Journal of the KSQC
Vol. 18, No. 1, June 1990



# A Nonparametric Test for the Equality of Several Regression Lines against Ordered Alternatives+

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## ABSTRACT

In this paper we propose a nonparametric test for testing the equality of several regression lines against ordered alternatives, when the independent variables are positive and all regression lines have a common intercept. The proposed test is based on a Jonckheere-type statistic applied to residuals. Under some conditions our proposed test statistic is asymptotically distribution-free. The small-sample powers of our test are compared with other tests by a Monte Carlo study. The simulation results show that the proposed test has significantly higher empirical powers than the other tests considered in this paper.

## 1. Introduction

Consider the linear regression model

$$Y_{ij}=\alpha_i+\beta_i x_{ij}+\varepsilon_{ij},\ j=1,\ \cdots,\ n_i\ ;\ i=1,\ \cdots,\ k, \tag{1.1}$$

+This research was supported by the Ministry of Education, through the Research Institute for Basic Sciences, SNU 1988-1989.

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where the $x_{ij}$'s are known constants, the $\alpha_i$'s are nuisance parameters, and the $\beta_i$'s are the slope parameters of interest. The $Y_{ij}$'s are observable while the $\varepsilon_{ij}$'s are mutually independent and identically distributed random variables with continuous cumulative distribution function(cdf) $F$.

We are interested in testing the equality of $k$ regression lines against ordered alternatives under the following assumptions.

A1. All $\alpha_i$'s are equal.
A2. The $x_{ij}$'s are positive.

Under the assumptions A1 and A2, we want to test

$$H_0 : \beta_1 = \cdots = \beta_k = \beta \text{ (unknown)} \quad (1.2)$$

against the ordered alternatives

$$H_1 : \beta_1 \leqq \cdots \leqq \beta_k, \quad (1.3)$$

wher at least one inequality is strict.

Tests for the parallelism of regression lines against the ordered alternatives(1.3) have been considered by Adichie(1976), Smith and Wolfe(1975), and Rao and Gore(1984), among others. Acichie(1976) has proposed parametric and nonparametric tests for the parallelism of several regression lines against ordered alternatives without the assumptions A1 and A2. These are the likelihood ratio(LR) test and a test based on linear combinations of least squares estimators. As nonparametric test, rank analogues of the two parametric test are proposed. Smith and Wolfe (1975) have investigated a distribution-free test, under the assumption that all $x_{ij}$'s are non-negative random variables. Rao and Gore(1984) have developed a distribution-free test. They have assumed that the independent variables $x_{ij}$ are equispaced.

The proposed test is an application of Jonckheere's procedure to regression problem. Jonckheere(1954) has proposed a distribution-free test for ordered alternatives in one-way analysis of variance model. The Jonckheere statistic is of the form $J = \sum_{u<v} U_{uv}$, where $U_{uv}$ is the Mann-Whitney statistic applied to $u$th and $v$th samples.

We consider the residuals defined by

$$Z_{ij} = Y_{ij} - \hat{\beta} x_{ij}$$

where $\hat{\beta}$ is an estimator of the common slope $\beta$ in(1.2). Then, assuming A1 and A2, we expect that for $u < v$, $Z_{vj}$ is stochastically larger than $Z_{uj}$ under ordered alternatives(1.3). We may thus construct a Jonckheere-type test statistic based on the residuals $Z_{ij}$. Asymptotic normality and asymptotic distribution-free properties of the test statistic can be obtained by using the results of Randles(1982, 1984). The asymptotic null distribution of the proposed test statistic is the same as that of the Jonckheere test statistic in location problem.

We also performed a small-sample Monte Carlo study to compare the empirical powers of the tests considered in this paper for various underlying distributions. The results show that the proposed test is more powerful than the other tests considered in the Monte Carlo study.

## 2. Preliminaries

In this section we introduce some notations and review the parametric and nonparametric tests for parallelism of regression lines against ordered alternatives. For $i=1, \cdots, k$, let

$$\bar{x}_i=\frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}\ ;\ \bar{Y}_i=\frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}$$

$$w_i^2=\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2\ ;\ W^2=\sum_{i=1}^{k} w_i^2\ ;\ \rho_i=w_i^2/W^2.$$

To introduce the LR test proposed by Adichie(1976), we use the following centered model :

$$Y_{ij}=\alpha_i+\beta_i(x_{ij}-\bar{x}_i)+\varepsilon_{ij},\ j=1, \cdots, n_i\ ;\ i=1, \cdots, k.$$

Assuming that the error terms $\varepsilon_{ij}$'s have a normal distribution with mean 0 and variance $\sigma^2$, the MLE of $\alpha_i$ and $\beta$, under $H_0$ are given as follows.

$$\hat{\beta}=\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)\, Y_{ij}/W^2.$$

$$\hat{\alpha}_i=\bar{Y}_i,\ i=1, \cdots, k.$$

Note that the estimator $\hat{\beta}$ can be written as

$$\hat{\beta}=\sum_{i=1}^{k}\rho_i\hat{\beta}_i,$$

where $\hat{\beta}_i$ is the MLE of $\beta_i$.

