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APPROXIMATE CONTROLLABILITY AND CONTROLLABILITY FOR DELAY VOLTERRA SYSTEM

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1. Introduction

We consider the following delay volterra control system

$$
\begin{aligned}
x(t) =& x_t(\phi)(0), \quad 0 < t \leq T \\
=& U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s, x_s(\phi)) + (Bv)(s)\}\, ds \qquad (1) \\
x_0(\theta) =& \phi \in C.
\end{aligned}
$$

Here, let X and V be Hilbert spaces. The state function $x(t)$, $0 \leq t \leq T$, takes values in X and the control function v is given in $L^2(0,T;V)$ and $U(t,s)$ is a linear evolution operator on X. Let C be a Banach space of all continuous functions from an interval of the form $I = [-h, 0]$ to X with the norm defined by supremum. If a function u is continuous from $I \cup [0,T]$ to X, then u_t is an element in C which has point-wise definition:

$$u_t(\theta) = u(t+\theta) \quad \text{for} \quad \theta \in I.$$

We assume that F is a nonlinear function from $[0,T] \times C$ to X and B is a bounded linear operator from $L^2(0,T;V)$ to $L^2(0,T;X)$.

The purpose of this paper is to prove the approximate controllability results, which were shown in [4] for the abstract semilinear control system, here, for the delay volterra system in the case of trajectories.

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2. Approximate Controllability

The norm of the space $L^2(0,T;X)$ or $L^2(0,T;V)$ is denoted by $\|\cdot\|$ and for the other spaces we use $\|\cdot\|_X$, $\|\cdot\|_C$ and so on. We assume the following hypotheses.

(A) There exist positive constants M', w such that

$$\|U(t,s)\| \le M'e^{w(t-s)}, \quad 0 \le s \le t \le T.$$

Here, we put $M = M'e^{wT}$.

(F) There exists a constant $L > 0$ such that

$$\|F(t,\varphi) - F(t,\psi)\|_X \le L\|\varphi - \psi\|_C,$$

$$\varphi,\ \psi \in C,\ 0 \le t \le T \quad \text{and} \quad F(t,0) = 0.$$

REMARK. To simplify the calculations and to give a simple inequality condition, we assume the above growth condition. Generally, it is sufficient to assume

$$\|F(t,\varphi) - F(t,\psi)\|_X \le L(1 + \|\varphi - \psi\|_C), \quad \varphi,\ \psi \in C.$$

We consider the nonlinear system

$$\dot{x}_t(\phi) = A(t)x_t(\phi) + F(t, x_t(\phi)) + (Bv)(t),$$

where the linear operator $A(t)$ generate a strongly continuous evolution system $\{U(t,s)\}$ on X and is continuously initially observable. Here a unique mild solution is given as, for each v in $L^2(0,T;V)$,

$$(2)\quad x_t(\phi;v)(0) = U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s, x_s(\phi;v)) + (Bv)(s)\}ds.$$

The solution mapping W from $L^2(0,T;V)$ to $C(0,T;C)$ can be defined by

$$(Wv)(t) = x_t(\phi;v)(\cdot).$$

We also define the continuous linear operator $\tilde{S}$ from $L^2(0,T;X)$ to $C(0,T;X)$ by

$$(\tilde{S}p)(t) = \int_0^t U(t,s)p(s)ds, \quad p \in L^2(0,T;X), \quad 0 \le t \le T.$$

The reachable sets of a nonlinear system are used to be compared to the reachable sets of its corresponding linear system ($F \equiv 0$ in (1)). Put

$$K_\alpha(0) = \{z \in C(\alpha,T;X) : z(t) = U(t,0)\phi(0) + \int_0^t U(t,s)(Bv)(s)\,ds, \quad v \in L^2(0,T;V)\}.$$

and define the reachable set $K_\alpha(F)$ in $C(\alpha,T;X)$ by

$$K_\alpha(F) = \{x_t(\phi;v)(0) \in C(\alpha,T;X) : x_t(\phi;v)(0) = U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s,x_s(\phi;v))_s) + (Bv)(s)\}ds, \ v \in L^2(0,T;V)\}.$$

LEMMA 1. *Let $v(\cdot) \in V$ and $\phi \in C$. Then under Hypothesis (F) the solution mapping $(Wv)(t) = x_t(\phi;v)$ of (2) satisfies*

$$\|x_t(\phi;v)\|_C \le (M\|\phi\|_C + M\|B\|\,\|u\|\sqrt{T})\exp(LMT),$$

where L and M are constants for $0 \le t \le T$.

