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A Study on Estimators of Parameters and $Pr[X<Y]$ in Marshall and Olkin's Bivariate Exponential Model+

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ABSTRACT

The objectives of this thesis are : first, to estimate the parameters and $Pr[X<Y]$ in the Marshall and Olkin's Bivariate Exponential Distribution ; and secondly, to compare the Bayes estimators of $Pr[X<Y]$ with maximum likelihood estimator of $Pr[X<Y]$ in the Marshall and Olkin's Bivariate Exponential Distribution.

Through the Monte Carlo Simulation, we observed that the Bayes estimators of $Pr[X<Y]$ perform better than the maximum likekihood estimator of $Pr[X<Y]$ and the Bayes estimator of $Pr[X<Y]$ with gamma prior distribution performs better than with vague prior distribution with respect to bias and mean squared error in the Marshall and Olkin's Bivariate Eponential Distributon.

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1. Introduction

Recently a number of papers dealt with the problem of estimating $P=Pr[X<Y]$ in the normal case. Church and Harris (1970) derived the Maximum Likelihood Estimator (M.L.E.) of P. while Downton (1973) derived the Uniformly Minimum Variance Unbiased Estimator (U.M.V.U.E.). Both papers are based on the assumption that the distribution of Y is known and a random sample X is observed.

In the problem of life testing and reliability analysis, the exponential distribution plays a central role as useful statistical model. The problem of estimating P in the exponential case has been considered in some papers. Tong (1974) derived two expressions for the M.V.U.E. of P. Kelly and Schucany (1976) derived the M.L.E. and U.M.V.U.E. for P.

However, in all the pervious studies, they have assumed the stochastic independence among the components of system.

But occasionally, independent assumption is not applicable in the practical situation. Naturally, it is more realistic to assume some forms of dependence among the components of system. This dependence among the components arise from common environmental shocks and stress, from components depending on common sources of power, and so on.

Awad, Azzam and Hamdan (1981) derived the M.L.E., moment type estimator and Mann-Whitney type estimator for P in Marshall and Olkin's Bivariate Exponential Model (BVE).

In this paper, we study the Bayes estimator of P when X and Y have a bivariate exponential distribution in Marshall and Olkin's BVE and compare with M.L.E. of P.

The Marshall and Olkin's BVE occupies an important place among bivariate life ditributions in that it has the bivariate loss of memory property and its marginals have the loss of memory property. Marshall and Olkin's BVE is a bivariate model for the life times of the components in two-components system.

It was assumed that the system is subject to shocks governed by three indepentent Poisson Processes with parameters λ_1, λ_2 and λ_0. It was further assumed that three types of shocks were fatal to the first component, the second component and to both components, respectively.

It X and Y are the life times of the two components, then

$$\begin{aligned} \bar{F}(x,\ y) &= Pr[X>x,\ Y>y] \\ &= \exp[-\lambda_1 x-\lambda_2 y-\lambda_0 \max(x,\ y)] \end{aligned} \qquad (1.1)$$

It follows that

$$P=Pr[X<Y]=\iint_{x<y} dF(x,y)=\frac{\lambda_1}{\lambda_0+\lambda_1+\lambda_2}$$

This model is applicable as a failure model for the systems where there exists positive probability of simultaneous failure of exponential components.

The structure of this thesis is as follows :

We introduce the M.L.E. of parameters derived by Proschan and Sullo (1976) and obtain the Bayes estimators of parameters in Chapter 2.

We introduce the M.L.E. of P and obtain the Bayes estimators of P in Chapter 3.

We compare the performance of the M.L.E. and Bayes estimators of P in a moderate sized samples through Monte Carlo Simulation in Chapter 4.

2. Estimators of Parameters

In this Chapter, we review the results of Proschan and Sullo (1976) on the M.L.E. of parameters and then consider the Bayes estimators of parameters.

2.1. Maximum Likelihood Estimators of Parameters

Consider a parallel system with two components where lifetimes are X and Y from (1.1). By Bemis, Bain and Higgins (1972), we have the likelihood function expressed as follows :

$$L(\underline{\lambda}|\underline{x}\ \underline{y})=\lambda_0^{n_0}\ \lambda_1^{n_1}\ \lambda_2^{n_2}\ (\lambda_2+\lambda_0)^{n_1}\ (\lambda_1+\lambda_0)^{n_2}$$
$$\exp[-\lambda_1\sum_{i=1}^{n}x_i-\lambda_2\sum_{i=1}^{n}y_i-\lambda_0\sum_{i=1}^{n}\max\ (x_i,\ y_i)], \qquad (2.1.1)$$

where

$$\underline{\lambda}=(\lambda_0,\ \lambda_1,\ \lambda_2),$$
$$\underline{x}=(x_1,\ x_2,\ \cdots,\ x_n),$$
$$\underline{y}=(y_1,\ y_2,\ \cdots,\ y_n),$$
$$n_1=\sum_{i=1}^{n}\ I(x_i<y_i),$$
$$n_2=\sum_{i=1}^{n}\ I(x_i>y_i),$$
$$n_0=\sum_{i=1}^{n}\ I(x_i=y_i),$$
$$n=n_0+n_1+n_2$$

