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ON A CERTAIN CLASS OF DIRICHLET SERIES

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1. Introduction

The Dirichlet series with which this paper is concerned are

$$L(s,a)=\sum_{n=1}^{\infty}\cos(2\pi na)n^{-s}$$

and

$$L^{*}(s,a)=\sum_{n=1}^{\infty}\sin(2\pi na)n^{-s}$$

with the condition $0<a<1$.
In particular for $a=1/2,1/3,1/4,1/6$ the series $L(n,a)$ and $L^{*}(n,a)$ can be expressed in L-series.

Dirichlet L-series are defined by

$$L_d=L_d(s)=\sum_{1}^{\infty}X_d(k)k^{-s},$$

where $X_d(k)$ is a character modulo d which will be referred to as the modulus and s as the degree or order if s is a positive integer n. Their properties have been summarised eleswhere [3]. If $X_d(k)$ is real, the L-series is divided into two types denoted by L_d if $X_d(d-1)=+1$ and L_{-d} if $X_d(d-1)=-1$. Three series will be referred here namely.

$$\begin{aligned}L_1 &= 1^{-s}+2^{-s}+3^{-s}+\cdots=\zeta(s)\\ L_{-3} &= 1^{-s}-2^{-s}+4^{-s}-5^{-s}+\cdots\\ L_{-4} &= 1^{-s}-3^{-s}+5^{-s}-7^{-s}+\cdots.\end{aligned}$$

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Thus for $a = 1/2, 1/3, 1/4, 1/6, L(n,a)$ and $L^*(n,a)$ have the following expressions (see also [4])

$$\begin{aligned}
L(n,1/2) &= -(1-2^{1-n})L_1(n) \\
L(n,1/3) &= -2^{-1}(1-3^{1-n})L_1(n) \\
L(n,1/4) &= -2^{n}(1-2^{1-n})L_1(n) \\
L(n,1/6) &= 2^{-1}(1-2^{1-n})(1-3^{1-n})L_1(n) \\
L^*(n,1/2) &= 0 \\
L^*(n,1/3) &= 2^{-1}\sqrt{3}L_{-3}(n) \\
L^*(n,1/4) &= L_{-4}(n) \\
L^*(n,1/6) &= 2^{-1}\sqrt{3}(1+2^{1-n})L_{-3}(n).
\end{aligned}$$

It is well known that $L_1(2n) = 0$ and $L_1(2n) = R\pi^{2n}$, where n is a positive integer and R is a rational number. Similarly, it may be shown [1] that $L_{-4}(1-2n) = 0$ and $L_{-4}(2n-1) = R'\pi^{2n-1}$ where R' is a rational number. These are special cases of the following theorem due to [3].

Theorem. *For a positive integer m*
(a) $L_{-d}(1-2m) = 0, \quad L_{-d}(2m-1) = R'd^{-\frac{1}{2}}\pi^{2m-1},$
$L_{-d}(-2m) = (-1)^m R'(2m)!/(2d)^{2m}$;
(b) $L_d(-2m) = 0, \quad L_d(2m) = Rd^{-\frac{1}{2}}\pi^{2m},$
$L_d(1-2m) = (-1)^m R(2m-1)!/(2d)^{2m-1}$;
where R and R' are rational numbers depending on m and d.

For $a \neq 1/2, 1/3, 1/4, 1/6$, $L(n,a)$ and $L^*(n,a)$ cannot, in general, be expressed in L–series. The aim of this paper is to extend the above results to the cases in which $a \neq 1/2, 1/3, 1/4, 1/6 (0 < a < 1)$.

We also give a recursion formulas for determining $F(-n,a)$, $n \in N$, where $F(s,a)$ is the Lerch's zeta function defined for $Re(s) > 0$ by the series

$$F(s,a) = \sum_{n=1}^{\infty} e^{2\pi n a i} n^{-s}.$$

This enables us to extend a result of Kai Wang [2] to obtain an exponential sums of certain recursion formulas by means of Bernoulli numbers and Bernoulli polynomials.

