Kyungpook Mathematical Journal
Volume 30, Number 2, December, 1990

A CLASS OF GENERALIZED TRANSFORM AND THEIR APPLICATION IN THE BOUNDARY VALUE PROBLEM OF HEAT CONDUCTION

A. Siddique

1. Definition and Inversion Formula

We consider the self adjoint Bessel differential equation as:

$$P(x)\frac{d}{dx}[Q(x)\frac{dy}{dx}] + [R(x) + S(x)]y = 0, \tag{1.1}$$

with the conditions

$$\begin{aligned} y(a) + h_1 y^1(a) &= 0 \\ y(b) + h_2 y^1(b) &= 0 \end{aligned} \tag{1.2}$$

where $P(x) = (x)^{1+2\alpha}$, $Q(x) = (x)^{1-2\alpha}$, $R(x) = \lambda^2\beta^2 x^{2\beta}$ $S(x) = (\alpha^2 - \nu^2\beta^2)$, and a, b are the inner and outer radii of the cylinder and h_1, h_2 are the independent radiation constant.

The general solution of (1.1) can be written as

$$y(x) = x^\alpha[C_1 J_\nu(\lambda x^\beta) + C_2 Y_\nu(\lambda x^\beta)] \tag{1.3}$$

where C_1 and C_2 are arbitrary constants and $J_\nu(\lambda x^\beta)$ and $Y_\nu(\lambda x^\beta)$ are the Bessel's functions of first and second kind respectively.

We want to obtain solutions of (1.1) which satisfies the conditions (1.2). Hence we have

$$\begin{aligned} &C_1[J_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha J_\nu(\lambda a^\beta) + \lambda\beta a^\beta J_{\nu'}(\lambda a^\beta)\}] + \\ &C_2[Y_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha Y_\nu(\lambda a^\beta) + \lambda\beta a^\beta Y_{\nu'}(\lambda a^\beta)\}] = 0 \end{aligned} \tag{1.4}$$

Received May 24, 1989.

and

$$C_1[J_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha J_\nu(\lambda b^\beta) + \lambda\beta b^\beta J_{\nu'}(\lambda b^\beta)\}] + \qquad (1.5)$$
$$C_2[Y_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha Y_\nu(\lambda b^\beta) + \lambda\beta b^\beta Y_{\nu'}(\lambda b^\beta)\}] = 0$$

From (1.4) and (1.5), we can deduce that

$$\begin{aligned}\frac{C_1}{C_2} &= -\frac{[J_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha J_\nu(\lambda a^\beta) + \lambda\beta a^\beta J_{\nu'}(\lambda a^\beta)\}]}{[Y_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha Y_\nu(\lambda a^\beta) + \lambda\beta a^\beta Y_{\nu'}(\lambda a^\beta)\}]} \qquad (1.6)\\ &= -\frac{[J_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha J_\nu(\lambda b^\beta) + \lambda\beta b^\beta J_{\nu'}(\lambda b^\beta)\}]}{[Y_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha Y_\nu(\lambda b^\beta) + \lambda\beta b^\beta Y_{\nu'}(\lambda b^\beta)\}]}\end{aligned}$$

Thus the function given by (1.3) is the solution of the equation (1.1), with the conditions (1.2), if λ_i is a root of the transcendental equation

$$\begin{aligned}&[J_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha J_\nu(\lambda a^\beta) + \lambda\beta a^\beta J_{\nu'}(\lambda a^\beta)\}]\\ &\quad[Y_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha Y_\nu(\lambda b^\beta) + \lambda\beta b^\beta Y_{\nu'}(\lambda b^\beta)\}] \qquad (1.7)\\ &-[Y_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha Y_\nu(\lambda a^\beta) + \lambda\beta a^\beta Y_{\nu'}(\lambda a^\beta)\}]\\ &\quad[J_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha J_\nu(\lambda b^\beta) + \lambda\beta b^\beta J_{\nu'}(\lambda b^\beta)\}] = 0\end{aligned}$$