From (2.1.1), Bemis, Bain and Higgins have obtained the maximum likelihood equations as follows :

$$\frac{n_2}{\hat{\lambda}_1+\hat{\lambda}_0}+\frac{n_1}{\hat{\lambda}_1}=\sum_{i=1}^{n}x_i$$

$$\frac{n_1}{\hat{\lambda}_2+\hat{\lambda}_0}+\frac{n_2}{\hat{\lambda}_2}=\sum_{i=1}^{n} y_i$$

$$\frac{n_1}{\hat{\lambda}_2+\hat{\lambda}_0}+\frac{n_2}{\hat{\lambda}_1+\hat{\lambda}_0}+\frac{n_0}{\hat{\lambda}_0}=\sum_{i=1}^{n}\max(x_i,\ y_j). \tag{2.1.2}$$

As noted by Proschan and Sullo (1976), the M.L.E. of $\underline{\lambda}=(\lambda_0,\ \lambda_1,\ \lambda_2)$ is as follows: For $n_i=0$, for some $i=0,1,2$, the M.L.E. of $\underline{\lambda}$ is given by

$$\hat{\lambda}_1=\begin{cases}\frac{n_1}{n-n_2}\ n/\sum_{i=1}^{n} x_i, & \text{if } n_2<n \\ n/\sum_{i=1}^{n} x_i, \cdots\cdots\cdots\cdots & \text{if } n_2=n\end{cases}$$

$$\hat{\lambda}_2=\begin{cases}\frac{n_2}{n-n_1}\ n/\sum_{i=1}^{n} y_i, & \text{if } n_2<n \\ n/\sum_{i=1}^{n} y_i, \cdots\cdots\cdots & \text{if } n_1=n\end{cases} \tag{2.1.3}$$

and

$$\hat{\lambda}_0=\begin{cases}\left[n-\left(\frac{n_1n_2}{n-n_2}+\frac{n_1n_2}{n-n_1}\right)\right]/\sum_{i=1}^{n}\max(x_i,y_i), & \text{if } n_1<n \text{ and } n_2<n \\ 0, & \text{if } n_1=n \text{ or } n_2=n\end{cases}$$

The M.L.E.'s given by (2.1.3) are unique with the following exception:

If $n_1=n$ or $n_2=n$, then $\hat{\lambda}_2$ and $\hat{\lambda}_0$, $\hat{\lambda}_1$ and $\hat{\lambda}_0$ are not unique respectively.

If $n_0=n$, the M.L.E.'s of $\underline{\lambda}$ do not exist.

If all $n_i>0$, $i=0,1,2$, the likelihood equations can be solved numerically using an iterative procedure as follows:

$$\lambda_1^{(m+1)}=(n_1+\xi_1^{(m)}n_2)/\sum_{i=1}^{n} x_i$$

$$\lambda_2^{(m+1)}=(n_2+\xi_2^{(m)}n_1)/\sum_{i=1}^{n} y_i$$

and

$$\lambda_0^{(m+1)}=[n-(n_2\xi_1^{(m)}+n_1\xi_2^{(m)})]/\sum_{i=1}^{n}\ \max(x_i,\ y_i).$$

Where

$$\xi_1^{(0)}=\frac{n_1}{n-n_2}\ ,$$

$$\xi_2^{(0)}=\frac{n_2}{n-n_1},$$

$$\xi_1^{(m)}=\frac{\lambda_1^{(m)}}{\lambda_1^{(m)}+\lambda_0^{(m)}}, \qquad (2.1.4)$$

and

$$\xi_2^{(m)}=\frac{\lambda_2^{(m)}}{\lambda_2^{(m)}+\lambda_0^{(m)}},$$

2.2 Bayes Estimators of Parameters

2.2.1. vague prior

We assume a quadratic loss function givn by

$$l(\lambda_i,\ \lambda_i^*)=(\lambda_i-\lambda_i^*)^2,\quad i=0,1,2 \qquad (2.2.1)$$

and the vague prior distribution for $\underline{\lambda}$ is as follows:

$$g(\lambda_0,\lambda_1,\lambda_2)\propto\frac{1}{\lambda_0^{c0}\ \lambda_1^{c1}\ \lambda_2^{c2}},\quad \lambda_i>0,\quad c_i>0,\quad i=0,1,2 \qquad (2.2.2)$$

To simplify the likelihood function of the n observations, let

$$t_1=\sum_{i=1}^{n} x_i,$$

$$t_2=\sum_{i=1}^{n} y_i,$$

and

$$t_0=\sum_{i=1}^{n}\max(x_i,\ y_i).$$

From (2.1.1), (2.2.1) and (2.2.2), we derived the joint posterior distribution of $\underline{\lambda}$ as follows:

$$\begin{aligned}&\pi(\lambda_0,\lambda_1,\lambda_2|\underline{d_1^*})\\&\propto g(\lambda_0,\lambda_1,\lambda_2)L(\underline{x},\underline{y}|\lambda_0,\lambda_1,\lambda_2)\\&\quad=K_1\lambda_0^{n_0-c_0}\ \lambda_1^{n_1-c_1}\ \lambda_2^{n_2-c_2}\ (\lambda_1+\lambda_0)^{n_2}\ (\lambda_2+\lambda_0)^{n_1}\end{aligned}$$

$$\cdot \exp[-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_0 t_0]$$

$$= K_1 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l} \binom{n_2}{m} \lambda_0^{n_0 - c_0 + l + m} \lambda_1^{n_1 + n_2 - c_1 - m} \lambda_2^{n_1 + n_2 - c_2 - l}$$

$$\cdot \exp[-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_0 t_0], \qquad (2.2.4)$$