2. Analytic behaviour of $L(s,a)$

Lemma 1.

$$L(s,a)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{\cos(2\pi a)-e^{-t}}{e^t+e^{-t}-2\cos(2\pi a)}t^{s-1}dt, \quad \mathrm{Re}(s)>1.$$

Proof. Make the substitution $u=nt$ in the Euler's integral for Γ

$$\Gamma(s)=\int_0^\infty e^{-u}u^{s-1}du, \quad Re(s)>0,$$

and we obtain for $Re(s)>1$

$$\sum_{n=1}^\infty \cos(2\pi na)n^{-s}=\frac{1}{\Gamma(s)}\sum_{n=1}^\infty\int_0^\infty \cos(2\pi na)e^{-nt}t^{s-1}dt.$$

Thus

$$L(s,a)=\frac{1}{\Gamma(s)}\int_0^\infty\sum_{n=1}^\infty \cos(2\pi na)e^{-nt}t^{s-1}dt.$$

The validity of the interchange of summation and integration is not difficult to establish. Let $H(t,a)$ be defined by

$$\begin{aligned} H(t,a) &= \sum_{n=1}^\infty \cos(2\pi na)e^{-nt}, Re(t)>0. \\ &= \frac{1}{2}\sum_{n=1}^\infty\{e^{-n(t-2\pi ai)}+e^{-n(t+2\pi nai)}\} \end{aligned}$$

Make use of $\sum_{n=1}^\infty r^{-n}=1/(r-1)$, to obtain

$$H(t,a)=\frac{1}{2}(\frac{1}{e^{t-2\pi ai}-1}+\frac{1}{e^{t+2\pi ai}-1}).$$

We have

$$H(t,a)=\frac{\cos 2\pi a-e^{-t}}{e^t+e^{-t}-2\cos 2\pi a}.$$

This completes the proof of the lemma.

The function $H(t,a), 0<a<1$, is analytic near $t=0$; therefore it can be expanded as a power series in t. We have

Lemma 2. *The Taylor series expansion of $H(t,a)$ is given by*

$$H(t,a)=-\frac{1}{2}+\sum_{n=0}^{\infty}\frac{a_n}{(2n+1)!}t^{2n+1}, |t|<2\pi\delta(\delta=\min\{a,1-a\}),$$

where the coefficient a_n satisfies the recurrence relation

$$a_0=\frac{1}{2(1-\cos 2\pi a)},$$

$$2(1-\cos 2\pi a)a_n=1-2\sum_{k=1}^{n}\binom{2n+1}{2k}a_{n-k}, n\geq 1.$$

Proof. It is to be noticed that $H(t,a)+\frac{1}{2}$ is an odd function, since

$$H(t,a)+\frac{1}{2}=\frac{e^t-e^{-t}}{2(e^t+e^{-t}-2\cos 2\pi a)}=-(H(-t,a)+\frac{1}{2}).$$

It follows that $H(t,a)+\frac{1}{2}=\sum_{n=0}^{\infty}\frac{a_n}{(2n+1)!}t^{2n+1}$ which is valid near zero (in fact valid in the disk $|t|<2\pi\delta$, where $\delta=\min\{a,1-a\}$, which extend to the nearest singularities $\pm 2\pi ai$ and $\pm 2\pi i(1-a)$ of $H(t,a)$). The relation $2(H(t,a)+\frac{1}{2})(e^t+e^{-t}-2\cos 2\pi a)=e^t-e^{-t}$ gives

$$\{2\sum_{n=0}^{\infty}\frac{a_n}{(2n+1)!}t^{2n+1}\}\{\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}-\cos 2\pi a\}=\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}.$$

Thus we obtain the formula

$$\sum_{n=0}^{\infty}\{2\sum_{k=0}^{n}\frac{a_{n-k}}{(2k)!(2n-2k+1)!}\}t^{2n+1}-2\cos 2\pi a\sum_{n=0}^{\infty}\frac{a_n}{(2n+1)}t^{2n+1}$$
$$=\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}.$$

which can be written in the form

$$\sum_{n=0}^{\infty}\{2\sum_{k=0}^{n}\binom{2n+1}{2k}\frac{a_{n-k}}{(2n+1)!}\}t^{2n+1}-2\cos 2\pi a\sum_{n=0}^{\infty}\frac{a_n}{(2n+1)}t^{2n+1}$$
$$=\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}.$$