Proof. From hypothesis and system (2) we have

$$\begin{aligned}
&\|x_{t+\theta}(\phi;v)(0)\|_X \\
&\le M\|\phi(0)\|_X + M\int_0^{t+\theta} \{\|F(s,x_s(\phi;v))\|_X + \|B\|\,\|v\|_X\}\,ds \\
&\le M\|\phi(0)\|_X + ML\int_0^{t+\theta} \|x_s(\phi;v)\|_C\,ds \\
&+M\|B\|\,\|v\|\sqrt{t+\theta}, \quad -h \le \theta \le 0.
\end{aligned}$$

Hence

$$\sup_{-h\le\theta\le 0}\|x_t(\phi;v)(\theta)\|_X \le M\|\phi\|_C + ML\int_0^t \|x_s(\phi;v)\|_C\,ds + M\|B\|\,\|v\|\sqrt{t}.$$

Thus we have

$$\|x_t(\phi;v)\|_C \le M\|\phi\|_C + M\|B\|\,\|v\|\sqrt{t} + ML\int_0^t \|x_s(\phi;v)\|_C\,ds.$$

From Gronwall's inequality,

$$\|x_t(\phi;v)\|_C \le (M\|\phi\|_C + M\|B\|\,\|v\|\sqrt{T})\exp(LMT)$$

for $0\le t\le T$.

LEMMA 2. *Let $v_1(\cdot)$ and $v_2(\cdot)$ be in V. Then under hypothesis (F) the solution mapping $(Wv)(t) = x_t(\phi;v)$ of (2) satisfies*

$$\|x_t(\phi;v_1)(\cdot) - x_t(\phi;v_2)(\cdot)\|_C \le M\sqrt{T}\exp(LMT)\|Bv_1(\cdot) - Bv_2(\cdot)\|_{L^2(0,T;X)}$$

Proof. From hypotheses and system (2) we have, for $-h\le\theta\le 0$,

$$\|x_t(\phi;v_1)(\theta) - x_t(\phi;v_2)(\theta)\|_X \le M\int_0^{t+\theta}\|Bv_1(s) - Bv_2(s)\|_{L^2(0,T;X)}\,ds + ML\int_0^{t+\theta}\|x_s(\phi;v_1) - x_s(\phi;v_2)\|_C ds.$$

Hence, by Gronwall's inequality,

$$\sup_{-h\le\theta\le 0}\|x_t(\phi;v_1)(\theta) - x_t(\phi;v_2)(\theta)\|_X = \|x_t(\phi;v_1) - x_t(\phi;v_2)\|_C \le M\sqrt{T}\|Bv_1(\cdot) - Bv_2(\cdot)\|_{L^2(0,T;X)}\exp(LMT).$$

Consequently, we have

$$\begin{aligned}&\|x_t(\phi;v_1)(\cdot)-x_t(\phi;v_2)(\cdot)\|_C\\ &\le M\sqrt{T}\exp(LMT)\|Bv_1(\cdot)-Bv_2(\cdot)\|_{L^2(0,T;X)}\end{aligned}$$

for $0\le t\le T$.

