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A Study of the Effective Approach Method for Median Control Chart of Non-Normally Distributed Process

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I. Introduction

Generally, a standard variable control chart refers to a traditional control chart such as $\bar{X}-R$, X or $\tilde{X}-R$ based on the 3σ principle of W. A. Shewhart. These charts represent typical SQC technique for process control used in a continuous production process, or in a process with a large lot, when the measurements of quality characteristics show a normal distribution or follow a similar distribution pattern.

If these control charts are to be applied as intended, normality of the subject variables distribution could be proved. When the variables show a non-normal distribution, the plots in the above mentioned control charts would often give a false alarm as if an abnormal condition had occurred. This reduces the effect of the SQC technique in application of the control charts for the following reasons.

First, a non-normal distribution generally shows higher probabilities at the tail portion compared with a normal distribution.

Second, when control charts commonly used in the traditional control process, like $\bar{X}$, are applied to a non-normal distribution, plots of samples tend to concentrate on either the upper limit or lower limit of the control line. This may be attributable to the fact that the sample median $\bar{X}$ is not the best indicator of the center in a non-normal distribution, particularly a non-symmetrical distribution.

In an attempt to resolve this problem, non-normal distribution control charts, such as Logarithm control chart, Mode control chart, Gram-Charlier and Pearsonian control charts, have been developed. In this regard, this study intended to develop a new median control chart

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which is easier to apply, more convenient and practical than the traditional 3σ control chart and which can control efficiently the non-normal distribution manufacturing processes common in small scale of intermittent production.

II. A Method of Study

In the practical application of already developed control charts for non-normal distribution, such as X, $\tilde{X}-R$, Logarithm and Gram-Charlier control charts, shortcomings were found ; they are applicable only in limited conditions and that they gave no consideration to diverse shapes of non-symmetrical distribution. Noting these problems, this study considered a control chart method based on the median, which is easiest to use and applicable irrespective of the type of distribution among the representative descriptive variables indicating the central tendency of the quality characteristics-the mean, median and mode.

By focusing on the instances of small sample sizes, intermittent rather than continuous production, or non-symmetrical distribution with a clear distribution of histogram produced from the production lot, this study attempted to design a model of a special control chart that has economic applicability.

The author decided to design a special control chart model using the median because, 1) the median(X) is easier to apply in the variable control chart than the mean(X), 2) it better identifies the average characteristic of the manufacturing process(population), and 3) the superiority of the median was proved in a computer simulation. For the convenience of practical application, the size of sample was limited to odd numbers. A probability density function(p.d.f) was induced from the cumulative probability distribution function(c.d.f), which was obtained from the probability sample of a certain production lot(population). These functions easily determine all the characteristics of the sample median. This study adopted Pearson's probability limit method and, from the Shewhart control chart, distinguished the control limit (upper control limit, lower control limit) into cases of bi-specification limit(upper limit, lower limit) and mono-specification limit based on the probability value beyond 3σ under normal distribution. The sample median($\tilde{X}$) was adopted as the central line.

The chart was designed in such a way that the probability P here is to become

$$P(\tilde{X} \geqq \mathrm{UCL}) + P(\tilde{X} \leqq \mathrm{LCL}) = 0.0027$$

when a bi-specification limit is given.

Since non-normal distributions are generally non-symmetrical, it is very difficult to find the values of UCL and LCL that satisfy the above equation. Therefore, their values were determined through an economic method satisfying the following equation.

$$P(\tilde{X} \geqq \mathrm{UCL}) = P(\tilde{X} \leqq \mathrm{LCL}) = 0.00135$$

In the case of a mono-specification limit, the values of UCL and LCL were determined to satisfy the following equations.

$P(\tilde{X} \leqq \text{LCL}) = 0.0027$

$P(\tilde{X} \geqq \text{UCL}) = 0.0027$

Based on this, a median control chart was designed for typical non-normal distributions such as Gamma, Beta, Log normal, Weibull, Pareto and Truncate distributions. The model chart's applicability to each of the distribution types was analyzed and reviewed and its superiority over them was proved. For broader application, a case study was done using a median control chart separately designed in this study to confirm its usefulness.