Where

$$K_1^{-1} = \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l} \binom{n_2}{m} \frac{\Gamma(n_0 - c_0 + l + m + l)}{t_0^{n_0 - c_0 + l + m + 1}}$$

$$\cdot \frac{\Gamma(n_1 + n_2 - c_1 - m + 1)}{t_1^{n_1 + n_2 - c_1 - m + 1}} \frac{\Gamma(n_1 + n_2 - c_2 - l + l)}{t_2^{n_1 + n_2 - c_2 - l + 1}}, \qquad (2.2.5)$$

$$\underline{d_1^*} = (\underline{d}, \underline{c}),$$

$$\underline{d} = (\sum_{i=1}^{n} \max(X_i, Y_i), \sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i, N_1, N_2),$$

and

$$\underline{C} = (C_0, C_1, C_2)$$

Then, the marginal posterior of $\underline{\lambda}$ is as follows :

$$\pi_1(\lambda_1 | \underline{d_1^*})$$

$$= \int_0^\infty \int_0^\infty \pi(\lambda_0, \lambda_1, \lambda_2 | \underline{d_1^*}) \, d\lambda_2 \, d\lambda_0$$

$$= K_1 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l} \binom{n_1}{m} \frac{\Gamma(n_0 - c_0 + l + m + 1)}{t_0^{n_0 - c_0 + l + m + 1}} \frac{\Gamma(n_1 + n_2 - c_2 - l + 1)}{t_2^{n_1 + n_2 - c_2 - l + 1}}$$

$$\cdot \lambda_1^{n_1 + n_2 + c_1 - m} e^{-\lambda_1 t_1}, \quad \lambda_1 > 0. \qquad (2.2.6)$$

Similarly,

$$\pi_2(\lambda_2 | \underline{d_1^*})$$

$$= \int_0^\infty \int_0^\infty \pi(\lambda_0, \lambda_1, \lambda_2 | \underline{d_1^*}) \, d\lambda_1 \, d\lambda_0$$

$$= K_1 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l} \binom{n_2}{m} \frac{\Gamma(n_0 - c_0 + l + m + 1)}{t_0^{n_0 - c_0 + l + m + 1}} \frac{\Gamma(n_1 + n_2 - c_1 - m + 1)}{t_1^{n_1 + n_2 - c_1 - m + 1}}$$

$$\cdot \lambda_1^{n_1 + n_2 - c_1 - l} e^{-\lambda_2 t_2}, \quad \lambda_2 > 0, \qquad (2.2.7)$$

and

$$\pi_0(\lambda_0|\underline{d_1^*})$$
$$=\int_0^\infty\int_0^\infty \pi(\lambda_0,\ \lambda_1,\ \lambda_2|\underline{d_1^*})\ d\lambda_1\ d\lambda_2$$
$$=K_1\sum_{l=0}^{n_1}\sum_{m=0}^{n_2}\binom{n_1}{l}\binom{n_2}{m}\frac{\Gamma(n_1-n_2-c_1-m+1)}{t_1^{n_1+n_2-c_1-m+1}}\ \frac{\Gamma(n_1+n_2-c_2-l+1)}{t_2^{n_1+n_2-c_2-l+1}}$$
$$\cdot\ \lambda_0^{n_0+c_0+l+m}\ e^{-\lambda_0 t_0},\ \ \lambda_0>0, \tag{2.2.8}$$

From (2.2.1), (2.2.6), (2.2.7) and (2.2.8), the Bayes estimator for $\underline{\lambda}$ is as follows:

$$\lambda_1^*=\int_0^\infty \lambda_1\ \pi_1(\lambda_1|\underline{d_1^*})\ d\lambda_1$$
$$=K_1\sum_{l=0}^{n_1}\sum_{m=0}^{n_2}\binom{n_1}{l}\binom{n_2}{m}\frac{\Gamma(n_0-c_0+l+m+1)}{t_0^{n_0-c_0+l+m+1}}\ \frac{\Gamma(n_1+n_2-c_1-m+2)}{t_1^{n_1+n_2-c_1-m+2}}$$
$$\cdot\ \frac{\Gamma(n_1+n_2-c_2-l+1)}{t_2^{n_1+n_2-c_2-l+1}}$$

Similarly,

$$\lambda_2^*=\int_0^\infty \lambda_2\ \pi_2(\lambda_2|\underline{d_1^*})\ d\lambda_2$$
$$=K_1\sum_{l=0}^{n_1}\sum_{m=0}^{n_2}\binom{n_1}{l}\binom{n_2}{m}\frac{\Gamma(n_0-c_0+l+m+1)}{t_0^{n_0-c_0+l+m+1}}\ \frac{\Gamma(n_1+n_2-c_1-m+1)}{t_1^{n_1+n_2-c_1-m+1}}$$
$$\cdot\ \frac{\Gamma(n_1+n_2-c_2-l+2)}{t_2^{n_1+n_2-c_2-l+2}}$$

and

$$\lambda_0^*=\int_0^\infty \lambda_0\ \pi_0(\lambda_0|\underline{d_1^*})\ d\lambda_0$$
$$=K_1\sum_{l=0}^{n_1}\sum_{m=0}^{n_2}\binom{n_1}{l}\binom{n_2}{m}\frac{\Gamma(n_0-c_0+l+m+2)}{t_0^{n_0-c_0+l+m+2}}$$
$$\cdot\ \frac{\Gamma(n_1+n_2-c_1-m+1)}{t_1^{n_1+n_2-c_1-m+1}}\ \frac{\Gamma(n_1+n_2-c_2-l+1)}{t_2^{n_1+n_2-c_2-l+1}}$$

2.2.2. gamma prior

We also assume quadratic loss function given by

$$l(\lambda_i,\ \lambda_i^{**}) = (\lambda_i - \lambda_i^{**})^2,\quad i=0,1,2 \tag{2.2.9}$$

Now, we consider a gamma prior distribution instead of vaque prior distribution.

