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Cost Limit Replacement Policy under Imperfect Repair with Inspection Error

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검사오차가 있는 불완전 수리에서의 비용한계 교체 정책

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Abstract

A replacement policy with repair cost limit is discussed. When a system fails, the repair cost is estimated by inspection and repair is then undertaken if the estimated cost is less than a predetermined limit L ; otherwise the system is replaced. After repair, the system is as good as new with probability$(1-p)$ or is minimally repaired with probability p. It is assumed that repair cost can not be estimated exactly because of inspection error. When the failure time follows a Weibull distribution and repair cost a normal distribution, the value of repair cost limit minimizing the expected cost rate is shown to be finite and unique.

1. Introduction

The repair cost limit method has been regarded as a good representation of the way people decide on whether to repair or replace. In the repair cost limit method, when a system fails, its repair cost is estimated by inspection. If the repair cost does not exceed the predetermined cost limit, the system is repaired ; otherwise it is replace. Hastings [5] considered the repair cost limit problem in the context of a Markov decision problem and applied dynamic programming techniques for obtaining the repair cost limits at each repair. Nakagawa and Osaki[8] studied a replacement policy with repair time limit. Nguyen and Murthy[9] showed that the result of Nakagawa and Osaki[10] is optimal over both deterministic and random repair time limit policies. Kaio and Osaki[6] discussed a repair limit policy with a cost constraint. In these models [5, 6, 8, 9], it is assumed that the system is

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as good as new upon repair. Park[10, 11, 12] proposed cost limit replacement policies under minimal repair. Bai and Yun[1] and Ċleroux et al.[4] studied age replacement policies with minimal repair cost limit. A generization of these cases is imperfect repair[see Berg et al.[2], Brown and Proschan[3], Yun and Bai[13]].

We study a repair cost limit replacement policy under imperfect repair and inspection. At the failure of a system, the repair cost is estimated by (imperfect) inspection and repair is undertaken if the estimated repair cost is less than a limit L; otherwise, the system is replaced. When a system is repaired at failure, it is returned to the good-as-new state with probability$(1-p)$ or to the good-as-old state with probability p. It is assumed that the repair cost is not estimated exactly(see[14]). For general failure and repair cost distributions, expected cost rate is obtained. When the failure time follows a Weibull distribution and repair cost a normal distribution, the optimal value of repair cost limit is shown to be finite and unique. The effect of various parameters to optimal repair cost limit is examined through a numerical example.

Basic Assumptions

1. Repair costs are i.i.d. r.v.ś, observable through inspection.
2. Hazard rate of the system is not disturbed by minimal repairs.
3. Replacements and repairs take only negligible time.
4. Planning horizon is infinite

Notation

T_n : r.v. denoting the nth failure time

$F(t)$, $R(t)$: Cdf, cummulative hazard of T_1

X_n : nth repair cost; a r.v.

x_n : realized value of X_n

Y_n : estimator of x_n

L : repair cost limit

$g(x_1)$, $G(x_1)$: pdf, cdf of X_1

$h(y_1 \mid x_1)$: conditional pdf of Y_1 given $X_1=x_1$

$K(\cdot)$, $\bar{K}(\cdot)$: Cdf, Sf of Y_1

$\phi(\cdot)$, $\Psi(\cdot)$: Pdf, Cdf of standard normal distribution

c_0 : replacement cost

p : Probability that the system after repair has the same failure rates as before failure

S_c : expected cost of a renewal period

S_d : expected duration of a renewal period

$C(L)$: expected cost rate

2. Model

Policy : When the system fails, its repair cost is estimated by inspection. If the estimated cost does not exceed a cost limit L, The systems is repaired. Otherwise, it is replaced. The repaired system is either as good as new with probability$(1-p)$ or as good as old with probability p. The expected cost of a renewal period is given by,

$$S_c=\sum_{n=1}^{\infty}[pK(L)]^{n-1}\{[(n-1)M_L+c_0]\bar{K}(L)+nM_L(1-p)K(L)\}$$

$$=[c_0\bar{K}(L)+M_Lk(L)]/[1-pK(L)] \quad \cdots\cdots\cdots (1)$$

where $M_L=E[X_1 \mid Y_1<L]$.