III. Improvements and Model Design in Applying Variable Control Chart to Non-Normal Distribution Process

1. Direction for Improvements

The manufacturing process with a non-normal distribution cannot be applied by a standard control chart with accuracy and speed and therefore it can not be controlled properly. To resolve this problem, control charts to be used for a non-normal distribution process should be applied under the following conditions ; even though the distribution of the sample population may be much closer to a normal distribution compared with that of a non-symmetrical population, if the population is considerably off the normal distribution, it does not satisfy the control limits based on the assumptions of a normal population.

In order to solve such problem, the following three solutions have been suggested.

First, to increase the sample size.

Second, to set non-symmetrical control limits, i.e., to make same the probabilities of plots appearing on the side of either the upper or lower control limit.

Third, to transform variables to a logarithm or other functions so that the data could show a standard distribution.

Here, this study developed a median control chart model aiming at solving the above problems by using the sample median, which is better applicable than $\bar{X}$, show good results even with a small sample size, is easy to identify the distributional characteristics and is independent of the distribution type.

2. Model Design for Median Control Chart

The quality characteristic values of n number of products randomly sampled from a certain population, i.e., from a production lost, were defined as X_1, X_2, $\cdots$, X_n and the order statistics derived from them as $X_{(1)}$, $X_{(2)}$, $\cdots$, $X_{(n)}$. (But $X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(n)}$)

For the convenience of structuring the logic, only odd numbers of sample were considered. Thus, $n=2k+1$, k being a positive whole number.

Here, the sample median$(\tilde{X})$ becomes the $(k+1)$th order statistic, $X_{(k+1)}$. Therefore, the design of the median control chart is determined by the characteristics of the order statistic.

$X_{(k+1)}$. If we define $F_{(x)}$, as the cumulative density function(c.d.f.) of the product property value X and $f_{(x)}$ as its probability density function(p.d.f.), the c.d.f. of $X_{(k+1)}$ becomes like equation(3-1) and the p.d.f. like equation(3-2).

$$P\{\tilde{X}\leq x\} = \sum_{i=k+1}^{n} P\{i\text{개의 } X\leq x,\ n-i\text{개의 } X>x\}$$

$$= \sum_{i=k+1}^{n} \binom{n}{i} (F(x)^{i}(1-F(x))^{n-1}) \tag{3-1}$$

$$f_{\tilde{x}}(x) = \frac{(2k+1)!}{k!\,k!} F(x)^{i}(1-F(x))^{k} f(x) \tag{3-2}$$

Therefore, the characteristic of the sample median $\tilde{X}$ can be easily obtained by using equations(3-1) and (3-2).

This study intended to design a control chart as follows based on this sample median. Since the sample median is a figure estimation the central tendency, its value could suddenly become large or small in the event of a shift in the production process from a stable condition to an unstable one. Upon such a phenomenon, the control chart should immediately indicate it. To be able to do so, the control chart should have an upper control limit(UCL) and a lower control limit(LCL) or, in the event of only one specification, either specification upper(S_u) or specification lower(S_L) must be given. The design of the median control chart is determined by the values of these control limits and the central line.

In this study, the control chart was designed in such a way that the UCL and LCL would satisfy the following equation when the two limits of ($S_L \sim S_u$) are given.

$$P(\tilde{X}\geq \text{UCL}) + P(\tilde{X}\leq \text{LCL}) = 0.0027 \tag{3-3}$$

Here, 0.0027 represents the probability of the traditional $\bar{X}$ control chart to go beyond 3σ in a normal distribution. However, since non-normal distributions generally show a non-symmetrical distribution, it is very difficult to find values of UCL and LCL that satisfy the equation(3-3). Therefore the values of UCL and LCL were determined to satisfy the equation

$$P(\tilde{X}\geq \text{UCL}) = P(\tilde{X}\leq \text{LCL}) = 0.00135$$

as is a common method used in equivalent sided test.