Thus for the coefficient a_n we have the recurrence formula

(1)
$$a_0 = \frac{1}{2(1-\cos 2\pi a)}, \quad 2(1-\cos 2\pi a)a_n = 1 - 2\sum_{k=1}^{n} \binom{2n+1}{2k} a_{n-k}, n \geq 1.$$

The first few terms are easily determined to be

$$a_1 = -\frac{2+\cos 2\pi a}{2(1-\cos 2\pi a)^2} \qquad a_2 = \frac{\cos^2 2\pi a + 13\cos 2\pi a + 16}{2(1-\cos 2\pi a)^3}$$

$$a_3 = -\frac{\cos^3 2\pi a + 60\cos^2 2\pi a + 297\cos 2\pi a + 272}{2(1-\cos 2\pi a)^4}.$$

Theorem 1. *The function $L(s,a)$ defined in lemma 1 by*

$$L(s,a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{\cos 2\pi a - e^{-t}}{e^t + e^{-t} - 2\cos 2\pi a} t^{s-1}dt, Re(s) > 1$$

has a holomorphic extension to the whole complex s–plane.

Proof. Recall from lemma 1 that $H(t,a) = \frac{\cos 2\pi a - e^{-t}}{e^t+e^{-t}-2\cos 2\pi a}$. Let us define the functions $P(s,a)$ and $Q(s,a)$ for $Re(s) > 1$ by

$$P(s,a) = \int_0^\delta H(t,a)t^{s-1}dt,$$

$$Q(s,a) = \int_\delta^\infty H(t,a)t^{s-1}dt \quad (\delta = \min\{a, 1-a\}).$$

The integral $\int_\delta^\infty H(t,a)t^{s-1}dt$ exists and converges uniformly in any finite region of the complex s–plane, since the function $e^{-t}t^{Re(s)+1}(e^t\cos(2\pi a) - 1)/(e^t + e^{-t} - 2\cos(2\pi a))$ is bounded for all values of $Re(s)$, and we can compare the integral with that of $1/t^2$. Thus $Q(s,a)$ is an entire function. Recall from lemma 2 that

$$H(t,a) = -\frac{1}{2} + \sum_{n=0}^{\infty} \frac{a_n}{(2n+1)!} 2n+1, \quad t \in [0,\delta]$$

the convergence being uniformly on $[0,\delta]$. We deduce for $Re(s) > 1$ that

$$P(s,a) = \int_0^\delta \{-\frac{1}{2} + \sum_{n=0}^{\infty} \frac{a_n}{(2n+1)!} t^{2n+1}\} t^{s-1}dt.$$

That is

$$\begin{aligned} P(s,a) &= -\frac{1}{2}\int_0^\delta t^{s-1}dt + \sum_{n=0}^{\infty}\{\frac{a_n}{(2n+1)!}\int_0^\delta t^{2n+s}dt\} \\ &= -\frac{\delta^s}{2s} + \sum_{n=0}^{\infty}\frac{\delta^{2n+s+1}}{(2n+1)!(2n+s+1)}a_n. \end{aligned}$$

Thus $P(s,a)$ is a meromorphic function on $\not{c}$ with simple poles at the pointes $0,-1,-3,-5,\cdots$. Since $1/\Gamma$ is an entire function, we may extend $L(s,a)$ to the whole of $\not{c}$ by

$$L(s,a) = \frac{P(s,a)}{\Gamma(s)} + \frac{Q(s,a)}{\Gamma(s)}. \tag{2}$$

Since $Q(s,a)$ and $1/\Gamma$ are entire functions, the singularities of $L(s,a)$ can only be those of $P(s,a)/\Gamma$. We have seen that $P(s,a)$ has simple poles at $0,-1,-3,-5,\cdots$. Since $1/\Gamma$ has simple zeros at $0,-1,-3,-5,\cdots$ it follows that $L(s,a)$ is regular for all values of s in the complex plane.