DEFINITION. The system (2) is called approximately controllable on $[0,T]$ if for any given $\varepsilon>0$ and $\xi_T\in L^2(0,T;X)$ there exists some control $v(\cdot)\in L^2(0,T;V)$ such that

$$\|\xi_T-U(T,0)\phi-\tilde{S}F(\cdot,x.(\phi;v))-\tilde{S}Bv\|<\varepsilon.$$

We assume the following hypotheses: (B) For any given $\varepsilon>0$ and $p(\cdot)\in L^2(0,T;X)$ there exists some $v(\cdot)\in L^2(0,T;V)$ such that

(B_1)
$$\|\tilde{S}p-\tilde{S}B_{(0,T)}v\|<\varepsilon;$$

(B_2)
$$\|B_{(0,T)}v(\cdot)\|_{L^2(0,T;X)}\le q_1\|p(\cdot)\|_{L^2(0,T;X)}$$

where q_1 is a positive constant independent of $p(\cdot)$;

(B_3) The constant q_1 satisfies $q_1LM\sqrt{T}\exp(LMT)<1$.

THEOREM 1. *Under hypothesis (B), the system (2) is ε-approximately controllable on* $[0,T]$.

Proof. Since the domain $D(A)$ is dense in $L^2(0,T;X)$, it is sufficient to prove

$$D(A)\subset\overline{K_\alpha(F)}\ ,$$

i.e., for any given $\varepsilon>0$ and $\xi_T\in D(A)$ there exists $v(\cdot)\in V$ such that

$$\|\xi_T-U(T,0)\phi-\tilde{S}F(s,x_s(\phi;v))-\tilde{S}Bv\|<\varepsilon,$$

where

$$x_t(\phi;v)=U(t,0)\phi+\int_0^t U(t,s)\{F(s,x_s(\phi;v))+Bv(s)\}ds.$$

As $\xi_T \in D(A)$ there exists some $p(\cdot) \in C(0,T;X)$ such that

$$\tilde{S}p = \xi_T - U(T,0)\phi.$$

Assume $v_1(\cdot) \in V$ is arbitrarily given. By hypothesis (B_1) there exists some $v_2(\cdot) \in V$ such that

$$\|\xi_T - U(T,0)\phi - \tilde{S}F(\cdot, x.(\phi; v_1)) - \tilde{S}Bv_2\| < \frac{\varepsilon}{2^2}.$$

For $v_2(\cdot)$ thus obtained, we determine $w_2(\cdot) \in V$ by hypothesis (B_1) and (B_2) such that

$$\|\tilde{S}[F(\cdot, x.(\phi; v_2)) - F(\cdot, x.(\phi; v_1))] - \tilde{S}Bw_2\| < \frac{\varepsilon}{2^3}$$

and by Lemma 2,

$$\begin{aligned}\|Bw_2(\cdot)\| &\le q_1 \|F(s, x_s(\phi; v_2)) - F(s, x_s(\phi; v_1))\|_{L^2(0,T;X)} \\ &\le q_1 L \|x_s(\phi; v_2) - x_s(\phi; v_1)\|_C \\ &\le q_1 LM\sqrt{T}\exp(LMT)\|Bv_2(\cdot) - Bv_1(\cdot)\|_{L^2(0,T;X)}.\end{aligned}$$

Thus we may define $v_3(\cdot) = v_2(\cdot) - w_2(\cdot)$ in V, which has the following property;

$$\begin{aligned}\|\xi_T &- U(T,0)\phi - \tilde{S}F(s, x_s(\phi; v_2)) - \tilde{S}Bv_3\| \\ &= \|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi; v_1)) - \tilde{S}Bv_2 + \tilde{S}Bw_2 \\ &\qquad - \tilde{S}[F(s, x_s(\phi; v_2)) - F(s, x_s(\phi; v_1))]\| \\ &< (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.\end{aligned}$$

By induction, it is proved that there exists a sequence $v_n(\cdot)$ in V such that

$$\begin{aligned}&\|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi; v_n)) - \tilde{S}Bv_{n+1}\| \\ &< (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1, 2, \cdots\end{aligned}$$

and

$$\begin{aligned}&\|Bv_{n+1}(\cdot) - Bv_n(\cdot)\|_{L^2(0,T;X)} \\ &\leq q_1 LM\sqrt{T}\exp(LMT)\|Bv_n(\cdot) - Bv_{n-1}(\cdot)\|_{L^2(0,T;X)}.\end{aligned}$$