Let the gamma prior distribution be

$$g(\lambda_0,\ \lambda_1,\ \lambda_2) \propto g_0(\lambda_0) g_1(\lambda_1) g_2(\lambda_2),\quad \lambda_i>0,\quad i=0,1,2 \tag{2.2.10}$$

where

$$g_i(\lambda_i) \propto \lambda_i^{a_i - 1}\ e^{-\beta_i \lambda_i},\quad a_i>0,\quad \beta_i>0,\quad \lambda_i>0,\quad i=0,1,2$$

From (2.1.1), (2.2.9) and (2.2.10), the joint posterior p.d.f. of λ is as follows :

$$\begin{aligned}
&\pi(\lambda_0, \lambda_1, \lambda_2 | \underline{d_2^*}) \\
&\propto g(\lambda_0, \lambda_1, \lambda_2) \cdot L(\underline{x}, \underline{y} | \lambda_0, \lambda_1, \lambda_2) \\
&\quad = K_2\ \lambda_0^{n_0 + a_0 - 1}\ \lambda_1^{n_1 + a_1 - 1}\ \lambda_2^{n_2 + a_2 - 1}\ (\lambda_1 + \lambda_0)^{n_2}\ (\lambda_2 + \lambda_0)^{n_1} \\
&\qquad \cdot \exp[-\lambda_1(t_1+\beta_1) - \lambda_2(t_2+\beta_2) - \lambda_0(t_0+\beta_0)] \\
&\quad = K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \lambda^{n_0+a_0+l+m-1}\ \lambda_1^{n_1+n_2+a_1-m-1}\ \lambda_2^{n_1+n_2+a_2-l-1} \\
&\qquad \cdot \exp[-\lambda_1(t_1+\beta_1) - \lambda_2(t_2+\beta_2) - \lambda_0(t_0+\beta_0)],
\end{aligned} \tag{2.2.11}$$

where

$$K_1^{-1} = \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l} \binom{n_2}{m} \frac{\Gamma(n_0+a_0+l+m)}{(t_0+\beta_0)^{n_0+a_0+l+m}} \frac{\Gamma(n_1+n_2+a_1-m)}{(t_1+\beta_1)^{n_1+n_2+a_1-m}} \cdot \frac{\Gamma(n_1+n_2+a_2-l)}{(t_2+\beta_2)^{n_1+n_2+a_2-l} (2.2.12)}$$

$$\begin{aligned}
\underline{d_2^*} &= (\underline{d}, \underline{c}), \\
\underline{d} &= \left(\sum_{i=1}^{n} \max(X_i,\ Y_i),\ \sum_{i=1}^{n} X_i,\ \sum_{i=1}^{n} Y_i,\ N_1,\ N_2\right), \\
\underline{c} &= (a_0, \beta_0,\ a_1,\ \beta_1,\ a_2,\ \beta_2).
\end{aligned} \tag{2.2.12}$$

Then, the marginal posterior distribution of λ is as follows :

$$\begin{aligned}
&\pi_1(\lambda_1 | \underline{d_2^*}) \\
&\quad = \int_0^\infty \int_0^\infty \pi(\lambda_0, \lambda_1, \lambda_2 | \underline{d_2^*})\ d\lambda_2\ d\lambda_0
\end{aligned}$$

$$= K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l}\binom{n_2}{m} \frac{\Gamma(n_0+\alpha_0+l+m)}{(t_0+\beta_0)^{n_0+\alpha_0+l+m}} \frac{\Gamma(n_1+n_2+\alpha_2-l)}{(t_2+\beta_2)^{n_1+n_2+\alpha_2-l}}$$

$$\cdot \lambda_1^{n_1+n_2+\alpha_1-m-1} \exp(-\lambda_1(t_1+\beta_1)), \quad \lambda_1>0. \qquad (2.2.13)$$

Similarly,

$$\pi_2(\lambda_2|\underline{d_2^*})$$

$$= K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l}\binom{n_2}{m} \frac{\Gamma(n_0+\alpha_0+l+m)}{(t_0+\beta_0)^{n_0+\alpha_0+l+m}} \frac{\Gamma(n_1+n_2+\alpha_1-m)}{(t_2+\beta_2)^{n_1+n_2+\alpha_2-m}}$$

$$\cdot \lambda_2^{n_1+n_2+\alpha_2-l-1} \exp(-\lambda_2(t_2+\beta_2)), \quad \lambda_2>0. \qquad (2.2.14)$$

and

$$\pi_0(\lambda_0|\underline{d_2^*})$$

$$= K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l}\binom{n_2}{m} \frac{\Gamma(n_1+n_2+\alpha_1-m)}{(t_1+\beta_1)^{n_1+n_2+\alpha_1-m}} \frac{\Gamma(n_1+n_2+\alpha_2-l)}{(t_2+\beta_2)^{n_1+n_2+\alpha_2-l}}$$

$$\cdot \lambda_0^{n_0+\alpha_0+l+m-1} \exp(-\lambda_0(t_0+\beta_0)), \quad \lambda_0>0. \qquad (2.2.15)$$