Now introducing the following notations

$$\begin{aligned}\Phi_{1,\nu}(\lambda a^\beta) &= [J_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha J_\nu(\lambda a^\beta) + \lambda\beta a^\beta J_{\nu'}(\lambda a^\beta)\}]\\ \Phi_{2,\nu}(\lambda a^\beta) &= [Y_\nu(\lambda a^\beta) + \frac{h_1}{a}\{\alpha Y_\nu(\lambda a^\beta) + \lambda\beta a^\beta Y_{\nu'}(\lambda a^\beta)\}]\\ \Omega_{1,\nu}(\lambda b^\beta) &= [Y_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha Y_\nu(\lambda b^\beta) + \lambda\beta b^\beta Y_{\nu'}(\lambda b^\beta)\}]\\ \Omega_{2,\nu}(\lambda b^\beta) &= [J_\nu(\lambda b^\beta) + \frac{h_2}{b}\{\alpha J_\nu(\lambda b^\beta) + \lambda\beta b^\beta J_{\nu'}(\lambda b^\beta)\}]\end{aligned}$$

(1.7) can be written as

$$\begin{aligned}&[J_\nu(\lambda a^\beta) + \Phi_{1,\nu}(\lambda a^\beta)][Y_\nu(\lambda b^\beta) + \Omega_{1,\nu}(\lambda b^\beta)] \qquad (1.8)\\ &\quad-[Y_\nu(\lambda a^\beta) + \Phi_{2,\nu}(\lambda a^\beta)][J_\nu(\lambda b^\beta) + \Omega_{2,\nu}(\lambda b^\beta)] = 0\end{aligned}$$

Let $\lambda_i (i = 1, 2, \cdots)$ be the positive roots of the equation (1.8). Then from (1.4) and (1.5), we have

$$y_i(x) = \frac{C_1 x^\alpha}{\Phi_{2,\nu}(\lambda_i a^\beta)}[J_\nu(\lambda_i x^\beta)\Phi_{2,\nu}(\lambda_i a^\beta) - Y_\nu(\lambda_i x^\beta)\Phi_{1,\nu}(\lambda_i a^\beta)]$$

and

$$y_i(x) = \frac{C_1 x^\alpha}{\Omega_{2,\nu}(\lambda_i b^\beta)}[J_\nu(\lambda_i x^\beta)\Omega_{2,\nu}(\lambda_i b^\beta) - Y_\nu(\lambda_i x^\beta)\Omega_{1,\nu}(\lambda_i b^\beta)] \qquad (1.9)$$

Then the following functions are the solution of the equation (1.1) with the conditions (1.2) :

$$\begin{aligned} C_\nu(h_1, h_2, \lambda_i x^\beta) &= [\Phi_{2,\nu}(\lambda_i a^\beta) + \Omega_{2,\nu}(\lambda_i b^\beta)]J_\nu(\lambda_i x^\beta) \\ &\quad -[\Phi_{1,\nu}(\lambda_i a^\beta) + \Omega_{1,\nu}(\lambda_i b^\beta)]Y_\nu(\lambda_i x^\beta) \end{aligned} \qquad (1.10)$$

Now according the theory of sturm–Liouville [1], the functions of the system (1.10) are orthogonal on the internal $[a, b]$ with weight function x, that is

$$\int_a^b x C_\nu(h_1, h_2, \lambda_i x^\beta) C_\nu(h_1, h_2, \lambda_j x^\beta) dx = 0 \quad i \neq j \qquad (1.11)$$

$$\int_a^b x C_\nu^2(h_1, h_2, \lambda_i x^\beta) dx = \|C_\nu(h_1, h_2, \lambda_i x^\beta)\|^2$$

Using some well known properties of the Bessel's functions [2, pp. 634, 968, 969] we can easily derive

$$\begin{aligned} \|C_\nu(h_1, h_2, \lambda_i x^\beta)\|^2 &= \frac{1}{2}M^2(\lambda_i, x^\beta, a, b)\{b^2 P(\lambda_i x^\beta; b; \nu) \\ &\quad -a^2 P(\lambda_i x^\beta; a; \nu) - N(\lambda_i x^\beta, a, b)N(\lambda_i x, a, b) \\ &\quad \{b^2 Q(\lambda_i x^\beta, b, \nu) - a^2 Q(\lambda_i x^\beta, a, \nu)\} \\ &\quad +\frac{N^2}{2}(\lambda_i x^\beta, a, b)\{b^2 R(\lambda_i x^\beta, b, \nu) - a^2 R(\lambda_i x^\beta, a, \nu)\} \end{aligned} \qquad (1.12)$$