By hypothesis (B_3) the sequence $\{Bv_n; n = 1, 2, \cdots\}$ is a Cauchy sequence in the Banach space $L^2(0,T;X)$ and there exists some $u(\cdot)$ in $L^2(0,T;X)$ such that

$$\lim_{n\to\infty} Bv_n(\cdot) = u(\cdot) \quad \text{in} \quad L^2(0,T;X).$$

Therefore, for any given $\varepsilon > 0$ there exists some integer N_ε such that

$$\|\tilde{S}Bv_{N_\varepsilon+1} - \tilde{S}Bv_{N_\varepsilon}\| < \frac{\varepsilon}{2}$$

and

$$\begin{aligned}\|\xi_T - &U(T,0)\phi - \tilde{S}F(s, x_s(\phi; v_{N_\varepsilon})) - \tilde{S}Bv_{N_\varepsilon}\| \\ \leq &\|\xi_T - U(T,0)\phi - \tilde{S}F(s, x_s(\phi; v_{N_\varepsilon})) - \tilde{S}Bv_{N_\varepsilon+1}\| \\ &+ \|\tilde{S}Bv_{N_\varepsilon+1} - \tilde{S}Bv_{N_\varepsilon}\| \\ < &(\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \leq \varepsilon.\end{aligned}$$

Thus the nonlinear system (2) is approximately controllable on $[0, T]$.

THEOREM 2. *Suppose the range Bv of the operator B is dense in $L^2(0,T;X)$. Then under hypothesis (F) the delay volterra system (2) is approximately controllable on $[0,T]$.*

Proof. For any given $\varepsilon > 0$ and $p(\cdot) \in X$ there exists some $v(\cdot) \in V$ such that if

$$\|Bv(\cdot) - p(\cdot)\|_{L^2(0,T;X)} < \delta\|p(\cdot)\|_{L^2(0,T;X)},$$

$$\|\tilde{S}p - \tilde{S}Bv\| < \varepsilon,$$

where $\delta > 0$ is any given constant. Thus we have

$$\|Bv(\cdot)\|_{L^2(0,T;X)} \leq \|p(\cdot)\|_{L^2(0,T;X)}(\delta + 1).$$

This satisfies the condition (B). By theorem 1,the system (2) is approximately controllable on $[0, T]$.

3. Delay Control System

We also consider the following delay volterra control system

$$(3)\quad x_t(\phi)(0) = U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s,x_s(\phi)) + (Bu)(s)\}\,ds$$
$$x_T(\phi)(0) = v.$$

Here $u \in L^2(0,T;U)$, a Banach space of possible control actions. We assume the following hypothesis;

(a) The linear pair (A,B) is exactly controllable to the subspace V.

(b) The nonlinear function $F; [0,T] \times C \longrightarrow X$ is continuous and satisfies a Lipschitz-type condition

$$\|F(t,\phi) - F(t,\psi)\| \le r(t)\|\phi - \psi\|_C$$

where $r(\|\phi\|,\|\psi\|) = r(t)$ is continuous on $[0,T]$, $r(t) \to 0$ as $t \to 0$ and $F(t,0) \equiv 0$, $0 \le t \le T$.

(c) The continuous linear evolution system generated by $A(t)$ satisfies $U(t,s)x \in X \cap V$ for all $x \in X$, $0 \le s \le t$ and

$$\|U(t,s)x\|_X \le p(t)\|x\|_X, \quad \|p\|_{L^2(0,T;X)} = c < \infty,$$
$$\|U(t,s)x\|_V \le q(t)\|x\|_X, \quad \|q\|_{L^2(0,T;X)} = d < \infty.$$

(d) There exists a positive $L > 0$ such that

$$\|\int_0^\cdot U(\cdot,s)Bu(s)ds\| \le L(\cdot)\|u\|_{L^2(0,T;V)},$$

where $L(\cdot)$ is increasing, $L(0) = 0$.