From (2.2.9), (2.2.13), (2.2.14) and (2.2.15), the Bayes estimator for λ is as follows :

$$\lambda_1^{**} = \int_0^\infty \lambda_1 \ \pi_1(\lambda_1|\underline{d_2^*}) \ d\lambda_1$$

$$= K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l}\binom{n_2}{m} \frac{\Gamma(n_0+\alpha_0+l+m)}{(t_0+\beta_0)^{n_0+\alpha_0+l+m}} \frac{\Gamma(n_1+n_2+\alpha_1-m+1)}{(t_1+\beta_1)^{n_1+n_2+\alpha_1-m+1}}$$

$$\cdot \frac{\Gamma(n_1+n_2+\alpha_2-l)}{(t_2+\beta_2)^{n_1+n_2+\alpha_2-l}}$$

Similarly,

$$\lambda_2^{**} = K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l}\binom{n_2}{m} \frac{\Gamma(n_0+\alpha_0+l+m)}{(t_0+\beta_0)^{n_0+\alpha_0+l+m}} \frac{\Gamma(n_1+n_2+\alpha_1-m)}{(t_1+\beta_1)^{n_1+n_2+\alpha_1-m}}$$

$$\cdot \frac{\Gamma(n_1+n_2+\alpha_2-l+1)}{(t_2+\beta_2)^{n_1+n_2+\alpha_2-l+1}}$$

and

$$\lambda_0^{**} = K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \binom{n_1}{l}\binom{n_2}{m} \frac{\Gamma(n_0+\alpha_0+l+m+1)}{(t_0+\beta_0)^{n_0+\alpha_0+l+m+1}} \frac{\Gamma(n_1+n_2+\alpha_1-m)}{(t_1+\beta_1)^{n_1+n_2+\alpha_1-m}}$$

$$\cdot \frac{\Gamma(n_1+n_2+\alpha_2-l)}{(t_2+\beta_2)^{n_1+n_2+\alpha_2-l}}$$

3. Estimators of $Pr[X<Y]$

In this Chapter, we obtain the M.L.E. and Bayes estimators under the vague and gamma prior distributions for P and quadratic loss function.

3.1. Maximum Likelihood Estimator of $Pr[X<Y]$

In section 2.1., we obtained the M.L.E.. Thus, using the invariance properties of M.L.E., we obtain the M.L.E. of $Pr[X<Y]$ as follows :

$$\hat{P}=\widehat{Pr}[X<Y]=\frac{\hat{\lambda}_1}{\hat{\lambda}_0+\hat{\lambda}_1+\hat{\lambda}_2} \quad ,$$

where $\hat{\lambda}_0, \hat{\lambda}_1$ and $\hat{\lambda}_2$ are the M.L.E.'s of λ_0, λ_1 and λ_2 of the Marshall and Olkin's BVE given in (2.1.3) and (2.1.4).

3.2. Bayes Estimators of $Pr[X<Y]$

3.2.1. vague prior
In this subsection, we consider the Bayes estimator for $Pr[X<Y]$ with vague prior. From (2.2.4), we obtain the Bayes estimator of $Pr[X<Y]$ as follows :

$$E[P(X<Y)]$$

$$=E\left[\frac{\lambda_1}{\lambda_0+\lambda_1+\lambda_2}\right]$$

$$=\int_0^\infty \int_0^\infty \int_0^\infty \frac{\lambda_1}{\lambda_0+\lambda_1+\lambda_2}\,\pi(\lambda_0,\lambda_1,\lambda_2|\underline{d_1^*})\,d\lambda_0\,d\lambda_1\,d\lambda_2$$

$$=K_1 \sum_{l=0}^{n_1}\sum_{m=0}^{n_2}\sum_{p=1}^{r_5}\sum_{q=0}^{p-1}\sum_{r=0}^{r_6}(-1)^{n_1+n_2-c_1-m-p+1}\frac{(p-1)!}{q!}\frac{1}{t_1^{p-q}}$$

$$\cdot\binom{n_1}{l}\binom{n_2}{m}\binom{n_1+n_2-c_1-m+1}{p}\binom{n_1+n_2+q-c_1-m-p+1}{r}$$

$$\cdot\frac{\Gamma(n_0-c_0+l+m+r+1)}{t_0^{n_0-c_0+l+m+r+1}}\frac{\Gamma(2(n_1+n_2)+q-c_1-c_2-l-m-p-r+2)}{t_2^{2(n_1+n_2)+q-c_1-c_2-l-m-p-r+2}}$$

$$+K_1\sum_{l=0}^{n_1}\sum_{m=0}^{n_2}\sum_{s=0}^{n_1+n_2-c_1-m+1}(-1)^{n_1+n_2-c_1-m+1}$$