where

$$\begin{aligned} M(\lambda_i x^\beta, a, b) &= \Phi_{1,\nu}(\lambda_i a^\beta) + \Omega_{1,\nu}(\lambda_i b^\beta) \\ N(\lambda_i x^\beta, a, b) &= [\Phi_{2,\nu}(\lambda_i a^\beta) + \Omega_{2,\nu}(\lambda_i b^\beta)] \\ P(\lambda_i x^\beta, \mu, \nu) &= [J_\nu^2(\lambda_i \mu^\beta) - J_{\nu-1}(\lambda_i \mu^\beta)J_{v+1}(\lambda_i \mu^\beta)] \\ Q(\lambda_i x^\beta, \mu, \nu) &= [J_{\nu'}(\lambda_i \mu^\beta)Y_{\nu-1}(\lambda_i \mu^\beta) \\ &\quad -\frac{1}{\lambda_i \mu^\beta}J_{\nu-1}(\lambda_i x^\beta)Y_\nu(\lambda_i \mu^\beta) \\ &\quad -J_{\nu'-1}(\lambda_i \mu^\beta)Y_\nu(\lambda_i \mu^\beta)] \\ R(\lambda_i x^\beta, \mu, \nu) &= [Y_\nu^2(\lambda_i \mu^\beta) - Y_{\nu-1}(\lambda_i \mu^\beta)Y_{\nu+1}(\lambda_i \mu^\beta)] \end{aligned} \qquad (1.13)$$

where $a, b = \mu$.

If a function $f(x)$ and its first derivative are piecewise continuous on the interval $[a, b]$, then the relation

$$\begin{aligned} T[f(x), a, b, \nu; \lambda_i] &= \bar{f}_\nu(\lambda_i) \\ &= \int_a^b x f(x) C_\nu(h_1, h_2, \lambda_i x^\beta) dx \end{aligned} \tag{1.14}$$

defines an integral transform, where λ_i are the positive roots of the equation (1.8). To obtain the inversion formula, let

$$f(x) = \sum_{j=1}^{\infty} a_j C_\nu(h_1, h_2, \lambda_i x^\beta) \tag{1.15}$$

multiplying both sides by $x C_\nu(h_1, h_2, \lambda_k x^\beta)$, ($k$ fixed), integrating with respect to x between a and b, we get

$$\begin{aligned} a_j &= \frac{\int_a^b x f(x) C_\nu(h_1, h_2, \lambda_j x^\beta) dx}{\|C_\nu(h_1, h_2, \lambda_j x^\beta)\|^2} C_\nu(h_1, h_2, \lambda_j x^\beta) \\ &= \frac{\bar{f}_\nu(\lambda_i x^\beta)}{\|C_\nu(h_1, h_2, \lambda_j x^\beta)\|^2}, \quad j = 1, 2, 3 \cdots \end{aligned} \tag{1.16}$$

Hence

$$f(x) = \sum_{j=1}^{\infty} \frac{\bar{f}_\nu(\lambda_j x^\beta)}{\|C_\nu(h_1, h_2, \lambda_i x^\beta)\|^2} C_\nu(h_1, h_2, \lambda_j x^\beta) \tag{1.17}$$

where summation is taken over all the positive roots of the equation (1.8).

2. Some Properties of the Generalized Integral Transforms

The following properties can be easily verified from the definition of the transform

(i)

$$\begin{aligned} &T[\alpha f(x) + \beta g(x), a, b, \nu; \lambda_i] \\ &= \alpha T[f(x), a, b, \nu : \lambda_i] + \beta T[g(x), a, b, \nu; \lambda_i] \end{aligned} \tag{2.1}$$

(ii)

$$T[f(\alpha x), a, b, \nu; \lambda_i] = \frac{1}{\alpha^2} T[f(x), a, b, \nu; \lambda_i] \tag{2.2}$$

(iii) Transform of

$$g(x) = \frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{\nu}{x^2} f \tag{2.3}$$