Adichie(1976) showed that the maximization problem under $H_1$ reduces to minimizing $\sum w_i^2(\hat{\beta}_i-\beta_i)^2$ over $\beta_i$'s satisfying the order restrictions $\beta_1\leqq\beta_2\leqq\cdots\leqq\beta_k$. Thus the MLE of $\alpha_i$ and $\beta_i$ can be obtained as follows.

$$\tilde{\alpha}_i=\bar{Y}_i,$$

$\tilde{\beta}_i$ = "the isotonic regression of $\hat{\beta}_i$ with weights $w_i^2$".

For the computation of the isotonic regression, we may use the "Pool-Adjacent-Violators" algorithm described in Bartholomew(1959) or in Barlow, et al. (1972).

When $\sigma^2$ is not known, the LR test is based on the test statistic

$$\bar{E}_k^2 = \frac{\sum_{i=1}^{k} w_i^2 (\tilde{\beta}_i - \hat{\beta})^2}{\sum_{i=1}^{k}\sum_{j=1}^{n_i} \{ Y_{ij} - \bar{Y}_i - \hat{\beta}(x_{ij} - \bar{x}_i) \}^2} \tag{2.1}$$

The statistic $\bar{E}_k^2$ has the E-bar-squared distribution whose table can be obtained in Chacko (1963) or in Barlow, et al. (1972).

The rank version of the LR test statistic, considered by Adichie (1976), is defined by

$$\bar{\chi}_R^2 = \sum_{i=1}^{k} w_i^2 (\tilde{\beta}_{R_i} - \hat{\beta}_R)^2 / A^2, \tag{2.2}$$

where

$$\hat{\beta}_R = \sum_{i=1}^{k} \rho_i \hat{\beta}_{R_i}$$

with

$$\hat{\beta}_{R_i} = \{ \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i) R_{ij}^* / (n+1) \} / w_i^2$$

where $R_{ij}^*$ is the rank of the $j$th residual $Y_{ij} - \hat{\beta}(x_{ij} - \bar{x}_i)$ among the $i$th sample with some estimator $\hat{\beta}$ of the common slope $\beta$, and $\tilde{\beta}_{R_i}$ is the isotonic regression of $\hat{\beta}_{R_i}$. We used the Wilcoxon scores in construction $\bar{\chi}_R^2$, and thus $A^2$ is defined as

$$A^2 = \int_0^1 u^2 du - (\int_0^1 u \, du)^2 = 1/12.$$

As an estimator $\hat{\beta}$, Adichie (1976) suggested to use the Hodges-Lehmann type estimator.

Rao and Gore (1984) dealt with several-sample linear regressions in which the independent variables are assumed to be equispaced. They also assumed $n_i = 2r$ on the $i$th line in order to propose an exactly distribution-free test for the problem of testing $H_0$ against the ordered alternatives (1.3).

Since the independent variables $x_{ij}$ are assumed to be equispaced, we may let

$$x_{ij} = (A_i + j), \quad i = 1, \cdots, k \; ; \; j = 1, \cdots, 2r,$$

where $A_i$ is the starting design-point for line $i$.

For each line $i$, to construct the slope estimators they adopted the pairing scheme of $x$ given by

$$(A_i + 1, \; A_i + r + 1), \; \cdots, \; (A_i + j, \; A_i + r + j), \; \cdots, \; (A_i + r, \; A_i + 2r).$$

For each such $x$ pair and the associated $Y$ pair they defined an independent slope estimator given by

$$b_{ij} = (Y_{i, A_i+r+j} - Y_{i, A_i+j})/r, \quad 1 \leq j \leq r.$$

Then, the Rao-Gore test statistic is defined by

$$G = \sum_{u<v}^{k} U_{uv}, \tag{2.3}$$

where $U_{uv}$ is the Mann-Whitney statistic given by

$$U_{uv} = \sum_{i=1}^{r} \sum_{j=1}^{r} \psi(b_{vi} - b_{uj})$$

where $\psi(\cdot)$ is an indicator function defined by

$$\psi(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Note that the Rao-Gore statistic $G$ has a usual null distribution of the Jonckheere statistic.