Theorem 2.
(i) $L(-2n,a) = 0, n \in N$.
(ii) $L(0,a) = -\frac{1}{2}$ *and*
(iii) $L(1-2n,a) = -a_{n-1}, n \in N$.

Proof. The proof depends on the partial fraction (2) of $L(s,a)$ and the fact that $1/\Gamma$ has simple zeros at $-n$ $(n = 0,1,2,\cdots)$. We have

$$L(-2n,a) = \frac{P(-2n,a)}{\Gamma(-2n)} + \frac{Q(-2n,a)}{\Gamma(-2n)} = 0, n \in N.$$

Thus the proof of (i) follows. To prove (ii) and (iii) we take the limit of (2) as $s \to 1-2n$. Therefore we have

$$\begin{aligned} L(1-2n,a) &= \lim_{s\to 1-2n}\{\frac{P(s,a)}{\Gamma(s)} + \frac{Q(s,a)}{\Gamma(s)}\} \\ &= \lim_{s\to 1-2n}\frac{P(s,a)}{\Gamma(s)} \end{aligned}$$

Thus

$$L(1-2n,a) = \lim_{s\to 1-2n}\frac{1}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{\delta^{2m+s-1}}{(2m+s-1)(2m-1)!}a_{m-1}.$$

Since Γ has simple poles at $-n(n=0,1,2\cdots)$ with residue $(-1)^n/(n)!$, we conclude that

$$\lim_{s\to 1-2n}\Gamma(s)(2n+s-1)=Res(\Gamma,1-2n)=-\frac{1}{(2n-1)!}.$$

Thus we obtain

$$\begin{aligned} L(1-2n,a) &= \lim_{s\to 1-2n}\frac{\delta^{2n+s-1}}{\Gamma(s)(2n+s-1)(2n-1)!}a_{n-1} \\ &= -a_{n-1}, n\in N, \end{aligned}$$

where a_n can be determined by (1). For $s=0$, we have

$$L(0,a)=\lim_{s\to 0}-\frac{\delta^s}{2s\Gamma(s)}=\lim_{s\to 0}-\frac{\delta^s}{2\Gamma(1+s)}=-\frac{1}{2}$$

This completes the proof of Theorem 2.

The first few values of $L(1-2n,a), n\in N$, are given by

$$\begin{aligned} L(-1,a) &= -\frac{1}{2(1-\cos 2\pi a)} \\ L(-3,a) &= \frac{2+\cos 2\pi a}{2(1-\cos 2\pi a)^2} \\ L(-5,a) &= -\frac{\cos^2 2\pi a+13\cos 2\pi a+16}{2(1-\cos 2\pi a)^3} \\ L(-7,a) &= \frac{\cos^3 2\pi a+60\cos^2 2\pi a+297\cos 2\pi a+272}{2(1-\cos 2\pi a)^4} \end{aligned}$$

3. Analytic continuation of $L^*(s,a)$

Similar results and statements of those of §2 can be obtained for $L^*(s,a)$. Here we state these main results without proof, since methods of proof used in §2 can be applied in analogous way here.

Lemma 3. $L^*(a,s)=\frac{1}{\Gamma(s)}\int_0^\infty H^*(t,a)t^{s-1}dt, Re(s)>1,$ *where*

$$H^*(t,a)=\frac{\sin 2\pi a}{e^t+e^{-t}-2\cos 2\pi a}.$$

Lemma 4. *The function* $H^*(t,a)$ *has the Taylor series expansion*

$$H^*(t,a) = \sum_{n=0}^{\infty} b_n t^{2n}, |t| < 2\pi\delta(\delta = \min\{a, 1, -a\}),$$

where the coefficient b_n *satisfies the recurrence relation*

$$(3) \quad b_0 = \frac{\sin 2\pi a}{2(1-\cos 2\pi a)}, (1-\cos 2\pi a)b_n + \sum_{k=1}^{n} \frac{1}{(2k)!} b_{n-k} = 0, n \geq 1.$$