Let

$$\begin{aligned}
I &= \int_a^b x[f''(x) + \frac{1}{x} f'(x)] C_\nu(h_1, h_2, \lambda_i x^\beta) dx \\
&= \int_a^b x f''(x) C_\nu(h_1, h_2, \lambda_i x^\beta) dx + \int_a^b f'(x) C_\nu(h_1, h_2, \lambda_i x^\beta) dx \\
&= \{x C_\nu(h_1, h_2, \lambda_i x^\beta) f'(x)\}_a^b - \lambda_i \beta \int_a^b x^\beta C'_\nu(h_1, h_2, \lambda_i x^\beta) f'(x) dx \\
&\quad - \int_a^b f'(x) C_\nu(h_1, h_2, \lambda_i x^\beta) dx + \int_a^b f'(x) C_\nu(h_1, h_2, \lambda_i x^\beta) dx \\
&= \{x[C_\nu(h_1, h_2, \lambda_i x^\beta) f'(x)] - [\lambda_i \beta x^\beta C'_\nu(h_1, h_2, \lambda_i x^\beta) f(x)]\}_a^b \\
&\quad + \int_a^b x^{-1}[\lambda_i x^{2\beta} C''_\nu(h_1, h_2, \lambda_i x^\beta) + \lambda_i C'_\nu(h_1, h_2, \lambda_i x^\beta)] f(x) dx
\end{aligned}$$

As the function $C_\nu(h_1, h_2, \lambda_i x^\beta)$ satisfies the Bessel's differential equation. We have

$$\begin{aligned}
\bar{g}_\nu(\lambda_i) &= \int_a^b x(f'' + \frac{1}{x} f' - \frac{\nu}{x^2} f) C_\nu(h_1, h_2, \lambda_i x^\beta) dx \qquad (2.4) \\
&= \{x[C_\nu(h_1, h_2, \lambda_i x^\beta) f'(x)] - [\lambda_i \beta C'_\nu(h_1, h_2, \lambda_i x^\beta) f(x)]\}_a^b \\
&\quad - \lambda_i^2 \bar{f}_\nu f(\lambda_i)
\end{aligned}$$

which on simplication gives

$$\begin{aligned}
\bar{g}_\nu(\lambda_i) &= \frac{b}{h} C_\nu(h_1, h_2, \lambda_i b^\beta)[f + h\frac{df}{dx}]_{x=b} \qquad (2.5) \\
&\quad + \lambda_i a C'_\nu(h_1, h_2, \lambda_i a^\beta) f(a) - \lambda_i^2 \bar{f}_\nu(\lambda_i)
\end{aligned}$$

Transform of x^ν. From the definition we have

$$T[x^\nu, a, b, \nu; \lambda_i] = \int_a^b x^{\nu+1} C_\nu(h_1, h_2, \lambda_i x^\beta) dx$$

using the result [2, p. 634]

$$\int x^{\rho+1} z(x) dx = X^{\rho+1} Z_{\rho+1}(x)$$

where $Z_\rho(x)$ is one of the Bessel functions, and after the simplification, we obtain

$$\begin{aligned} T[x^\nu, a, b, \nu; \lambda_i] &= \frac{b^{\nu+1}}{\lambda_i^2}[\frac{\nu}{b} + \frac{1}{h}]C_\nu(h_1, h_2, \lambda_i b^\beta) \\ &\quad + \frac{a^{\nu+1}}{\lambda_i} C_\nu(h_1, h_2, \lambda_i b^\beta) \end{aligned} \tag{2.6}$$

(v) Transform of a constant. We can easily derive

$$\begin{aligned} T[\delta, a, b, 0, \lambda_i] &= \frac{\delta}{\lambda_i^2}[\frac{b}{h} C_0(h_1, h_2, \lambda_i b^\beta)] \\ &\quad + a\lambda_i C_0(h_1, h_2, \lambda_i a^\beta). \end{aligned}$$

3. Particular Cases

If we take $\alpha = 0$ and $\beta = 1$ in (1.1) and $h_1 = 0$ in (1.2), then it would correspond to a known result [3, pp. 149–154]. Several other cases can be derived by specializing the parameters in (1.1) with respect to the choice of the boundary conditions.

4. Problem of heat conduction in the cylinder

Let us consider a cylinder of radii a, b, height h and symmetrical along z–axis, having a heat source inside which leads axially symmetrical temperature distribution. Let $(r, \emptyset, z)$ be the cylindrical coordinate system and the heat is conducted symmetrically with respect to z–axis. The temperature function θ is the function of space and time.