As result, based on the equation(3-1), the UCL and LCL become

$$P(\tilde{X}\geq \text{UCL}) = 1 - P(\tilde{X}\leq \text{LCL})$$

$$= 1 - \sum_{i=k+1}^{n} \binom{n}{i} (F(\text{UCL}))^{i}(1-F(\text{UCL}))^{n-i}$$

$$= 0.00135 \tag{3-4}$$

$$P(\tilde{X} \leq \mathrm{LCL}) = \sum_{i=k+1}^{n} \binom{n}{i} (F(\mathrm{LCL}))^{i} (1-F(\mathrm{LCL}))^{n-i}$$
$$= 0.00135 \qquad (3\text{-}5)$$

The values of UCL and LCL that satisfy the equations(3-4) and (3-5) are found easily using simple numerical analysis methods such as the bisection method. Concerning the central line of the control chart, since the sample median gives a good estimated values of the population median, this study adopted the sample median as the central line. In the case of having only one of the upper of lower limits of the quality characteristic values (S_L or S_u), the limits are defined as follows.

$$\text{Lower limit} : P(\tilde{X} \leq \mathrm{LCL}) = 0.0027 \qquad (3\text{-}6)$$

$$\text{Upper limit} : P(\tilde{X} \geq \mathrm{UCL}) = 0.0027 \qquad (3\text{-}7)$$

Therefore, if the distribution function of the quality characteristic values can be determined, the control limits of the median control chart can be designed based on the above functions(3-4, 3-5) and (3-6, 3-7).

Ⅳ. Design of the Median Control Chart in Various Types of Distribution

This chapter will be devoted to the designing of the control limits of the median control chart for non-symmetrical distributions by using the theories presented in section 2 of the previous chapter.

In order to use the median control chart, estimated values of the population are needed in each distribution. And many estimate values with a good indication of the property have already been developed for that purpose. Therefore, this study intended to design the equation of the control limits for the median control chart model to be developed for the following non-symmetrical distributions.

1. Gamma Distribution

When the quality characteristic values show a Gamma distribution the p.d.f. of the distribution is as follows.

$$f_x(x) = \frac{(x-\delta)^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-(x-\delta)/\beta}, \quad x > \delta$$

Through variable transformation, the following control limit line can be obtained.

$$\{\mathrm{LCL} \leq \tilde{X} \leq \mathrm{UCL}\} \equiv \left\{ \frac{\mathrm{LCL}-\delta}{\beta} \leq \frac{\tilde{X}-\delta}{\beta} \leq \frac{\mathrm{UCL}-\delta}{\beta} \right\}$$

If we assume $\frac{CL-\delta}{\beta}=\frac{\tilde{X}-\delta}{\beta}=CL_G$, set the $\frac{\tilde{X}-\delta}{\beta}$ as UCL_G for UCL and LCL_G for LCL, we can design the followings.

$$\begin{cases} CL=\hat{\beta}CL_G+\hat{\delta} \\ UCL=\hat{\beta}UCL_G+\hat{\delta} \\ LCL=\hat{\beta}LCL_G+\hat{\delta} \end{cases} \quad (4\text{-}1)$$

Here, $\hat{\beta}$, and $\hat{\delta}$ represent estimates of β and δ, respectively, and CL_G, UCL_G and LCL_G control limit coefficients for the median control chart in the gamma distribution. Since $F_Y(y)$ is c.d.f. of the Gamma distribution and $F_{\tilde{Y}}(y)$ is c.d.f. of the median for the data from the Gamma distribution, the equation becomes

$$F_{\tilde{Y}}(y)=\sum_{i=k+1}^{n}\binom{n}{i}(F_Y(y))^i(1-F_Y(y))^{n-i}$$

Therefore, in accordance with the equations(3-1) to (3-7), the control limit coefficients CL_G, LCL_G and UCL_G of the median control chart were assumed to have the control line of $F_y(y)$ = 0.5, which was considered the median of the population, and it was defined that

$$F_Y(CL_G)=0.5$$

The lower control limit is a value that satisfies

$$P\{LCL\geq\tilde{X}\}=0.00135$$

$$F_{\tilde{Y}}(LCL_G)=0.00135$$

while the upper control limit value should satisfy

$$P\{UCL\leq\tilde{X}\}$$

$$=1-P\{\tilde{X}\leq UCL\}=0.00135$$

$$F_{\tilde{Y}}(UCL_G)=1-0.00135$$

But, $F_Y(y)=\int_0^y \frac{t^{\alpha-1}}{\Gamma(x)}e^{-t}dt$

Tables 4-1 and 4-2 show values of CL_G, UCL_G, and LCL_G in cases of Two-sided Specification and Single-sided Specification assignment