## 3. The Proposed Test and its Properties

In this section we construct a nonparametric test statistic for testing the equality of several regression lines against ordered alternatives under the assumptions A1 and A2. Let $\hat{\beta}$ be a consistent estimator of the common slope $\beta$ under $H_0$ such as the Hodges-Lehmann type estimators. For example, we may use the median or the weighted median of the set of slope estimators

$$S_{ist} = \frac{Y_{it} - Y_{is}}{x_{it} - x_{is}}, \quad 1 \leq s < t \leq n_i \; ; \; i = 1, \cdots, k. \tag{3.1}$$

Jee(1989) has shown that the weighted median $\hat{\beta}$ of $S_{ist}$ with weights $x_{it} - x_{is}$ is a $\sqrt{n}$-consistent estimator of the common slpe $\beta$.

Let $Z_{ij}(\hat{\beta})$ denote the residuals defined by

$$Z_{ij}(\hat{\beta}) = Y_{ij} - \hat{\beta} x_{ij}, \quad j = 1, \cdots, n_i \; ; \; i = 1, \cdots, k.$$

For $u < v$, we define the Mann-Whitney type statistic $U_{uv}$ based on the residuals from the $u$th and $v$th lines as follows.

$$U_{uv}(\hat{\beta}) = \sum_{s=1}^{n_u} \sum_{t=1}^{n_v} \psi(Z_{vt}(\hat{\beta}) - Z_{us}(\hat{\beta})).$$

The proposed test statistic is then defined by

$$J(\hat{\beta}) = \sum_{u>v}^{k} U_{uv}(\hat{\beta}). \tag{3.2}$$

Under the ordered alternatives $H_1$ in (1.3) the values of $J(\hat{\beta})$ is expected to be large. We thus reject $H_0$ in favor of $H_1$ for large values of $J(\hat{\beta})$.

Note that if $\hat{\beta}$ is replaced by the true common slope $\beta$ in $J(\hat{\beta})$, then the distribution of $J(\beta)$ is the same as that of Jonckheere statistic, which has a distribution-free property. The exact distribution of $J(\hat{\beta})$ may be too complicated to be useful. However, the asymptotic equivalence of $J(\hat{\beta})$ and $J(\beta)$ can be proved using the results of Randles(1982, 1984), who investigated the asymptotic properties of statistics based on residuals.

Note that the proposed statistic $J(\hat{\beta})$ in(3.2) is the sum of

$$U_{uv}(\hat{\beta}) = \sum_{s=1}^{n_u}\sum_{t=1}^{n_v} \psi(Z_{vt}(\hat{\beta}) - Z_{us}(\hat{\beta})), \tag{3.3}$$

which is the Mann-Whitney statistic applied to the residuals from the $u$th and $v$th regression lines. While, if $\hat{\beta}$ is replaced by $\beta$ in (3.3), the statistic becomes

$$U_{uv}(\beta) = \sum_{s=1}^{n_u}\sum_{t=1}^{n_v} \psi(Z_{vt}(\beta) - Z_{us}(\beta)),$$

which is the Mann-Whitney statistic applied to independent observations.

We now want to prove asymptotic equivalence of $U_{uv}(\hat{\beta})$ and $U_{uv}(\beta)$ for each $(u, v)$ to show that $J(\hat{\beta})$ and $J(\beta)$ have the same limiting distribution. Here we assume that

$$\lim_{N\to\infty}\frac{n_i}{N} = \lambda_i, \quad 0<\lambda_i<1, \quad i=1, \cdots, k, \tag{3.4}$$

with

$$N = \sum_{i=1}^{k} n_i.$$

The $U$ statistics corresponding to $U_{uv}(\hat{\beta})$ and $U_{uv}(\beta)$ are respectively given by

$$U_{uv}^{*}(\hat{\beta}) = \frac{1}{n_u n_v} U_{uv}(\hat{\beta})$$

and

$$U_{uv}^{*}(\beta) = \frac{1}{n_u n_v} U_{uv}(\beta),$$

which are two-sample $U$ statistics. The following theorem gives some conditions under which the two-sample $U$ statistics $U_{uv}^*(\hat{\beta})$ and $U_{uv}^*(\beta)$ are asymptotically equivalent.

