Kyungpook Mathematical Journal
Volume 30, Number 2, December, 1990

ON PERTURBATIONS OF NONLINEAR SYSTEMS OF VOLTERRA INTEGRAL EQUATIONS

A.A.S. Zaghrout

1. Introduction

The mathematical literature on this subject provided a good information concerning the existence, uniqueness, stability, and continuous dependence of solutions of various classes of Volterra integro-differential equations, see for example, ([4,5,7]). Once the existence and uniqueness have been established, a quite different analysis is required for finding asymptotic properties of the solutions. We shall discuss and compare the boundeness and asymptotic behaviour of solutions of the perturbed Volterra integral system with that of the corresponding unperturbed system.

Consider the perturbed nonlinear system of Volterra integral equations:

$$x(t) = f(t) + \int_{t_0}^{t} g(t,s,x(s))ds + \int_{t_0}^{t} h(t,s,x(s))ds \qquad (1.1)$$

with the initial condition

$$x(t_0) = f(t_0) = x_0 \neq 0$$

and the corresponding unperturbed nonlinear system

$$y(t) = f(t) + \int_{t_0}^{t} g(t,s,y(s))ds, \qquad (1.2)$$

with the initial equation

$$y(t_0) = f(t_0) = x_0 \neq 0$$

Received September 27, 1988

where $x, y \in R^n$, an n–dimensional Euclidean space.
We use $x(t) = x(t; t_0, x_0)$ to denote the solution of (1.1) passing through x_0 at $t = t_0$ and $y(t) = y(t, t_0, x_0)$ to denote a solution of (1.2) passing through x_0 at $t = t_0$. The symbol $|\cdot|$ will denote some convient norm on R^n as well as corresponding consistent matrix norm.

Our main hypotheses are :

i) $f \in R^n$ is a continuous function and has continuous derivatives on $J = (0, \infty)$.

ii) $g \in R^n$ is a continuous function and the partial derivatives g_t, g_x, g_{tx} are continuous for all $0 \leq s \leq t < \infty$.

iii) $h \in R^n$ is a continuous and has continuous derivatives with repect to t on $0 \leq s \leq t < \infty$.

The nonlinear variation of constants formula of Brauer [2] has been used to obtain various results on the effect of a perturbation of Volterra integral system. He has shown that if $y(t)$ is a solution of (1.2), the corresponding variational system of (1.2) is

$$z(t) = U + \int_{t_0}^{t} g_y(t; s, y(s)) z(s) ds, z(t_0) = U \tag{1.3}$$

where U is the unit matrix. Let $\Phi(t, t_0 x_0)$ be the solution matrix of (1.3) with respect to the solution $y(t)$ of (1.2). Then the solutions of (1.1) and (1.2) with the same initial values are related by

$$\begin{aligned} x(t) &= y(t) + \int_{t_0}^{t} \Phi(t, s, x(s))[h(s, s, x(s)) \\ &\quad + \int_{t_0}^{t} \frac{\partial h}{\partial s}(s, \tau, x(\tau)) d\tau] ds, \end{aligned} \tag{1.4}$$

For more details see Brauer [2].

We need the following definitions :

Definition 1. The solution $y(t)$ of (1.2) is said to be globally uniform stable in variation if there exists a constant M such that

$$|y(t, t_0, x_0)| \leq M |x_0|$$

$$|\Phi(t, t_0, x_0)| \leq M,$$

for all $t \geq t_0$ and $|x_0| < \infty$.

Definition 2. The solution $y(t)$ of (1.2) is said to be exponentially asymptotic stable in variation if there exist constants $M > 0$, $\alpha > 0$ such that $|y(t, t_0, x_0)| \leq M|x_0|.\exp[-\alpha(t-t_0)]$ and $|\Phi(t, t_0, x_0)| \leq M \exp[-\alpha(t-t_0)]$ for all $0 \leq t_0 \leq t$ and $|x_0|$ is sufficiently small.

Definition 3. The solution $y(t)$ of (1.2) is said to be uniformly slowly growing in variation if and only if for every $\alpha > 0$, there exists a constant M such that

$$|y(t; t_0, x_0)| \leq M|x_0| \exp[\alpha(t - t_0)]$$
$$|\Phi(t; t_0, x_0)| \leq M \exp[\alpha(t - t_0)]$$

for all $0 \leq t_0 \leq t$, $|x_0| < \infty$.