The first few terms are found to be

$$\begin{aligned} b_1 &= -\frac{\sin 2\pi a}{4(1-\cos 2\pi a)^2} \\ b_2 &= \frac{\cos 2\pi a + 5}{48(1-\cos 2\pi a)^3} \sin 2\pi a \\ b_3 &= -\frac{\cos^2 2\pi a + 28\cos 2\pi a + 61}{1440(1-\cos 2\pi a)^4} \sin 2\pi a. \end{aligned}$$

Theorem 3. *The function* $L^*(s,a)$ *can be extend to a holomorphic function in the whole complex s–plane. Moreover the partial fraction expansion of* $L^*(s,a)$ *is given by*

$$(4) \qquad L^*(s,a) = \frac{P^*(s,a)}{\Gamma(s)} + \frac{Q^*(s,a)}{\Gamma(s)}$$

where $P^*(s,a)$ *and* $Q^*(s,a)$ *are defined for* $Re(s) > 1$ *by*

$$P^*(s,a) = \int_0^{\delta} H^*(t,a)t^{s-1}dt, \quad Q^*(s,a) = \int_{\delta}^{\infty} H^*(t,a)t^{s-1}dt,$$

where the function $H^*(t,a)$ *appears in the integrand is given in lemma 4.*

Proof. It can be shown, as in Theorem 1, that the function $Q^*(s,a)$ is an entire function and that $P^*(s,a)$ is a meromorphic function with simple poles at $0, -2, -4, -6, \cdots$. Since $1/\Gamma$ has simple zeros at $0, -2, -4, -6, \cdots$, it follows from (4) that $L^*(s,a)$ is regular for all $s \in \not{C}$.

Theorem 4. (i) $L^*(1-2n, a) = 0$, (ii) $L^*(-2n, a) = (2n)!b_n$ $(n = 0, 1, 2, \cdots)$, *where* b_n *is given by (3).*

Here are some values of $L^*(-2n,a)\,(n=0,1,2,\cdots)$

$$\begin{aligned}
L^*(0,a) &= \frac{\sin 2\pi a}{2(1-\cos 2\pi a)}\\
L^*(-2,a) &= -\frac{\sin 2\pi a}{2(1-\cos 2\pi a)^2}\\
L^*(-4,a) &= \frac{\cos 2\pi a+5}{2(1-\cos 2\pi a)^3}\sin 2\pi a\\
L^*(-6,a) &= -\frac{\cos^2 2\pi a+28\cos 2\pi a+61}{2(1-\cos 2\pi a)^4}\sin 2\pi a.
\end{aligned}$$

4. Exponential sums of certain recursion formulas

In this section let $B_n(x)$ be the nth Bernoulli polynomials; defined by the power series

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}$$

and $B_n=B_n(0)$ is the nth Bernoulli number.

For the special case $a=\frac{\gamma}{m}(r<m)$ the coefficients a_n and b_n in the recursion formulas (1) and (3) will be denoted by $a_n(\frac{\gamma}{m})$ and $b_n(\frac{\gamma}{m})$ respectively.

In this section we will prove the following result:

Theorem 5. *Let $m>2$ be a fixed positive integer and let α be an integer such that $\alpha\not\equiv 0$ (mod m). Then for $n\geq 1$,*

$$\sum_{\gamma=1}^{m-1}e^{-2\pi i\alpha\gamma/m}a_n(\frac{\gamma}{m})=-\frac{1}{2n}(B_{2n}-m^{2n}B_{2n}(\frac{\alpha}{m}-[\frac{\alpha}{m}]))$$

and

$$\sum_{\gamma=1}^{m-1}e^{-2\pi i\alpha\gamma/m}b_n(\frac{\gamma}{m})=\frac{-i}{(2n+1)!}(B_{2n+1}-m^{2n+1}B_{2n+1}(\frac{\alpha}{m}-[\frac{\alpha}{m}])),$$

where $a_n(\frac{\gamma}{m})$ and $b_n(\frac{\gamma}{m})$ are given recursively by (1) and (3).

For the proof we shall need the following theorem :

Theorem 6. *For n a positive integer*

(a) $F(1-2n,a) = -a_{n-1}$
(b) $F(-2n,a) = (2n)!b_n i$,

where $F(s,a)$ is the Lerch zeta-function as defined in the Introduction.