**Theorem** 3.1 Assume that the density $f$ of the error terms and the design points $x_{ij}$ satisfy the following conditions :

$C_1$ : f is bounded by $M_2$ and symmetric about zero.
$C_2$ : There exists an $M_3>0$ such that for each $(u, v)$ $\max_{s,t}|x_{vt}-x_{us}|\leq M_3$.
$C_3$ : Let $\bar{x}_{i\cdot}=\frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}$, $i=1, \cdots, k$. Then, for each $(u, v)$, $\bar{x}_{v\cdot}-\bar{x}_{u\cdot}\to 0$ as $N\to\infty$

Then, under $H_0$,

$$\sqrt{N}\,[U_{uv}^*(\hat{\beta})-U_{uv}(\beta)]\xrightarrow{P}0,$$

where $\hat{\beta}$ is a $\sqrt{N}$-consistent estimator of $\beta$.

**Proof.** The proof is similar to that of Theorem 3.5 of Jee(1989).

From Theorem 3.1 we have the following theorem which indicates the asymptotic equivalence of $J(\hat{\beta})$ and $J(\beta)$.

**Theorem** 3.2. Assume that the conditions $C_1-C_3$ are satisfied for every $(u, v)$. Assume also that the sample sizes satisfy the condition(3.4). Then, under $H_0$,

$$N^{-3/2}[J(\hat{\beta})-J(\beta)]\xrightarrow{P}0,$$

where $\hat{\beta}$ is a $\sqrt{N}$-consistent estimator of $\beta$.

Since the null distribution of $J(\beta)$ is the same as that of the Jonckheere statistic, we have the following corollary.

**Corollary** 3.3. Under the conditions in Theorem 3.2, the limiting distribution of

$$[J(\hat{\beta})-E_0(J(\beta))]/[\mathrm{Var}_0(J(\beta))]^{1/2}$$

is standard normal when $H_0$ is true, where

$$E_0(J(\beta))=\frac{1}{4}[N^2-\sum_{i=1}^{k}n_i^2]$$

and

$$\mathrm{Var}_0(J(\beta)) = \frac{1}{72}[N^2(2N+3) - \sum_{i=1}^{k} n_i^2[2n_i+3)].$$

## 4. Small Sample Monte Carlo Study

### 4.1. Design of the Experiment

A series of Monte Carlo simulations was conducted to compare the empirical powers and the empirical significance levels of our proposed test statistic $J$ with the Adichie's test statistics $\bar{E}_k^2$ and $\bar{\chi}_k^2$, and with the Rao-Gore test statistic $G$, which are defined by (2.1), (2.2), and (2.3), respectively. In the statistic $J(\hat{\beta})$, the common slope $\beta$ is estimated by the median of $\{S_{ist} : s<t,\ i=1, \cdots, k\}$ in (3.1).

In our Monte Carlo study the powers and significance levels are compared for various underlying distributions such as the uniform, normal, double exponential, Cauchy and contaminated normal distributions. The cdf of an $\varepsilon$-contaminated normal distribution is given by

$$F(x) = (1-\varepsilon)\Phi(x) + \varepsilon\Phi(x/\sigma), \tag{4.1}$$

where $\Phi(x)$ is the cdf of the standard normal. To design the simulation experiment, we mostoy applied the methodology in Lo, Simkin and Worthley(1978).

The number of regression lines and observations per line are $k=5$ and $n_i=10$, respectively. In Jee(1989), other combinations of $k$ and $n_i$ are considered. But, the results are very similar to the case presented in this paper. The design points $x_{ij}$'s are fixed with (1, 2, ⋯, 10), and $Y_{ij}$'s are obtained from the model(1.1) with $\alpha_i=0$. For the choice of $\beta_i$'s under the ordered alternatives(1.3), we considered the equally-spaced slopes given by

$$\beta_i = \beta_0 + (i-1)m\Delta,\quad i=1, \cdots, k,$$

where $\Delta$ is the standard deviation of the least squares estimator of $\beta$ for the combined sample. The initial value $\beta_0$ was set to be 1. The values of $m$ indicate the significance of ordering of the slope parameters. We have chosen the values of $m$ as $m=0$, 1, or 2.

The error terms $\varepsilon_{ij}$ were generated from teh uniform, normal, double exponential, Cauchy, and contaminated normal distributions. For the contaminated normal distributions, two cases were considered, i.e., ($\varepsilon=0.05$, $\sigma=3$) and ($\varepsilon=0.10$, $\sigma=5$) in (4.1).