The heat conduction equation is given as

$$\rho c \frac{\partial \theta}{\partial t} = k \nabla^2 \theta + \xi(r, z, t, \theta) \tag{4.1}$$

where $\xi(r, z, t, \theta)$ is a source function.
The use of substitutions

$$\xi(r, z, t, \theta) = \emptyset(r, z, t) + \epsilon(t)\theta(r, z, t) \tag{4.2}$$

$$u(r, z, t) = \theta(r, z, t) \exp\{-\int_0^t \epsilon(y) dy\} \tag{4.3}$$

$$\varphi(r,z,t) = \emptyset(r,z,t)\exp\{-\int_0^t \epsilon(y)dy\} \tag{4.4}$$

the heat conduction equation (4.1) reduces to

$$\frac{\partial u}{\partial t} = k\nabla^2 u + \frac{\psi(r,z,t)}{\rho} \tag{4.5}$$

where $k = K/\rho C$, k the diffusivity, K the thermal conductivity, ρ the density and C is the specific heat, and boundary conditions are

$$\begin{aligned} u(a,z,t) + h_1\frac{\partial}{\partial r}u(a,z,t) &= \eta_a(z,t) \text{ for all } 0 < z < h, t > 0 \\ u(b,z,t) + h_2\frac{\partial}{\partial r}u(b,z,t) &= \eta_b(z,t) \text{ for all } 0 < z < h, t > 0 \end{aligned} \tag{4.6}$$

where h_1 and h_2 are independent radiation constants. The initial conditions are

$$\begin{aligned} u(r,h,t) &= 0 \text{ for all } a < r < b, \quad t > 0 \\ u(r,o,t) &= 0 \text{ for all } a < r < b, \quad t > 0 \end{aligned} \tag{4.7}$$

and

$$u(r,z,0) = u_0(r,z), \text{ for all } a < r < b, 0 < z < h \tag{4.8}$$

where $\eta_a(z,t), \eta_b(z,t)$ and $u_0(r,z)$ are the known functions.

First phase. Let us consider that the density ρ of the cylinder is constant. Then equation (4.5) becomes

$$K(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}) + \xi(r,z,t) - \frac{\partial u}{\partial t} = 0 \tag{4.9}$$

Applying (1.14) in (4.9) with respect to r and taking $a = 0 = \nu$ due to the choice of the boundary conditions

$$K\frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial \bar{u}}{\partial t} + \xi(\lambda_i, z, t) - K\lambda_i^2\bar{u} + k\psi(z,t) \tag{4.10}$$

where

$$\psi(z,t) = \frac{aC_0(h_1,h_2,\lambda_i a^\beta)\eta_a(z,t)}{h_1} - \frac{bC_0(h_1,h_2,\lambda_i b^\beta)\eta_b(z,t)}{h_2}$$

Now using finite Fourier sine transform to equation (4.10) with respect to z, using (4.7), we have

$$\frac{d\bar{u}_s}{dt} + k(\frac{m^2\pi^2}{h^2} + \lambda_i^2)\bar{u}_s(\eta, m, t) = -k\psi_s(m, t) + \xi(\lambda_i, m, t) \tag{4.11}$$

where

$$\int_0^h \frac{\partial^2 u}{\partial z^2}\sin(\frac{m\pi z}{h})dz = -\frac{m^2\pi^2}{h^2}\bar{u}_s(n, m, t)$$

$$\psi_s(m, t) = \int_0^h \psi(z, t)\sin(\frac{m\pi z}{h})dz$$

and

$$\xi(\lambda_i, m, t) = \int_0^h \xi(\lambda_i, z, t)\sin(\frac{m\pi z}{h})dz$$

Again Laplace transform in variable t to equation (4.11) and using (4.8) gives

$$\begin{aligned} L[\bar{u}_s(n, m, t)] &= \frac{[\bar{u}_{0,s}(n, m)]}{[p + k(\lambda_i^2 + \frac{m^2\pi^2}{h^2})]} - \frac{kL[\psi_s(m, t)]}{[p + k(\lambda_i^2 + \frac{m^2\pi^2}{h^2})]} \\ &+ \frac{L[\xi(\lambda_i, m, t)]}{[p + k(\lambda_i^2 + \frac{m^2\pi^2}{h^2})]} \end{aligned} \tag{4.12}$$

On using inverse Laplace transform in (4.12), then applying convolution theorem of Laplace transform to it and again using the inversion theorem of finite Fourier since transform to this result, we have

$$\bar{u}(n, z, t) = \frac{2}{h}\sum_{m=1}^{\infty} \bar{u}_s(n, m, t)\sin(\frac{m\pi z}{h}) \tag{4.13}$$

And lastly using (1.17) to (4.13), we have

$$\begin{aligned} u(r, z, t) &= \frac{2}{h}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\|C_\nu(h_1, h_2, \lambda_i x^\beta)\|^2}[u_0(n, m) \\ &\cdot \exp\{-k(\lambda_i^2 + \frac{m^2\pi^2}{h^2})\}t - \int_0^t k\psi_s(m, u) - \xi(\lambda_i, m, t)\} \\ &\cdot \exp\{-k(\lambda_i^2 + \frac{m^2\pi^2}{h^2})\}(t - u)du]\sin(\frac{m\pi z}{h})C_0(h_1, h_2, \lambda_i \nu^\beta) \end{aligned} \tag{4.14}$$

where $\|C_\nu(h_1, h_2, \lambda_i x^\beta)\|^2$ is given in (1.12).