$$\begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} n_2 \\ m \end{bmatrix} \begin{bmatrix} n_1+n_2-c_1-m+1 \\ s \end{bmatrix}$$
$$\cdot \Gamma(n_0-c_0+l+m+s+1) \ \Gamma(2(n_1+n_2)-c_1-c_2-l-m-s+2)$$
$$\cdot \int_0^1 x^{r_2+r_4-1} \ (r_1x+t_1)^{-r_2} \ (r_3x+t_1)^{-r_4} \ dx, \tag{3.2.1}$$

where

$$r_1 = t_0 - t_1$$
$$r_2 = n_0 - c_0 + l + m + s + 1,$$
$$r_3 = t_2 - t_1,$$
$$r_4 = 2(n_1+n_2) - c_1 - c_2 - l - m - s + 2,$$
$$r_5 = n_1 + n_2 - c_1 - m + 1$$
$$r_6 = n_1 + n_2 + q - c_1 - m - p + 1,$$

and K_1^{-1} is given in (2.2.5).

3.2.2 gamma prior

Now, we consider the Bayes estimator of $Pr[X<Y]$ with gamma prior. From (2.2.11), we obtain the Bayes estimator of $Pr[X<Y]$ as follows :

$$E[P(X<Y)]$$
$$= E\left[\frac{\lambda_1}{\lambda_0+\lambda_1+\lambda_2}\right]$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\lambda_1}{\lambda_0+\lambda_1+\lambda_2} \ \pi(\lambda_0, \lambda_1, \lambda_2 | d_2^*) \ d\lambda_0 \ d\lambda_1 \ d\lambda_2$$
$$= K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \sum_{p=1}^{r_6} \sum_{q=0}^{p-1} \sum_{r=0}^{r_7} (-1)^{n_1+n_2+a_1-m-p} \ \frac{(p-1)\,!}{q\,!} \ \frac{1}{(t_1+\beta_1)^{p-q}}$$
$$\cdot \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} n_2 \\ m \end{bmatrix} \begin{bmatrix} n_1+n_2+a_1-m \\ p \end{bmatrix} \begin{bmatrix} n_1+n_2+a_1+q-m-p \\ r \end{bmatrix}$$
$$\cdot \frac{\Gamma(n_0+a_0+l+m+r)}{(t_0+\beta_0)^{n_0+a_0+l+m+r}} \ \frac{\Gamma(2(n_1+n_2)+a_1+a_2+q-l-m-p-r)}{(t_2+\beta_2)^{2(n_1+n_2)+a_1+a_2+q-l-m-p-r}}$$
$$+ K_2 \sum_{l=0}^{n_1} \sum_{m=0}^{n_2} \sum_{s=0}^{n_1+n_2+a_1-m} (-1)^{n_1+n_2+a_1-m} \begin{bmatrix} n_1 \\ l \end{bmatrix} \begin{bmatrix} n_2 \\ m \end{bmatrix} \begin{bmatrix} n_1+n_2+a_1-m \\ s \end{bmatrix}$$
$$\cdot \Gamma(n_0+a_0+l+m+s) \ \Gamma(2(n_1+n_2)+a_1+a_2-l-m-s)$$
$$\cdot \int_0^1 x^{r_3+r_5-1} \ (r_2x+r_1)^{-r_3} \ (r_4x+r_1)^{-r_5} \ dx, \tag{3.2.2}$$

where

$$r_1 = t_1 + \beta_1,$$
$$r_2 = t_0 + \beta_0 - t_1 - \beta_1,$$
$$r_3 = n_0 + \alpha_0 + l + m + s,$$
$$r_4 = t_2 + \beta_2 - t_1 - \beta_1,$$
$$r_5 = 2(n_1 + n_2) + \alpha_1 + \alpha_2 - l - m - s,$$
$$r_6 = n_1 + n_2 + \alpha_1 - m,$$
$$r_7 = n_1 + n_2 + \alpha_1 + q - m - p,$$

and K_2^{-1} is given in (2.2.12).

4. Empirical Comparison

In the previous Chapter, even though we obtain the M.L.E. and Bayes estimators of $P = Pr[X < Y]$ for the Marshall and Olkin's BVE, the exact distributions of such estimators are very difficult to derive analytically. Thus, through the Monte Carlo simulation, we compared the performances of the M.L.E. and Bayes estimators of P.

The efficiency of the estimators of P is measured in terms of the estimates of the bias and MSE. The estimates of the bias and MSE were obtained from 1000 trials and $n=5$ under the vague prior distribution with $c_0 = c_1 = c_2 = 1$, and under the gamma prior distribution with $\alpha_0 = \alpha_1 = \alpha_2 = \beta_0 = \beta_1 = \beta_2 = 1$.

To generate the exponential random variates in our simulation study, we used the subroutine GGEXN of the package IMSL (International Mathematical and Statistical Libraries) on Vax-780 at Seoul National University. In each situation generating two dependent exponential random variables of the Marshall and Olkin's BVE, we use the method proposed by Friday and Patil (1977).

From Table 1−6, we observe the following facts for the Marshall and Olkin's BVE.

(1) The Bayes estimators of P performs better than the M.L.E. of P with respect to bias and MSE in most cases.

(2) The bias of Bayes estimator of P with vague prior distribution is less than the bias of Bayes estimator of P with gamma prior distribution for fixed $\lambda_0=1$, $0<\lambda_1$, $\lambda_2 \le 1$.