2. Other Non-Symmetrical Distribution

In case of other non-symmetrical normal distributions, such as Beta, Log normal, Weibul and

Table 4-1. Values of Control Limits for Gamma Distribution (Doubly Specification Assignment)

a	LCL_G		CL_G	LCL_G	
	$n=3$	$n=5$		$n=3$	$n=5$
1.0	0.0199	0.0511	0.6931	3.9267	2.9981
2.0	0.2130	0.3548	1.6783	5.8510	4.7467
3.0	0.5639	0.8168	2.6740	7.5356	6.2990
4.0	1.0116	1.3652	3.6720	9.1047	7.7572
5.0	1.5236	1.9687	4.6709	10.6024	9.1573
6.0	2.0820	2.6113	5.6701	12.0502	10.1571
8.0	3.2977	3.9787	7.6629	14.8422	13.1527
10.0	4.6070	5.4229	9.6687	17.5374	15.7102

Table 4-2. Values of Control limits for Gamma Distribution (Single Specification Assignment)

a	LCL_G		CL_G	LCL_G	
	$n=3$	$n=5$		$n=3$	$n=5$
1.0	0.0283	0.0653	0.6931	3.5773	2.7601
2.0	0.2573	0.4064	1.6783	5.4398	4.4570
3.0	0.6464	0.9025	2.6740	7.0775	5.9707
4.0	1.1291	1.4810	3.6720	8.6073	7.3968
5.0	1.6733	2.1116	4.6709	10.0703	8.7687
6.0	2.2614	2.7789	5.6701	11.4868	10.1031
8.0	3.5309	4.1903	7.6692	14.2230	12.6936
10.0	4.8881	5.6736	9.6687	16.8691	15.2114

Truncated normal distributions, the control limits and central lines can be obtained, like in the Gamma distribution, by using the p.d.f. of the concerned distribution and by transforming variables.

For the practical application of a median control chart that has this type of control line and control limits, this study worked out the coefficient tables(Tables 4-1, 4-2) for each design parameter.

In order to identify the characteristics of the designed median control chart, 1,000 samples were taken out of the instances when the production process changed to a non-symmetrical distribution and were analyzed to check on the instances of going beyond the control limits. The analysis, done by a computer simulation proved that the shape of the $\bar{X}$ control chart shifted much depending on the change in distribution, while the median control chart had more applicability with less susceptibility to the assumed distribution.

Therefore, it was confirmed that the model of median control chart developed in this study was much more useful than the $\bar{X}$ control chart.

V. Control Limit Equation of the Median Control Chart by Distribution Type

In order to control the process where the quality characteristic values are variables and their distribution shows a non-symmetrical normal shape, a model of median control chart was developed based on the sample median in the case of a small number of samples size(3 or 5).

It is a well-known fact that in a non-symmetrical normal distribution the sample median is for less susceptible than the mean value of the sample to the basic assumptions for the development of the model. Taking advantage of such characteristics, this study designed the median control chart in such a way that the probability of going beyond the control limit will become 0.0027 in widely known, easily applicable non-symmetrical normal distributions such as Gamma, Beta, Log normal, Weibul, Pareto and Truncated normal distribution.

Table 5-1 shows equations of the central line and the control limits for the median control chart model designed in this study.

Table 5-1. Equation of Control Limits for Non-Normal Distributions.