Remark 1. We say that a continuous function $z(t)$ is slowly growing if and if for every $\alpha > 0$, there exists a constant M, which may depend on α such that

$$|z(t)| \leq M \exp[\alpha t], \quad t \geq 0.$$

2. Main Results

In this section we state and prove our main results on the boundedness, stability and asymptotic behaviour of the solutions of the perturbed of Volterra integral equation (1.1) under some suitable assumptions on the perturbation term. Our results, in this section depend on the following two lemmas (Pachpatte [6]) :

Lemma 1. *Let $u(t), p(t)$ and $q(t)$ be real-valued nonnegative continuous functions defined on J, for which the inequality*

$$u(t) \leq u_0 + \int_{t_0}^{t} p(s)u(s)ds + \int_{t_0}^{t} p(s)(\int_{t_0}^{s} q(r)u(r)dr)ds, t \in J,$$

holds, where u_0 is a nonnegative constant. Then

$$u(t) \leq u_0[1 + \int_{t_0}^{t} p(s) \exp(\int_{t_0}^{s} [p(r) + q(r)]dr)ds, t \in J,$$

Lemma 2. *Let $u(t), p(t)$, and $g(t)$ be real-valued nonnegative continuous functions defined on J, for which the inequality*

$$u(t) \leq u_0 + \int_{t_0}^{t} p(s)u(s)(u(s) + \int_{t_0}^{s} q(r)u(r)dr)ds, t \in J,$$

holds, where u_0 is a positive constant. Then

$$u(t) \le u_0 \exp(\int_{t_0}^{t} p(s) u_0 \exp(\int_{t_0}^{s} q(r)dr)/R(s)ds, t \in J,$$

where

$$R(s) = 1 - u_0 \int_{t_0}^{t} p(r) \exp(\int_{t_0}^{r} q(k)dk)dr, \quad t \in J.$$

$$|p(r) \exp(\int_{t_0}^{t} q(k)dk)dr| \le u_0^{-1}, \quad t \in J.$$

Theorem 1. *Assume the followings :*

i) *The solution $y(t)$ of (1.2) is uniformly slowly growing in variation.*

ii) *The function $h(t,s,x)$ in (1.1) and its derivative $h_t(t,s,x)$ satisfy*

$$|h(t,s,x)| \le p(t)|x|, \quad t \in J$$

$$|h_t(t,s,x)| \le p(t) \cdot \exp(\alpha t) \cdot q(s) \cdot |x|, \quad 0 \le s \le t < \infty$$

where α is a positive constant, p and q are continuous functions defined on J such that

$$\int_{t_0}^{\infty} Mp(s) \exp(\int_{t_0}^{s} [Mp(r) + q(r) \exp \alpha r] dr)\, ds \le k,$$

where M is a positive constant. Then all solution of (1.1) are slowly growing.

Proof. Using the nonlinear variation of constants formula developed by Brauer [2], the solutions of (1.1) and (1.2) with the same initial values are related by (1.4).

Using the assumptions (i), (ii) and (1.4) together with the definition of uniformly slowly growing in variation of the solution $y(t)$ of (1.2), we get

$$\begin{aligned} |x(t)| \;\le\; & M|x_0| \exp \alpha(t - t_0) + \int_{t_0}^{t} M \cdot \exp \alpha(t-s) \cdot p(s)|x(s)|ds \\ & + \int_{t_0}^{t} M \cdot \exp \alpha(t-s) \cdot \int_{t_0}^{s} p(s) \cdot \exp \alpha s \cdot q(r)|x(r)|drds. \end{aligned}$$

Multiplying both sides of the above inequality by $\exp(-\alpha t)$ and applying Lemma 1, with $u(t) = X(t) \exp(-\alpha t)$, we have

$$\begin{aligned} |x(t)| \;\le\; & M|x_0| \exp \alpha(t - t_0)[1 + \int_{t_0}^{s} \{Mp(s) \exp \int_{t_0}^{s} [Mp(r) + \exp(\alpha r). \\ & q(r)]dr\}\, ds, \\ \le\; & M|x_0| \exp \alpha(t - t_0)[1 + k]. \end{aligned}$$

The above estimate yields the desired result if we choose M and $|X_0|$ small enough. This completes the proof.