Since the system is minimally repaired at all the failure until a replacement, the failure process is NHPP with mean value function, $R(t)$(see[6]). Expected duration of a renewal period is given by

$$S_d=\sum_{n=1}^{\infty} [pK(L)]^{n-1}[\overline{K}(L)+K(L)(1-p)]E(T_n)$$

$$=[1-pK(L)]\int_0^{\infty}\sum_{n=1}^{\infty}\{(pK(L))^{n-1}R(t)^j e^{-R(t)}/j!\}dt$$

$$=\int_0^{\infty}\exp[-R(t)(1-pK(L))]dt. \quad \cdots\cdots\cdots\cdots (2)$$

From (1) and (2), the expected cost rate is

$$C(L)=[c_0\overline{K}(L)+M_LK(L)]/[1-pK(L))\int_0^{\infty}e^{-R(t)(1-pK(L))}dt]. \quad \cdots\cdots\cdots\cdots (3)$$

3. Analysis

To obtain the optimal policy, we seek the values of L which minimizes C(L). However, the optimal values are difficult to obtain in general case. Therefore, the special cases of C(L) are examined.

1. $p=0$(After repair, the system is as good as new)

$$C_1=[c_0\overline{K}(L)+M_LK(L)][\int_0^{\infty}e^{-R(t)}dt]^{-1}.$$

In this case, the optimal L, $L^*=0$(When system fails, the system always be repaired).

2. $p=1$(After repair, the system is as good as old)

$$C_2(L)=[c_0\overline{K}(L)+M_L K(L)][(1-K(L))\int_0^{\infty}e^{-R(t)(1-K(L))}dt]^{-1}.$$

which agrees with Yun and Bai[14].

3. The conditional distribution $h_y(y\mid x)$ is degenerate(The repair cost is estimatable exactly).

$$C_3(L)=[c_0\overline{G}(L)+E_L G(L)]/[(1-pG(L))\int_0^{\infty}e^{-R(t)(1-pG(L))}dt]$$

which agrees with Yun and Bai[13].

4. It is difficult to analyze the behavior of C(L) for general distributions. Hence, we discuss a Weibull failure distribution and a normal repair cost distribution ;

$F(t)=1-\exp(-(t/\lambda)^{\beta})$, $X_1\sim N(\mu,\ \sigma^2_1)$ and the conditional distribution of Y_1 given $X_1=x_1$, is $N(x_1,\ \sigma^2_2)$.

It can be shown that the marginal distribution of Y_1 is $N(\mu,\ \sigma_1^2+\sigma_2^2)$. The following results can be easily derived(see[14]).

Result 1. When $X_1\sim N(\mu_1,\ \sigma_1^2)$, $m(t)=\int_0^{\infty}x_1 g(x_1)dx_1=\mu\ \Psi(\frac{t-\mu}{\sigma_1})-\sigma_1\ \phi(\frac{t-\mu}{\sigma_1})$.

Result 2. $\int_0^{\infty}\int_0^{\infty}x_1 g(x_1)h(y_1\mid x_1)dx_1dy_1=\mu\ \Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\ \phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})$.

Using the results 1, 2, we obtain

$$K(L)=\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}),$$

$$M_L=\{\mu\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})-\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})\}(\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))^{-1}.$$

Hence,

$$S_c=\{c_0[1-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})]+\mu\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\ \Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})\}\ \{1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})\}^{-1}$$

$$S_d=\int_0^{\infty}e^{-R(t)(1-pK(L))}dt=\int_0^{\infty}e^{-\lambda t^{\beta}(1-pK(L))}dt$$

$$=\lambda\Gamma(1/\beta+1)(1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))^{-1/\beta}$$

Therefore, the expected cost rate is as follows :

$$C_4(L)=\frac{1}{\lambda\Gamma(1+1/\beta)}\ \{(1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))^{1/\beta-1}$$

$$(c_0(1-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))+\mu\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})$$

$$-\frac{\sigma^2_1}{\sigma_1^2+\sigma_2^2}\ \phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))\}. \quad \cdots\cdots\cdots\cdots(4)$$

Lemma 1. If $A(L)=$

$$\mu\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\ \phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})\quad ,$$

then $A(L)$ is an increasing function of L.