(3) The MSE of Bayes estimator of P with gamma prior distribution is less than the MSE of Bayes estimator of P with vague prior distribution.

(4) The MSE of Bayes estimators of P increase as P increase.

5. Concluding Remarks

In this thesis, we considered the problems of the estimation of parameters and $Pr[X<Y]$ in Marshall and Olkin's BVE.

In Chapter 2, we introduce the M.L.E. of parameters and obtain the Bayes estimators of parameters under the quadratic loss function and the vague, gamma prior distributions, we observed the M.L.E. of parameters cannot be found explicitly in general case.

In Chapter 3, we introduce the M.L.E. of P and obtain the Bayes estimators of P under the quadratic loss function and vague, gamma prior distributions.

In Chapter 4, through the Monte Carlo simulation, we observed the following facts :

(1) The Bayes estimators of P performs better than the M.L.E. of P with respect to bias and MSE in most cases.

(2) The Bayes estimator of P with gamma prior distribution performs better than the Bayes estimator of P with vague prior distribution with respect to bias and MSE in most cases.

Table I. Estimates of Bias of $Pr[X<Y]$ for $n=5$ in Marshall and Olkin's Bivariate Exponential Model

(λ_0	λ_1	λ_2)	$Pr[X<Y]$	M.L.E.	BE(vague)	BE(gamma)
1.00	0.25	0.25	0.1667	0.0775	0.0454	0.0581
1.00	0.25	0.50	0.1429	−0.0051	0.0575	0.0678
1.00	0.25	0.75	0.1250	−0.0024	0.0619	0.0757
1.00	0.25	1.00	0.1111	0.0923	0.0646	0.0816
1.00	0.50	0.25	0.2857	−0.0129	0.0054	0.0125
1.00	0.50	0.50	0.2500	0.0897	0.0166	0.0249
1.00	0.50	0.75	0.2222	−0.0062	0.0255	0.0371
1.00	0.50	1.00	0.2000	0.0883	0.0286	0.0438
1.00	0.75	0.25	0.3750	−0.0109	−0.0107	−0.0176
1.00	0.75	0.50	0.3333	−0.0092	−0.0026	−0.0044
1.00	0.75	0.75	0.3000	0.0664	0.0033	0.0064
1.00	0.75	1.00	0.2727	0.0638	0.0024	0.0122
1.00	1.00	0.25	0.4444	−0.0050	−0.0239	−0.0413
1.00	1.00	0.50	0.4000	0.0061	−0.0223	−0.0291
1.00	1.00	0.75	0.3636	0.0233	−0.0174	−0.0190
1.00	1.00	1.00	0.3333	0.0421	−0.0131	−0.0092

Table 2. Estimates of Bias of $Pr[X<Y]$ for $n=5$ in Marshall and Olkin's Bivariate Exponential Model

$(\lambda_0$	λ_1	$\lambda_2)$	$Pr[X<Y]$	M.L.E.	BE(vague)	BE(gamma)
0.25	1.00	0.25	0.6667	−0.1143	−0.0682	−0.0781
0.25	1.00	0.50	0.5714	0.0047	−0.0697	−0.0547
0.25	1.00	0.75	0.5000	0.0079	−0.0655	−0.0387
0.25	1.00	1.00	0.4444	−0.0531	−0.0559	−0.0277
0.50	1.00	0.25	0.5714	−0.0147	−0.0559	−0.0693
0.50	1.00	0.50	0.5000	−0.0717	−0.0464	−0.0485
0.50	1.00	0.75	0.4444	−0.0082	−0.0453	−0.0377
0.50	1.00	1.00	0.4000	−0.0342	−0.0346	−0.0289
0.75	1.00	0.25	0.5000	−0.0156	−0.0390	−0.0564
0.75	1.00	0.50	0.4444	−0.0083	−0.0318	−0.0387
0.75	1.00	0.75	0.4000	−0.0149	−0.0296	−0.0302
0.75	1.00	1.00	0.3636	−0.0030	−0.0223	0.0203
1.00	1.00	0.25	0.4444	−0.0050	−0.0239	−0.0413
1.00	1.00	0.50	0.4000	−0.0061	−0.0223	−0.0291
1.00	1.00	0.75	0.3636	0.0233	−0.0174	−0.0190
1.00	1.00	1.00	0.3333	0.0421	−0.0131	−0.0092

Table 3. Estimates of Bias of $Pr[X<Y]$ for $n=5$ in Marshall and Olkin's Bivariate Exponential Model

$(\lambda_0$	λ_1	$\lambda_2)$	$Pr[X<Y]$	M.L.E.	BE(vague)	BE(gamma)
0.25	0.25	1.00	0.1667	−0.0158	0.0073	0.0263
0.25	0.50	1.00	0.2857	−0.0310	−0.0270	−0.0029
0.25	0.75	1.00	0.3750	−0.0097	−0.0432	−0.0166
0.25	1.00	1.00	0.4444	−0.0531	−0.0559	−0.0277
0.50	0.25	1.00	0.1429	−0.0192	0.0373	0.0542
0.50	0.50	1.00	0.2500	0.0177	0.0020	0.0162
0.50	0.75	1.00	0.3333	−0.0184	−0.0234	−0.0094
0.50	1.00	1.00	0.4000	−0.0342	−0.0346	−0.0289
0.75	0.25	1.00	0.1250	−0.0107	0.0533	0.0697
0.75	0.50	1.00	0.2222	−0.0164	0.0161	0.0310
0.75	0.75	1.00	0.3000	0.0237	−0.0066	0.0025
0.75	1.00	1.00	0.3636	−0.0030	−0.0223	−0.0203
1.00	0.25	1.00	0.1111	0.0923	0.0646	0.0816
1.00	0.50	1.00	0.2000	0.0883	0.0286	0.0438
1.00	0.75	1.00	0.2727	0.0638	0.0024	0.0122
1.00	1.00	1.00	0.3333	0.0421	−0.0131	−0.0092