Now we apply the above technique to investigate the nonlinear system of Volterra integral system of the form

$$\begin{aligned} x(t) &= f(t) + \int_{t_0}^{t} g(t,s,x(s))ds \\ &\quad + \int_{t_0}^{t} x(s)h(s,x(s),\int_{t_0}^{t} k(s,\tau,x(\tau))d\tau)ds, \end{aligned} \tag{2.1}$$

$$x(t_0) = f(t_0) = x_0$$

as a perturbation of the system (1.2). $h(t,x,z)$ and $k(t,s,x)$ are continuous on $0 \le s \le t < \infty, |x| < \infty$. The solution of (1.2) and (2.1) with the same initial values are related by (Brauer [2])

$$\begin{aligned} x(t) &= y(t) + \int_{t_0}^{t} \Phi(t,s,x(s))x(s)h(s,x(s), \\ &\quad \int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds \end{aligned} \tag{2.2}$$

Theorem 2. *Assume*

i) *The solution $y(t)$ of (1.2) is globally uniformly stable in variation*

ii) *The functions $h(t,x,z)$ and $K(t,s,x)$ in (2.1) satisfy*

$$|h(t,x,z)| \le p(t)(|x| + |z|), \quad t \in J$$

$$|k(t,s,x)| \le q(s)|x| \quad 0 \le s \le t < \infty.$$

where p and q are continuous functions defined on J such that

$$\int_0^{\infty} \{Mp(s)M|x_0| \exp(\int_{t_0}^{t} q(r)dr)/R(s)\}ds < \infty \tag{2.3}$$

where

$$R(t) = 1 - M|x_0| \int_{t_0}^{t} Mp(R)\exp(\int_{t_0}^{r} q(\tau)d\tau)dr, \tag{2.4}$$

$M > 0, x_0 \ne 0$ are constants. Then all solutions of (2.1) are bounded on J.

Proof. Using (2.1) and the assumption (i), (ii) we have

$$|x(t)| \le M|x_0| + \int_{t_0}^{t} Mx(s)p(s)(|x(s)| + |\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau|)ds. \tag{2.5}$$

Applying lemma 2 and using (2.3) and (2.4), inequality (2.5) reduced to

$$|x(t)| \leq M|x_0| \exp \int_{t_0}^{t} \{Mp(x) \cdot M|x_0| \exp(\int_{t_0}^{s} q(r)dr)/R(s)\}ds.$$

The above estimation in view of the assumption (2.3) implies the boundedness of all solutions of (2.1). This completes the proof.

Remark 2. We note that theorem 1 implies not only the boundedness, but the stability of the solution $x(t)$ of (2.1) if $|x_0|$ is small enough. However the above estimation does not prove the asymptotic stability.

Theorem 3. *Assume*

i) *The solution $y(t)$ of (1.2) is uniformly slowly growing in variation.*
ii) *The functions h and k in (2.1) satisfy*

$$|h(t,x,z)| \leq p(t)(|x| + |z|), \quad t \in J$$

$$|k(t,s,x)| \leq \exp(\alpha t) \cdot q(s)|x|, \quad 0 \leq s \leq t < \infty,$$

where α is a positive constant and p and q are continuous function defined on J such that

$$\int_{t_0}^{\infty} \{Mp(s)\exp(\alpha s) M x_0 \exp(-\alpha t_0) \cdot \exp(\int_{t_0}^{s} q(r)\exp(\alpha r)dr)/R(s)\}ds < \infty, \tag{2.6}$$

where

$$R(t) = 1 - M|x_0|\exp(-\alpha t_0) \int_{t_0}^{t} Mp(\tau)\exp(\alpha\tau) \cdot \exp(\int_{t_0}^{\tau} q(r)\exp(\alpha r)dr)d\tau, \quad M > 0, x_0 \neq 0$$

are constant. Then all solutions of (2.1) are slowly growing.