The random variates were generated using the IMSL subroutines on VAX 780 at Seoul National University. The subroutine GGUBT was used to generate uniform random variates. The subroutine GGNPM was used to generate normal random variates with and without contaminations. The inverse integral transformation was applied to generate double exponential and Cauchy random variates. For each distribution, the random variates were scaled to have a unit variance. For the Cauchy distribution, where the second moment does not exist, the random variates were devided by the value of $F^{-1}(0.84) - F^{-1}(0.5) = 1.8326$. Note that for the standard normal distribution the value of $\Phi^{-1}(0.84) - \Phi^{-1}(0.5)$ corresponds to the standard deviation.

The total number of replications in each case was 1000. As level of significance, we used $\alpha=0.05$. The simulated proportions of rejecting $H_0$ in favor of the ordered alternatives $H_1$ among 1000 replications are presented in Table 1.

Table 1. Empirical Powers

($k=5$, $n_i=10$, $\alpha=0.05$, 1,000 replications)

| distribution | $m$ | $\bar{E}_k^2$ | $\bar{\chi}_R^2$ | $G$ | $J$ |
|---|---|---|---|---|---|
| uniform | 0 | 0.047 | 0.034 | 0.050 | 0.040 |
| | 1 | 0.373 | 0.296 | 0.319 | 0.786 |
| | 2 | 0.808 | 0.674 | 0.707 | 1.000 |
| normal | 0 | 0.038 | 0.032 | 0.048 | 0.050 |
| | 1 | 0.343 | 0.272 | 0.307 | 0.831 |
| | 2 | 0.836 | 0.766 | 0.744 | 1.000 |
| double exponential | 0 | 0.053 | 0.038 | 0.048 | 0.063 |
| | 1 | 0.366 | 0.371 | 0.354 | 0.931 |
| | 2 | 0.850 | 0.856 | 0.803 | 1.000 |
| contaminated normal ($\varepsilon=0.05$, $\sigma=3$) | 0 | 0.062 | 0.046 | 0.056 | 0.059 |
| | 1 | 0.382 | 0.333 | 0.359 | 0.893 |
| | 2 | 0.859 | 0.837 | 0.787 | 1.000 |
| contaminated normal ($\varepsilon=0.10$, $\sigma=5$) | 0 | 0.052 | 0.031 | 0.045 | 0.052 |
| | 1 | 0.418 | 0.548 | 0.511 | 0.985 |
| | 2 | 0.820 | 0.936 | 0.922 | 1.000 |
| Cauchy | 0 | 0.060 | 0.042 | 0.061 | 0.049 |
| | 1 | 0.107 | 0.253 | 0.197 | 0.753 |
| | 2 | 0.201 | 0.632 | 0.468 | 0.985 |

### 4.2. Simulation Results

To compute the empirical powers of the tests, we count the number of times that the null hypothesis $H_0$ is rejected for each test. Then the empirical power is the number of times of rejecting $H_0$ divided by 1000. When $m=0$, the empirical power is the empirical significance level. The empirical powers and significance levels of the tests considered in this paper are presented in Table 1.

Jee (1989) performed a more extensive simulation study for other values of $k$, $n$ and $\alpha$. She also included the Adichie's tests based on scores in the simulation. The results show that the general trend of the simulation is very similar to those presented in Table 1.

The proposed rank test $J$ has higher empirical powers than any other tests considered in this paper. The simulation results also show that nonparametric tests $\bar{\chi}_R^2$ and $G$ have lower empirical powers than the parametric test $\bar{E}_k^2$ for the light-tailed and normal distributions. For the moderately heavy-tailed distributions such as double exponential and contaminated normal with $\varepsilon=0.05$, $\sigma=3$, the parametric test $\bar{E}_k^2$ is better than the nonparametric tests $\bar{\chi}_R^2$ and $G$.

The Rao-Gore test $G$, which is also a Jonckheere type statistic, is much worse than the proposed test $J$ for all distributions considered. It is because the statistic $G$ loses too much information in pairing the points to obtain the distribution-free property. Hence, the powers of the Rao-Gore test $G$ are significantly lower than the other tests especially for small samples.

The proposed test $J$ was developed under the assumptions A1 and A2, which are rather strong assumptions. Thus, it may not be fair to compare the proposed test with other tests which were developed under more general conditions. But, in some experiments such as the investigaton of the effects of exposure for rats of increasing age groups, we may assume the conditions A1 and A2. In these cases the proposed test in significantly more powerful than the other tests.

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