Proof. Using $d\Psi(x)/dx=\phi(x)$ and $d\phi(x)/dx=-x\phi(x)$, we can obtain that

$$dA(L)/dL=\frac{\sigma_1^2L+\sigma_2^2\mu}{[\sigma_1^2+\sigma_2^2]^{3/2}}\ \phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}).$$

Thus $A(L)>0$ and is increasing function of L.

Theorem 1. The optimal repair cost limit L^* minimizing the expected cost rate function (4) is finite and unique.

Proof. see Appendix.

Example 1.

Suppose that $\lambda=1$, $\sigma_2=1$ and $\mu=10$. Table 1 gives optimal repair cost limits for selected values of β, σ_1, p and c_0. it indicates that the optimal repair cost limit is decreasing in p but increasing in c_0. However, we cannot find the general trend of the optimal repair cost limit to β and σ_1 in this example.

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Table 1. Optimal values of minimal repair cost limit

	40			60			80		
	2	3	4	2	3	4	2	3	4
2	18.8	25.0	31.3	33.8	47.5	47.9	48.8	70.0	80.2
4	18.5	22.8	28.1	30.2	41.9	53.6	42.9	61.0	79.1
2	13.1	13.3	13.4	17.5	22.5	27.5	27.5	37.5	47.5
4	14.9	15.3	15.6	18.2	21.0	24.9	24.9	33.4	41.9
2	11.8	11.5	11.3	12.6	12.5	12.3	13.5	13.4	13.4
4	13.0	12.8	12.5	14.5	14.3	14.2	15.7	15.8	15.9

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Appendix

Proof of Theorem 1 ;

If let $A(L)=\mu\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})$

$$C_4(L)=\frac{1}{\lambda\Gamma(1+1/\beta)}\quad\{(1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))^{1/\beta-1}$$

$$(c_0(1-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))+A(L)\}$$

A necessary condition for L to minimize $C_4(L)$ is $dC_4(L)/dL=0$. From Lemma 1, we can obtain

$$dC_4(L)/dL=\frac{1}{\lambda\Gamma(1+1/\beta)}\{(1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))^{1/\beta-2}\phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})$$

$$[c_0(1-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))+A(L))p(1-1/\beta)+$$

$$(1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))(-c_0+\frac{\sigma_1^2L+\sigma_2^2\mu}{\sigma_1^2+\sigma_2^2})].$$

Hence, the necessary condition can be modified as follows :

$$[(\beta-1)pA(L)+\frac{\sigma_1^2L+\sigma_2^2\mu}{\sigma_1^2+\sigma_2^2}(1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))]$$

$$[(1-p)\beta+p(1-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))]^{-1}=c_0. \cdots\cdots (5)$$

Let the l.h.s. of (5) be Q(L). First, we obtain the derivative of Q(L).

$$dQ(L)/dL=\frac{1}{\sigma_1^2+\sigma_2^2}\phi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}})$$

$$[(1-p)\beta+p(-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))]^{-2}$$

$$\{p(\beta-1)\frac{\sigma_1^2L+\sigma_2^2\mu}{\sigma_1^2+\sigma_2^2}((1-p)(\beta-1)$$

$$+p(1-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))$$

$$+\frac{\sigma_1^2}{\sigma_1^2+\sigma_1^2}(1-p\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))$$

$$((1-p)\beta+p(1-\Psi(\frac{L-\mu}{\sqrt{\sigma_1^2+\sigma_2^2}}))$$

$$+(\beta-1)A(L)p^2\}>0.$$

Therefore, Q(L) is increasing in L. Further,

$Q(\infty)=\infty$. If $Q(0)>c_0$, $L^*=0$(the system is always replaced at failure). Otherwise, L^* is unique and finite.