Proof. As in Theorem 2, the solutions of (2.1) and (1.2) with the same initial values are related by (2.2). Using the assumptions (i), (ii) and (2.2) we have

$$|x(t)| \leq |y(t)| + \int_{t_0}^{t} |\Phi(t,s,x(s))|x(s_0) \\ h(s,x(s), \int_{t_0}^{s} |k(s,\tau,x(\tau)|d\tau)ds$$

i.e.

$$
\begin{aligned}
|x(t)| \quad \leq \quad & M|x_0| \cdot \exp \alpha(t-t_0) \\
& + \int_{t_0}^{t} M_t \exp \alpha(t-s) \cdot x(s) \cdot p(s) \cdot (|x(s)| + \int_{t_0}^{t} \exp \alpha s \cdot q(r)|x(r)|dr)ds.
\end{aligned} \tag{2.7}
$$

Multiplying (2.7) by $e^{-\alpha t}$ and applying lemma 2 with $u(t) = |x(t)| \exp(-\alpha t)$, we obtain

$$
\begin{aligned}
|x(t)| \exp(-\alpha t) \quad \leq \quad & M|x_0| \exp(-\alpha t) \cdot \exp\{Mp(s) \exp(\alpha s) \cdot M|x_0| \exp(-\alpha t) \\
& \cdot \exp(\int_{t_0}^{s} \exp(\alpha\tau) \cdot q(\tau)d\tau)/R(s)\}ds.
\end{aligned}
$$

The above estimation yields the desired result if we choose M and $|x_0|$ sufficiently small and all solutions of (2.1) grow more slowly than any positive exponential. This completes the proof.

The next theorem shows that under some suitable conditions on the functions h, the exponential asymptotic stability in variation of the solutions of (1.2) implies that all the solutions of (2.1) approach zero as $t \to \infty$.

Theorem 4. *Assume*

i) *The solution $y(t)$ of (1.2) is exponentially asymptotically stable in variation*

ii) *The functions h and k in (2.1) satisfy*

$$|h(t,x,z)| \leq p(t)(|x| + |z|), \quad t \in J,$$

$$|k(t,s,x)| \leq \exp(-\alpha t)q(s)|x(s)|, \quad 0 \leq s \leq t < \infty$$

where α is a positive constant, p and q are continuous function defined on J such that

$$
\begin{aligned}
\int_{t_0}^{\infty} & \{Mp(s) \exp(-\alpha s) \cdot M|x_0| \exp(\alpha t_0) \\
& \cdot \exp(\int_{t_0}^{s} q(r) \exp(-\alpha r)dr)/R(s)\}ds < \infty,
\end{aligned} \tag{2.8}
$$

where

$$
\begin{aligned}
R(s) \quad = \quad & 1 - M|x_0| \exp(\alpha t_0) \int_{t_0}^{t} Mp(\tau) \exp(-\alpha\tau) \\
& \cdot \exp(\int_{t_0}^{\tau} q(r) \exp(-\alpha r)dr)d\tau.
\end{aligned}
$$

Then all solutions of (2.1) approach zero as $t \to \infty$. The proof is similar to that of Theorem 3, and so is omitted.

Now we shall study the asymptotic behaviour of the perturbed Volterra integral equations allowing more general perturbations than we previously allowed.

Consider the system

$$x(t) = f(t) + \int_0^t a(t,s)[x(s), g(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau)]ds, \qquad (2.9)$$

as a perturbation of the system

$$y(t) = f(t) + \int_0^t a(t,s)y(s)ds, \qquad (2.10)$$

where $a(t,s)$ is an $n \times n$ continuous matrix, x, y, f, g and k are as defind before.

It is known [5] that the resolvent system corresponding to the system (2.10) is

$$r(t,s) = a(t,s) + \int_s^t a(t,u)r(u,s)du, \quad 0 \leq s \leq t < \infty, \qquad (2.11)$$

and its solution is called the resolvent Kernel. If $a(t,s)$ is locally L^1 in (t,s) and if $r(t,s)$ exists and is locally L^1 in (t,s), then the system (2.9) may be written in the equivalent from "variation of constants formula"

$$x(t) = y(t) + \int_0^s r(t,s)g(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau)ds, \qquad (2.12)$$

where $y(t)$ is the solution of the linear system (2.10) given by

$$y(t) = f(t) + \int_0^t r(t,s)f(s)ds, t \geq 0. \qquad (2.13)$$

Our next theorem shows that under some suitable conditions on the perturbation term g and on the function k, the uniform slowly groving of (2.10) relative to its resolvant kernel implies that all solutions of (2.10) are slowly growing.