(e) γ is chosen so that the following conditions hold

$$\sup_{\|\phi(t)\|\le\gamma} (c + L(t)d)r(\|\phi(t)\|,0) \le k < 1,$$

$$c \sup_{0\le\phi(t),\psi(t)\le\gamma} r(t) \le k < 1.$$

(f) The evolution system $\{U(t,s)|0 \le s \le t \le T\}$ is compact mapping X to X.

Throughout this paper, we consider the case where the initial function ϕ satisfies $\phi(\theta) \equiv 0$, $\theta \in I$.

THEOREM 3([3]). *Suppose that S is a closed, bounded convex subset of a Banach space X. Suppose that Φ_1, Φ_2 are continuous mappings from S into X such that*

(i) $(\Phi_1 + \Phi_2)S \subset S$,

(ii) $\|\Phi_1 x - \Phi_1 x'\|_X \leq k\|x - x'\|$ for all x, $x' \in S$ where k is constant and $0 \leq k \leq 1$,

(iii) $\overline{\Phi_2(S)}$ is compact. Then the operator $\Phi_1 + \Phi_2$ has a fixed point in S.

THEOREM 4. *Hypothesis (a)-(f) are satisfied. Then the state of the system (3) can be steered from the ϕ to any final state v, satisfying*

$$\|v\|_V \leq \frac{(1-k)\gamma}{L}$$

in the time interval $[0, T]$.

Proof. We define the linear operator G from U to X by

$$Gu = \int_0^T U(T,s)Bu(s)\,ds.$$

We can assume, without loss of generality that $Range\, G = V$ and we can construct an invertible operator $\tilde{G}$ defined on $L^2(0,T;U)/ker\, G$ ([1]). Then, the control can be introduced

$$u(s) = \tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))\,ds\Big](s).$$

This control is substituted into equation (3) to provide the operator

$$\begin{aligned}\Phi x_t(\phi)(0) = &\int_0^t U(t,s)F(s,x_s(\phi))\,ds\\ &+ \int_0^t U(t,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))\,ds\Big](s)ds\end{aligned}$$

Notice that $\Phi_{x_T}(\phi)(0) = v$, which means that the control u steers the nonlinear system from the origine to v in time T provided we can obtain a fixed point of the nonlinear operator Φ. We now define

$$\Phi_1 x_t(\phi)(0) = \int_0^t U(t,s)F(s,x_s(\phi))\,ds$$

and

$$\Phi_2 x_t(\phi)(0) = \int_0^t U(t,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))\,ds\Big](s)\,ds.$$

We can now employ Theorem 3 with

$$S = \{x_t(\phi)(\cdot) \in C : \|x_t(\phi)\| \le \gamma\}.$$

Then the set S is closed, bounded and convex. From the definition,

$$\begin{aligned}\Phi x_t(\phi)(0) &= \Phi_1 x_t(\phi)(0) + \phi_2 x_t(\phi)(0)\\ &= \int_0^t U(t,s)F(s,x_s(\phi))\,ds\\ &\quad + \int_0^t U(t,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))\,ds\Big](s)\,ds.\end{aligned}$$

Thus for any $x_t(\phi)(\cdot) \in S$,

$$\begin{aligned}&\|\Phi x_t(\phi)(\theta)\|_X = \|\Phi x_{t+\theta}(\phi))(0)\|\\ &= \Big\|\int_0^{t+\theta} U(t+\theta,s)F(s,x_s(\phi))\,ds\\ &\quad + \int_0^{t+\theta} U(t,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))\,ds\Big](s)\,ds\Big\|\\ &\le \|p\|_{L^2(0,T;X)}\|F(s,x_s(\phi)\|_{L^2(0,T;X)} + L\|v\|_{L^2(0,T;U)}\\ &\quad + L\|q\|_{L^2(0,T;X)}\|F(s,x_s(\phi))\|\\ &\le (c+Ld)r(t)\|x_s(\phi)\|_C + L\|v\|_{L^2(0,T;X)}\\ &\le k\gamma + (1-k)\gamma = \gamma, \quad -h \le \theta \le 0.\end{aligned}$$