Proof. Recall that $F(s,a)$ satisfies the following functional equation

$$F(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}(e^{\pi i(1-s)/2}\zeta(1-s,a) + e^{\pi i(s-1)/2}\zeta(1-s,1-a)),$$

where $\zeta(s,a)$ is the Hurwitz zeta–function, defined for $Re(s) > 1$ by the series

$$\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}, \quad 0 < a < 1.$$

The functional equation of $F(s,a)$ can be written in the form

$$(5) \quad F(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{\sin(\frac{1}{2}\pi s)(\zeta(1-s,a)+\zeta(1-s,1-a)) + i\cos(\frac{1}{2}\pi s)(\zeta(1-s,a)-\zeta(1-s,1-a))\}.$$

By observing that

$$F(s,a) = L(s,a) + iL^*(s,a),$$

we find that

$$(6) \qquad L(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\sin(\frac{1}{2}\pi s)\{\zeta(1-s,a)+\zeta(1-s,1-a)\}$$

and

$$(7) \qquad L^*(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\cos(\frac{1}{2}\pi s)\{\zeta(1-s,a)-\zeta(1-s,1-a)\}.$$

The functional equation (6), with $s = 1-2n$, and value obtained in Theorem 2 for $L(1-2n,a)$ give the following identity :

$$(8) \qquad \zeta(2n,a)+\zeta(2n,1-a) = (-1)^{n+1}\frac{(2\pi)^{2n}}{(2n+1)!}a_{n-1}, n \in N$$

Similarly, the functional relation (7), with $s = -2n$, and the value obtained in Theorem 4 for $L^*(-2n,a)$ give the identity

$$(9) \ \zeta(2n+1,a) - \zeta(2n+1,1-a) = (-1)^n(2\pi)^{2n+1}b_n, n = 0,1,2,\cdots.$$

Now the first part of the Theorem follows from the functional relation (5), with $s = 1 - 2n$, and the identity in (8). The second part of the Theorem follows from the functional equation (5), with $s = -2n$, and the identity in (9).

Proof of Theorem 5. The proof of Theorem (5) follows now from Theorem (6) and the following result of K. Wang [2].

Theorem. *Let $m > 2$ be a fixed positive integer and let α be an integer such that $\alpha \not\equiv 0 \pmod m$. Then for $n \geq 1$,*

$$\sum_{\gamma=1}^{m-1} e^{-2\pi i\alpha\gamma/m} F(1-n, \frac{\gamma}{m}) = \frac{1}{n}(B_n - m^n B_n(\frac{\alpha}{m} - [\frac{\alpha}{m}])).$$

5. Evaluation of $L(2n, a)$ and $L^*(2n+1, a)$

Values of $L(2n, a)$ and $L^*(2n+1, a)$, $n \in N$, depends on the determination of $\zeta(-n, a)$, which we try to find in this section.

Recall that

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-at}}{1 - e^{-t}} t^{s-1} dt, Re(s) > 1.$$

The function $te^{-at}/(1-e^{-t})$ is analytic near zero. Let $\sum_{n=0}^{\infty}((-1)^n/n!)c_n t^n$ be its expansion about zero, which is vaild for $|t| < 2\pi$. Let $G(t) = e^{-at}/(1 - e^{-t})$. The relation

$$(G(t))(1 - e^{-t})/t = e^{-at}$$

gives

$$(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_n t^n)(\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} t^n) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} t^n.$$

i.e

$$\sum_{n=0}^{\infty} (\sum_{k=0}^{n} \frac{(-1)^n}{(k+1)!(n-k)!} c_{n-k}) t^n = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} t^n.$$

Therefore we obtain the recurrence formula

$$c_0 = 1, \quad c_n = a^n - \sum_{k=1}^{n} \frac{1}{k+1} \binom{n}{k} c_{n-k}, n \geq 1. \tag{10}$$

The first few c_n are found to be

$$\begin{aligned} c_1 &= -\frac{1}{2} + a \\ c_2 &= \frac{1}{6} + a(a-1) \\ c_3 &= \frac{1}{2}a(2a^2 - 3a + 1) \\ c_4 &= -\frac{1}{30} + a^2(a^2 - 2a + 1). \end{aligned}$$

It is well known that, the function $\zeta(s,a)$ can be extend as a meromorphic function to the entire s-plane by the contour integral

$$\zeta(s,a) = \frac{\Gamma(1-s)e^{-i\pi s}}{2\pi i} \int_c \frac{t^{s-1}e^{-at}}{1-e^{-t}} dt \tag{11}$$

where C is a loop around the positive real axis.