Table 4. Estimates of MSE of $Pr[X<Y]$ for $n=5$ in
Marshall and Olkin's Bivariate Exponential Model

$(\lambda_0$	λ_1	$\lambda_2)$	$Pr[X<Y]$	M. L. E.	BE(vague)	BE(gamma)
1.00	0.25	0.25	0.1667	0.0650	0.0205	0.0134
1.00	0.25	0.50	0.1429	0.0221	0.0153	0.0133
1.00	0.25	0.75	0.1250	0.0199	0.0139	0.0132
1.00	0.25	1.00	0.1111	0.0680	0.0127	0.0133
1.00	0.50	0.25	0.2857	0.0346	0.0333	0.0149
1.00	0.50	0.50	0.2500	0.0627	0.0248	0.0132
1.00	0.50	0.75	0.2222	0.0285	0.0208	0.0125
1.00	0.50	1.00	0.2000	0.0641	0.0179	0.0124
1.00	0.75	0.25	0.3750	0.0358	0.0412	0.0164
1.00	0.75	0.50	0.3333	0.0346	0.0322	0.0146
1.00	0.75	0.75	0.3000	0.0500	0.0271	0.0138
1.00	0.75	1.00	0.2727	0.0537	0.0230	0.0131
1.00	1.00	0.25	0.4444	0.0365	0.0423	0.0184
1.00	1.00	0.50	0.4000	0.0385	0.0366	0.0170
1.00	1.00	0.75	0.3636	0.0412	0.0314	0.0155
1.00	1.00	1.00	0.3333	0.0454	0.0273	0.0145

Table 5. Estimates of MSE of $Pr[X<Y]$ for $n=5$ in
Marshall and Olkin's Bivariate Exponential Model

$(\lambda_0$	λ_1	$\lambda_2)$	$Pr[X<Y]$	M. L. E.	BE(vague)	BE(gamma)
0.25	1.00	0.25	0.6667	0.0814	0.0379	0.0232
0.25	1.00	0.50	0.5714	0.0323	0.0412	0.0218
0.25	1.00	0.75	0.5000	0.0396	0.0410	0.0219
0.25	1.00	1.00	0.4444	0.0542	0.0379	0.0225
0.50	1.00	0.25	0.5714	0.0320	0.0414	0.0231
0.50	1.00	0.50	0.5000	0.0520	0.0371	0.0206
0.50	1.00	0.75	0.4444	0.0396	0.0346	0.0202
0.50	1.00	1.00	0.4000	0.0456	0.0321	0.0200
0.75	1.00	0.25	0.5000	0.0358	0.0433	0.0212
0.75	1.00	0.50	0.4444	0.0370	0.0367	0.0187
0.75	1.00	0.75	0.4000	0.0398	0.0324	0.0177
0.75	1.00	1.00	0.3636	0.0418	0.0294	0.0166
1.00	1.00	0.25	0.4444	0.0365	0.0423	0.0184
1.00	1.00	0.05	0.4000	0.0385	0.0366	0.0170
1.00	1.00	0.75	0.3636	0.0412	0.0314	0.0155
1.00	1.00	1.00	0.3333	0.0454	0.0273	0.0145

Table 6. Estimates of MSE of $Pr[X<Y]$ for $n=5$ in Marshall and Olkin's Bivariate Exponential Model

(λ_0	λ_1	λ_2)	$Pr[X<Y]$	M.L.E.	BE(vague)	BE(gamma)
0.25	0.25	1.00	0.1667	0.0279	0.0094	0.0103
0.25	0.50	1.00	0.2857	0.0394	0.0214	0.0175
0.25	0.75	1.00	0.3750	0.0449	0.0305	0.0215
0.25	1.00	1.00	0.4444	0.0542	0.0379	0.0225
0.50	0.25	1.00	0.1429	0.0207	0.0112	0.0112
0.50	0.50	1.00	0.2500	0.0426	0.0196	0.0139
0.50	0.75	1.00	0.3333	0.0404	0.0259	0.0174
0.50	1.00	1.00	0.4000	0.0456	0.321	0.0200
0.75	0.25	1.00	0.1250	0.0193	0.0119	0.0122
0.75	0.50	1.00	0.2222	0.0305	0.0178	0.0126
0.75	0.75	1.00	0.3000	0.0444	0.0245	0.0145
0.75	1.00	1.00	0.3636	0.0418	0.0294	0.0166
1.00	0.25	1.00	0.1111	0.0680	0.0127	0.0133
1.00	0.50	1.00	0.2000	0.0641	0.0179	0.0124
1.00	0.75	1.00	0.2727	0.0537	0.0230	0.0131
1.00	1.00	1.00	0.3333	0.0454	0.0273	0.0145

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