\exp\left\{\frac{(\xi_j-\alpha(t_j)-\xi_{j-1}+\alpha(t_{j-1}))^2}{2(\beta(t_j)-\beta(t_{j-1}))}\right\}\\
&\quad\int_{\mathbf{R}} u^2\sqrt{(2\pi)^2(\beta(t)-\beta(t_{j-1}))(\beta(t_j)-\beta(t))}\\
&\quad\exp\left\{-\frac{1}{2}\left[\frac{(u-\alpha(t)-\xi_{j-1}+\alpha(t_{j-1}))^2}{\beta(t)-\beta(t_{j-1})}\right.\right.\\
&\qquad\left.\left.+\frac{(\xi_j-\alpha(t_j)-u+\alpha(t))^2}{\beta(t_j)-\beta(t)}\right]\right\}du\\
&=\alpha^2+\frac{\beta(t)-\beta(t_{j-1}))(\beta(t_j)-\beta(t))}{(\beta(t_j)-\beta(t_{j-1})},
\end{aligned}$$

where

$$a = \left\{ \frac{\beta(t_i) - \beta(t)}{\beta(t_i) - \beta(t_{i-1})} \{\xi_{i-1} - \alpha(t_{i-1})\} + \frac{\beta(t) - \beta(t_{i-1})}{\beta(t_i) - \beta(t_{i-1})} \{\xi_i - \alpha(t_i)\} \right\} + \alpha(t).$$

In particular, if $\alpha(t) = 0$ and $\beta(t) = t$, then

$$\begin{aligned} &E\left(\int_0^T x(t)^2 dt \mid X(x) = \vec{\xi}\right) \\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \frac{(t - t_{j-1})(t_j - t)}{(t_j - t_{j-1})} + \left\{ \frac{(t_{j-1} - t)\xi_{j-1} + (t - t_{j-1})\xi_j}{(t_j - t_{j-1})} \right\}^2 dt \\ &= \sum_{j=1}^n \frac{(t_j - t_{j-1})^2}{6} + \frac{1}{3} \sum_{j=1}^n (\xi_j^2 + \xi_j \xi_{j-1} + \xi_{j-1}^2)(t_j - t_{j-1}) \end{aligned}$$

which is a result in [4].

(3) Let $F(x) = \exp\left\{\int_0^T x(t)\, dt\right\}$, $x \in C_0\,[0,T]$. Then

$$\begin{aligned} &E\left(\exp\left\{\int_0^T x(t)dt\right\} \mid X(x) = \vec{\xi}\right) \\ &\quad = \int_{C_{\vec{t},\vec{\xi}}} \exp\left\{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} x(t)dt\right\} dm_{\vec{t},\vec{\xi}}(x) \\ &\quad = \int_{C_{\vec{t},\vec{\xi}}} \prod_{i=1}^n \exp\left\{\int_{t_{i-1}}^{t_i} x(t)dt\right\} dm_{\vec{t},\vec{\xi}}(x) \\ &\quad = \int_{\prod_{i=1}^n C_{t_{i-1},\xi_{i-1};t_i,\xi_i}} \prod_{i=1}^n \exp\left\{\int_{t_{i-1}}^{t_i} x_i(t)dt\right\} dm^*(x_1, \ldots, x_n) \\ &\quad = \prod_{i=1}^n \int_{C_{t_{i-1},\xi_{i-1};t_i,\xi_i}} \exp\left\{\int_{t_{i-1}}^{t_i} x_i(t)dt\right\} dm_{t_{i-1},\xi_{i-1};t_i,\xi_i}(x_i). \end{aligned}$$

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