Theorem 5. *Assume*

i) *The solution $y(t)$ of (2.10) is uniformly slowly growing relative to its resolvent kernel.*

ii) *The perturbation* $g(t,x,z)$ *satisfies the inequality*

$$|g(t,x,z)| \leq p(t)(|x| + \exp(\varepsilon t)|z|), \quad t \in J$$

and $\int_0^\infty p(s)ds < \infty, \varepsilon > 0$ *is a constant.*

iii) *The function* $k(t,s,x)$ *satisfies the inequality*

$$|k(t,s,x)| \leq q(s)|x|, \quad t,s, \in J$$

and

$$\int_0^\infty q(s)ds < \infty.$$

iv) *There exists a constant* N *such that*

$$\int_0^\infty Mp(s)\exp(\int_0^s [M(p(\tau) + q(\tau)\exp(\varepsilon\tau)]d\tau)ds \leq N$$

where M *is a constant in definition 3. Then all solutions of (2.9) are slowly growing.*

Proof. As before, by using the variation of constants formula, the solution of (2.9) and (2.10) are related by (1.12). Using the assumptions (ii), (iii), (2.12) together with the uniformly slowly growing of (2.10) relative to its resolvent kernel, we obtain

$$|x(t)| \leq |y(t)| + \int_0^t |s,(t,s)||g(x,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)|ds,$$

i.e.

$$\begin{aligned} |x(t)| \;\; &\leq \;\; M|x_0|\exp(\varepsilon t) + \int_0^t M\exp\varepsilon(t-s)p(s)(|x| \\ &\qquad +\varepsilon s\int_0^s q(\tau)|x(\tau)|d\tau ds. \end{aligned} \tag{2.14}$$

Multiplying both sides of (2.14) by $\exp(-\varepsilon t)$ and applying lemma 2 with $u(t) = x(t)\exp(-\varepsilon t)$ we have

$$\begin{aligned} \exp(-\varepsilon t)\cdot|x(t)| &\leq M|x_0|M\exp(-\varepsilon s)\cdot p(s)(|x| \\ &+\exp(\varepsilon s)\int_0^s q(r)|x(r)|dr)ds. \end{aligned}$$

i.e.

$$\exp(-\varepsilon t)|x(t)| \leq M|x_0|[1 + \int Mp(s)\exp(\int_0^s (Mp(r) + \exp(\varepsilon r)q(r))dr)ds].$$

Hence

$$|x(t)| \leq M|x_0| \exp(\varepsilon t)[1 + N].$$

The above estimation yields the desired result if we choose M and $|x_0|$ small enough, and the proof of the theorem is complete.

References

[1] Agarwal, P. and Thandapani, E., *Remarks on Generalizations of Gronwall's Inequalities*, Chinese J. of Math. Vol. 9 (1981), 1-22.

[2] Brauer, F., *A nonlinear variation of constant formula for Volterra equations*, Math. System Theory B (1972), 226-234.

[3] Grace, S. and Lalli, B., *On boundedness and asymptotic behaviour of certain second order integra-differential equations*, J. Math. Phy. Sci. 14(1980), 191-203.

[4] Lakshmikantham, V. and Leela, S., *Differential and Integral ineqalities*, Academic Press, London, (1969).

[5] Miller, R., *Nonlinear Volterra Integral Equations*, W.A. Benjamin Inc., Menle Park, California, (1971).

[6] Pachpatte, B., *On some new integral inequalities for differential and integral equations*, J. Math. Phys. Sci. 10 (1976), 101-116.

[7] Walter, W., *Differential and integral inequalities*, Springer-Verlag, (1970).

[8] Zaghrout, A. and Aly, I., *On inequalities of Gronwall-Reid-Bellman type I*, Tamkang Math. J. (1987), to appear, Vol. 18, No.2.

[9] ________, *On inequalities of Gronwall-Reid-Bellman type II*, Kyungpook Math. J. (to appear) (1988).

[10] ________, *On nonlinear integral equations*, (Arch. Math. (Brno), (to appear), (1988).

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AL-AZHAR UNIVERSITY, NASR-CITY, CAIRO-EGYPT