Hence

$$\sup_{-h\le\theta\le 0} \|\Phi x_t(\phi)(\theta)\|_X = \|\Phi x_t(\phi)\|_C \le \gamma$$

using (b), (c), (d), and (e). Hence $\Phi_1 x_t(\phi)(0) + \Phi_2 x_t(\phi)(0) \in S$ for all $x_t(\phi) \in S$, which means that part (i) of Theorem 3 is satisfied.

To show that Φ_1 and Φ_2 are completely continuous. We consider

$$\begin{aligned}
&\|\Phi_1(x_t(\phi)+\eta)(\theta)-\Phi_1 x_t(\phi)(\theta)\|_X \\
&\quad=\|\Phi_1 x_{t+\theta}(\phi)+\eta)(0)-\Phi_1 x_{t+\theta}(\phi)(0)\|_X \\
&\quad=\left\|\int_0^{t+\theta} U(t+\theta,s)[F(s,x_s(\phi)+\eta)-F(s,x_s(\phi))]ds\right\|_X \\
&\quad\le \|p\|_{L^2(0,T;X)}\|F(s,x_s(\phi)+\eta)-F(s,x_s(\phi))\|_{L^2(0,T;X)} \\
&\quad\le \|p\|_{L^2(0,T;X)} r(t)\|\eta\|_C,\ -h\le\theta\le 0,\ 0\le t\le T.
\end{aligned}$$

Hence

$$\begin{aligned}
\sup_{-h\le\theta\le 0}&\|\Phi_1(x_t(\phi)+\eta)(\theta)-\Phi_1(x_t(\phi)(\theta)\|_X \\
&=\|\Phi_1(x_t(\phi)+\eta)-\Phi_1(x_t(\phi)\|_C \\
&\le \|p\|_{L^2(0,T;X)} r(t)\|\eta\|_C \longrightarrow 0
\end{aligned}$$

as $\eta\to 0$. Thus, we have

$$\begin{aligned}
&\|\Phi_2(x_t(\phi)+\eta')(\theta)-\Phi_2(x_t(\phi))(\theta)\|_X \\
&=\|\Phi_2(x_{t+\theta}(\phi)+\eta')(0)-\Phi_2(x_{t+\theta}(\phi)(0)\|_X \\
&=\left\|\int_0^{t+\theta} U(t+\theta)B\tilde{G}^{-1}[v-\int_0^T U(T,s)F(s,x_s(\phi)+\eta')\,ds](s)\,ds\right. \\
&\qquad\left.-\int_0^{t+\theta} U(t+\theta,s)B\tilde{G}^{-1}[v-\int_0^T F(s,x_s(\phi))\,ds](s)\,ds\right\|_X \\
&\le L\|q\|_{L^2(0,T;X)}\|F(s,x_s(\phi)+\eta')-F(s,x_s(\phi))\|_{L^2(0,T;X)} \\
&\le L\|q\|_{L^2(0,T;X)} r(t)\|\eta'\|_C,\quad -h\le\theta\le 0,\ 0\le t\le T.
\end{aligned}$$

Consequently

$$\begin{aligned}
\sup_{-h\le\theta\le 0}&\|\Phi_2(x_t(\phi)+\eta')(\theta)-\Phi_2(x_t(\phi)(\theta)\|_X \\
&=\|\Phi_2(x_t(\phi)+\eta')-\Phi_2(x_t(\phi))\|_C \\
&\le L\|q\|_{L^2(0,T;X)} r(t)\|\eta'\|_C \longrightarrow 0
\end{aligned}$$

as $\eta'\to 0$. Thus Φ_1 and Φ_2 are continuous.