Second phase. Here we take the composite cylinder of variable density and suppose $\rho = \rho_0 r^2$, ρ_0 is constant.

The equation (4.5) reduces to

$$K\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\xi(r,z,t)}{\rho_0 r^2}-\frac{\partial u}{\partial t}=0 \tag{4.15}$$

Using (1.14) to the equation (4.15) with respect to r, we have

$$K\frac{d^2\bar{u}}{dz^2}-\frac{d\bar{u}}{dt}+\frac{\bar{G}(\lambda_i,z,t)}{\rho_0}-\lambda_i^2 k\bar{u}=k\psi(z,t) \tag{4.16}$$

where $\psi(z,t)$ is given in first phase analysis and

$$\bar{G}(\lambda_i,z,t)=\int_a^b \frac{\xi(r,z,t)}{r}C_0(h_1,h_2,\lambda_i r^\beta)dr$$

Now applying Fourier sine transform in (4.16) with respect to z as in first phase due to (4.7), we get

$$\frac{d\bar{u}_s}{dt}+k\left(\frac{m^2\pi}{h^2}+\lambda_i^2\right)\bar{u}_s(n,m,t)=k\psi_s(m,t)+\frac{\bar{G}_s(\lambda_i,m,t)}{\rho_0} \tag{4.17}$$

where $k\psi_s(m,t)$ is given in the first phase and

$$\bar{G}_s(\lambda_i,m,t)=\int_0^h \frac{\xi(r,z,t)}{r}\sin(m\pi z/h)dz$$

Further, the Laplace transform of (4.17) with respect to t, due to (4.8), gives

$$\begin{aligned} L[\bar{u}_s(n,m,t)] &= \frac{[\bar{u}_{0,s}(n,m)]}{[p+k(\lambda_i+\frac{m^2\pi^2}{h^2})]}-\frac{kL[\psi_s(m,t)]}{[p+k(\lambda_i^2+\frac{m^2\pi^2}{h^2})]} \\ &+\frac{L[\bar{G}_s(\lambda_i,m,t)]}{[p+k(\lambda_i^2+\frac{m^2\pi^2}{h^2})]} \end{aligned} \tag{4.18}$$

On using inverse Laplace transform in (4.18), then apply convolution theorem of Laplace transform to it and further applying the inversion theorem of finite Fourier sine transform to this result, gives

$$u(n,z,t)=\frac{2}{h}\sum_{m=1}^{\infty}u_0(n,m,t)\sin(m\pi z/h) \tag{4.19}$$

And finally using (1.17) in (4.19), we have

$$u(r,z,t) = \frac{2}{h}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{\|C_\nu(h_1,h_2,\lambda_i x^\beta)\|^2\}^{-1}u_0(n,m)\exp\{-k(\lambda_i^2+\frac{m^2\pi^2}{h^2})\}t$$
$$-\int_0^t\{k\psi_s(m,u)-\bar{G}_s\frac{(\lambda_i,m,t)}{\rho_0}\}\exp\{-k(\lambda_i^2+\frac{m^2\pi^2}{h^2})\}(t-u)du]]$$
$$\sin(\frac{m\pi z}{h})C_0(h_1,h_2,\lambda_i r^\beta) \quad (4.20)$$

References

[1] Birkhoof, G. and Rota, G.C., *Ordinary differential equations*, Bhaisdell publishing CP., Waltham, Mass., 1969.

[2] Gradohtein, I.S. and Ryzhik, I.W., *Table of Integrals, series and products.* Academic Press, New York, 1965.

[3] Kalla, S.L. and Alfredo Villalobos, *On a new integral Transform*, Jnanabha, Vol. 9/10, pp. 149-154, 1980.

DEPARTMENT OF MATHEMATICS, KENYATTA UNIVERSITY, P.O.BOX 43844, NAIROBI, KENYA.