If s is an integer, then the integrand in (11) is single valued and we obtain

$$\begin{aligned} \zeta(-n,a) &= \frac{e^{i\pi n} n!}{2\pi i} \int_C \frac{t^{-n-1}e^{-at}}{1-e^{-t}} dt \\ &= \frac{(-1)^n n!}{2\pi i} \int_{|t|=\delta} \left(\frac{te^{-at}}{1-e^{-t}}\right) t^{-n-1} \frac{dt}{t} \\ &= (-1)^n n! \frac{1}{2\pi i} \int_{|t|=\delta} \left(\sum_{m=0}^{\infty} \frac{(-1)}{m!} c_m t^m\right) t^{-n-1} \frac{dt}{t} \\ &= (-1)^n n! \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} c_m \left(\frac{1}{2\pi} \int_0^{2\pi} t^{m-n-1} dt\right). \end{aligned}$$

Hence for negative integers, the value of $\zeta(s,a)$ is given by

$$\zeta(-n,a) = -\frac{1}{n+1} c_{n+1} \quad (n = 0, 1, 2, \cdots). \tag{12}$$

We find from (10) and (12) that

$$\begin{aligned} \zeta(0,a) &= \frac{1}{2} - a \\ \zeta(-1,a) &= -\frac{1}{12} - \frac{1}{2}a(a-1) \\ \zeta(-2,a) &= -\frac{a}{6}(2a^2 - 3a + 1) \\ \zeta(-3,a) &= \frac{1}{120} - (a^2/4)(a^2 - 2a + 1). \end{aligned}$$

Lemma 5. *For n a positive integer,*

$$\zeta(-n, 1-a) = (-1)^{n+1}\zeta(-n, a)$$

Proof. Let $s = 1 + 2n$ in the functional relation (6). Since the left–hand side of (6) is regular and since $\Gamma(1-s)$ is regular in the s–plane except for simple poles when s is a positive integer, we must have

$$\zeta(-2n, 1-a) = -\zeta(-2n, a). \tag{13}$$

Similarly, let $s = 2n$ in (7). The same argument shows that

$$\zeta(1-2n, 1-a) = \zeta(1-2n, a). \tag{14}$$

Theorem 7.

(a) $L(2n, a) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}c_{2n}$,

(b) $L^*(2n-1, a) = (-1)^{n}\frac{(2\pi)^{2n-1}}{2(2n-1)!}c_{2n-1}$.

Proof. As $\Gamma(s)\Gamma(1-s) = csc(\pi s)$ for non–integral s then the function relations (6) and (7) can be written in the form

$$L(s, a) = \frac{(2\pi)^s}{4\Gamma(s)}\sec(\frac{1}{2}\pi s)\{\zeta(1-s, a) + \zeta(1-s, 1-a)\} \tag{15}$$

and

$$L^*(s, a) = \frac{(2\pi)^s}{4\Gamma(s)}\csc(\frac{1}{2}\pi s)\{\zeta(1-s, a) - \zeta(1-s, 1-a)\} \tag{16}$$

respectively.
Let $s = 2n$ in (15) and using the relation in (13), we obtain

$$L(2n, a) = (-1)^n\frac{(2\pi)^{2n}}{2(2n-1)!}\zeta(1-2n, a).$$

The first part of the theorem follows now from the relation in (12). Similarly, the relation in (16), with $s = 2n - 1$, and the relation in (14) give

$$L^*(2n-1, a) = (-1)^{n+1}\frac{(2\pi)^{2n-1}}{2(2n-2)!}\zeta(-2n+2, a).$$

The second part of the theorem now follows from (12).

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