Using the Arzela-Ascoli Theorem we show that Φ_2 maps S into a precompact subset of S. We consider

$$\Phi_2 x_t(\phi)(\theta) = \int_0^{t+\theta} U(t+\theta, s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds$$

We now define

$$\begin{aligned}&\Phi_{2-\varepsilon} x_t(\phi)(\theta)\\ &= \int_0^{t+\theta-\varepsilon} U(t+\theta, s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds\end{aligned}$$

for all $(x(\phi))_t \in S$. Then

$$\begin{aligned}\Phi_{2-\varepsilon} x_t(\phi)(\theta) = U(t+\theta, t+\theta-\varepsilon)\int_0^{t+\theta-\varepsilon} & U(t+\theta-\varepsilon, s)B\tilde{G}^{-1}\Big[v \\ & - \int_0^T F(s,x_s(\phi))ds\Big](s)ds.\end{aligned}$$

By hypothesis (f), $U(t+\theta, t+\theta-\varepsilon)$ is a compact operator. Thus the set

$$K_2[x_t(\phi)(\theta)] = \{\Phi_{2-\varepsilon} x_t(\phi)(\theta);\ x_t(\phi) \in S\}$$

is precompact. Also

$$\begin{aligned}&\|\Phi_2 x_t(\phi)(\theta) - \Phi_{2-\varepsilon} x_t(\phi)(\theta)\|_X\\ &= \Big\|\int_{t+\theta-\varepsilon}^{t+\theta} U(t+\theta, s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds\Big\|_X\\ &\le L(\varepsilon)\{\|v\|_{L^2(0,T;X)} + dr(t)\|x_s(\phi)\|_C\} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{-h\le\theta\le 0} &\|\Phi_2 x_t(\phi)(\theta) - \Phi_{2-\varepsilon} x_t(\phi)(\theta)\|_X\\ &= \|\Phi_2 x_{t+\theta}(\phi)) - \Phi_{2-\varepsilon} x_{t+\theta}(\phi)\|_C\\ &\le L(\varepsilon)\{\|v\|_{L^2(0,T;X)} + dr(t)\|x_s(\phi)\|_C\} \longrightarrow 0\end{aligned}$$

as $\varepsilon \to 0$. Thus there are precompact sets arbitrarily close to the set

$$K_2[x_t(\phi)(\theta)] = \{\Phi_2 x_t(\phi)(\theta);\ x_t(\phi) \in S\}$$

and therefore $K_2[x_t(\phi)(\theta)]$ is precompact.

We next show that Φ_2 maps the function in S into an equicontinuous family of functions. For equicontinuity from the left we take $t > \varepsilon > t' > 0$ then

$$\begin{aligned}
&\|\Phi_2 x_t(\phi)(\theta) - \Phi_2 x_{t-t'}(\phi)(\theta)\|_X \\
&= \Big\| \int_0^{t+\theta} U(t+\theta,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \\
&\quad - \int_0^{t-t'+\theta} U(t-t'+\theta,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \Big\|_X \\
&\le \Big\| \int_0^{t+\theta-\varepsilon} U(t+\theta,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \\
&\quad - \int_0^{t+\theta-\varepsilon} U(t-t'+\theta,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \Big\|_X \\
&\quad + \Big\| \int_{t+\theta-\varepsilon}^{t+\theta} U(t+\theta,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \Big\| \\
&\quad + \Big\| \int_{t+\theta-\varepsilon}^{t+\theta-t'} U(t+\theta-t',s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \Big\|_X \\
&\le \Big\| U(t+\theta,t-t'+\theta) \int_0^{t+\theta-\varepsilon} U(t-t'+\theta,s)B\tilde{G}^{-1} \\
&\qquad \cdot \Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \\
&\quad - \int_0^{t+\theta-\varepsilon} U(t-t'+\theta,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \Big\|_X \\
&\quad + \Big\| \int_{t+\theta-\varepsilon}^{t+\theta} U(t+\theta,s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \Big\|_X \\
&\quad + \Big\| \int_{t+\theta-\varepsilon}^{t+\theta-t'} U(t+\theta-t',s)B\tilde{G}^{-1}\Big[v - \int_0^T U(T,s)F(s,x_s(\phi))ds\Big](s)ds \Big\|_X
\end{aligned}$$