Distribution Type	CL	UCL	LCL
1. Gamma	$\hat{\beta}\,CL_G+\hat{\delta}$	$\hat{\beta}\,UCL_G+\hat{\delta}$	$\hat{\beta}\,LCL_G+\hat{\delta}$
2. Beta	$\hat{a}+(\hat{b}-\hat{a})CL_B$	$a+(b-a)UCL_B$	$a+(b-a)LCL_B$
3. Normal	$\bar{\bar{X}}$	$\bar{\bar{X}}+UCL_N A_2\bar{R}$	$\bar{\bar{X}}+LCL_N A_2\bar{R}$
4. Log-normal	$\exp(\bar{\bar{X}})$	$\exp(\bar{\bar{X}}+UCL_N A_2\bar{R})$	$\exp(\bar{\bar{X}}+LCL_N A_2\bar{R})$
5. Weibull	$\hat{\delta}+\left(\frac{CL_W}{\hat{\alpha}}\right)^{1/\beta}$	$\hat{\delta}+\left(\frac{UCL_W}{\hat{\alpha}}\right)^{1/\beta}$	$\hat{\delta}+\left(\frac{LCL_W}{\hat{\alpha}}\right)^{1/\beta}$
6. Pareto	$\hat{k}\exp(CL_P/\hat{a})$	$\hat{k}\exp(UCL_P/\hat{a})$	$\hat{k}\exp(LCL_P/\hat{a})$
7. Truncate			
1) Left sided	$\hat{\mu}+CL_T\hat{\sigma}$	$\hat{\mu}+UCL_{LT}\hat{\sigma}$	$\hat{\mu}+LCL_{LT}\hat{\sigma}$
2) Right sided	$\hat{\mu}+CL_T\hat{\sigma}$	$\hat{\mu}+UCL_{RT}\hat{\sigma}$	$\hat{\mu}+LCL_{RT}\hat{\sigma}$
3) Two-sided	$\hat{\mu}$	$\hat{\mu}+UCL_{DT}\hat{\sigma}$	$\hat{\mu}+LCL_{DT}\hat{\sigma}$

VI. Application Procedure for the Median Control Chart in a Non-Normal Distribution Process

The median control chart developed in this study aims at controlling efficiently the manufacturing process that shows a non-normal distribution. In order to draw a control chart that will fulfill this purpose, it is essentiual to know how the quality characteristic values distribute, which can be told by drawing a histogram using various data obtained from the manufacturing process.

In drawing a histogram, the first task is to determine the number of Cells and the Cell interval them because the shape of the histogram can change widely depending on their values.

This study made a histogram by employing the methods suggested by H. A. Sturge's Formula when the number of data is n, the number of Cells $k=1+3.3\log_{10}^{n}$ and the Cell interval as $(X_{max}-X_{min})/k$.

Another method of estimating the distribution without drawing the histogram, is to use the probability paper. There are various kinds of probability paper, including normal, Log normal and Weibul probability paper, and in most papers, the distribution is estimated when the samples and the plotted sample distribution functions reveal lineal relations.

Once the distribution of the quality characteristic values and its shape are estimated, the population parameter of the distribution is estimated by using various estimation techniques. Following the estimation of the distribution and the population parameter is the verification of whether the current data of quality characteristic values are extracted from the estimated distribution or not through a goodness fit test.

There are various different methods of testing the goodness fit. Yet, most typical ones are χ^2 and Kolmogorov-Smirnov test methods. Once the distribution of the quality characteristic values are confirmed by the estimation and test, the central line and control limits should be established from the table of control limit coefficients of the relevant distribution by using the estimated value of the population parameter.

VII. Conclusion

The median control chart model developed in this study proved its better efficiency over the traditional $\bar{X}$ control chart in the event of a shift in the manufacturing process to a process that shows a non-symmetrical distribution, or when the production process is not sure to show a normal distribution. Manufacturing processes in which the quality characteristic values define either the specification upper(S_u)or specification lower(S_L), or, even if both S_u and S_L are defined, when the process is an unstable one, one, is of an intermittent production system or is a small quantity, diverse-item basis production, are all typical of the small-and medium-size businesses. Compared with the traditional variable control chart, the $\bar{X}$ control chart based on the assumption of a normal distribution, the median control chart designed in this study can be very efficient in controlling all the processes that show a non-normal distribution, such as a continuous, large-quantity production system on a limited number of items.

The chart can also be very useful in a production process of a small-and medium-size companies operating under a diverse-item, small-quantity production system. Particularly, when the process involves an unstable, small quantity, or intermittent production, which shows a non-symmetrical distribution, or when only one of the specifications are defined$(S_u$ or $S_L)$, the median control chart can replace the shortcomings of the standard variable control chart($\bar{X}$ control chart). The median control chart also proved excellence in convenience and applicability over the existing control charts and therefore could be applied into practical operations.

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