$$
\begin{aligned}
&\le \|U(t+\theta, t-t'+\theta) - I\| \int_0^{t+\theta-\varepsilon} \Big\| U(t-t'+\theta, s)B\tilde{G}^{-1} \\
&\qquad \cdot \Big[v - \int_0^T U(T,s)F(s, x_s(\phi))ds \Big](s) \Big\|_X ds \\
&\quad + \int_{t+\theta-\varepsilon}^{t+\theta} \Big\| U(t+\theta, s)B\tilde{G}^{-1} \Big[v - \int_0^T U(T,s)F(s, x_s(\phi))ds \Big](s) \Big\|_X ds \\
&\quad + \int_{t+\theta-\varepsilon}^{t+\theta-t'} \Big\| U(t+\theta-t', s)B\tilde{G}^{-1} \Big[v - \int_0^T U(T,s) \\
&\qquad\qquad \cdot F(s, x_s(\phi))ds \Big](s) \Big\|_X ds \\
&\le \|U(t+\theta, t-t'+\theta) - I\| L(t+\theta-\varepsilon)\|u\|_X \\
&\quad + L(\varepsilon)\|u\|_X + L(\varepsilon - t')\|u\|_X \longrightarrow 0
\end{aligned}
$$

as $\varepsilon \to 0$, by $L(t) \to 0$ as $t \to 0$ and $U(t,s)$ is continuous in s and t. Thus we have

$$
\begin{aligned}
&\sup_{-h \le \theta \le 0} \|\Phi_2 x_t(\phi)(\theta) - \Phi_2 x_{t-t'}(\phi)(\theta)\|_X \\
&= \|\Phi_2 x_t(\phi) - \Phi_2 x_{t-t'}(\phi)\|_C \longrightarrow 0 \ \ as\ t' \to 0.
\end{aligned}
$$

The equicontinuity from the right is similar. Finally we must have a Lipschitz condition for the operator Φ_1. Consider for $x_t(\phi)$, $\hat{x}_t(\phi) \in S$,

$$
\begin{aligned}
&\|\Phi_1 x_t(\phi)(\theta) - \Phi_1 \hat{x}_t(\phi)(\theta)\|_X \\
&= \Big\| \int_0^{t+\theta} U(t,s)[F(s, x_s(\phi)) - F(s, \hat{x}_s(\phi))]ds \Big\|_X \\
&\le \|p\|_{L^2(0,T;X)} \|F(s, x_s(\phi)) - F(s, \hat{x}_s(\phi))\|_X \\
&\le cr(t)\|x_s(\phi) - \hat{x}_s(\phi)\|_C.
\end{aligned}
$$

Consequently,

$$
\|\Phi_1 x_t(\phi) - \Phi_1 \hat{x}_t(\phi)\|_C \le cr(t)\|x_s(\phi) - \hat{x}_s(\phi)\|_C.
$$

Therefore, by Theorem 3, the proof of Theorem 4 is complete.

References

1. Carmichael, N. and Quinn, M.D., *An Approach to Nonlinear Control problems using Fixed-Point Methods, Degree Theory and Pseudo-Inverses,* Numer. Funct. Anal. and Optimiz. **7**(2&3), 197–219 (1984–85).
2. Naito, K. and Park, J.Y., *Approximate Controllability for Trajectories of a Delay Volterra Control System,* Journal of Optimization Theory and Applications, Vol.**61**, No.2, May 1989.
3. Nussbaum, R.P., *The Fixed Point Index and Asymptotic Fixed Point Theorems for K-set Contractions,* Bull. Amer. Math. Soc., Vol **75**, 490–495, 1969.
4. Zhou, H.X., *A Note an Approximate Controllability for Semilinear One-Dimensional Heat Equation,* Appl. Math. Optim. **8**, 275–285, 1982.

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