Kyungpook Mathematical Journal
Volume 30, Number 2, December, 1990

ON CYCLE DOUBLE COVER CONJECTURE

Yongju Bae and Chan-Young Park

1. Introduction

A *graph* G consists of a finite nonempty *vertex* set $V(G)$ together with an *edge* set $E(G)$. By a *topological* graph G, we shall mean a realization of the graph G as a 1-dimensional CW-complex. Two graphs G and H are said to be *homeomorphic* if they are homeomorphic as CW-complexes. In this paper a graph will always mean a topological graph.

The *degree* of a vertex v is the number of edges meeting v. Given a connected graph G, a *cut vertex* is a vertex $v \in V(G)$ such that $G - \{v\}$ is disconnected. A connected graph is called a *block* if it has no cut vertex. An edge $e \in E(G)$ is called a *bridge* (or *cut edge*) if $G - \{e\}$ is disconnected. A graph G is said to be *n-connected* $(n > 0)$ if the removal of fewer than n vertices from G neither disconnects nor reduces G to the trivial graph K_1, where K_1 is the graph with one vertex and with no edges. Note that the statement that a graph G has no cut vertices is equivalent to that G is 2-connected.

An *embedding* of G into a closed surface S is a homeomorphism of G into S. An embedding $i : G \to S$ is called a *2-cell embedding* if every component of $S - i(G)$, called a *region*, is 2-cell. A 2-cell embedding $i : G \to S$ is called a *closed 2-cell embedding* if every boundary walk of its regions is a simple closed cycle. An embedding $i : G \to S_k$ of G into an orientable [non-orientable] surface of genus k is *minimal* if G can not be embedded into an orientable [non-orientable] surface of genus less than k. It is well-known that every minimal embedding of a connected graph is a 2-cell embedding.

Received August 20, 1990.
The second author was partially supported by a grant from TGRC-KOSEF 1990.

A connected graph G is said to be *planar* if there is an embedding $i : G \to S^2$ of G into the sphere S^2. Let G be a 2–connected graph. A subgraph H of G is called a *maximal planar subgraph* of G if it is a planar subgraph of G such that for every planar subgraph K of G, $|E(K)| \leq |E(H)|$.

Recently, many authors have studied various topics on 2–cell embeddings of graphs. Their topics include the following:

(1) Finding algorithms for embedding problems ([GRT], [L]).
(2) Covering graphs or bundle graphs ([GT], [L]).
(3) Finding embedding distributions (minimal or maximal genus of graphs) ([GF], [GRT], [GT], [L], [W]).
(4) Determining the genus of groups ([W]).

On closed 2–cell embeddings, one can ask the same questions as one can do for 2–cell embeddings once the existence of closed 2–cell embeddings of graphs is guaranteed. But the existence problem for closed 2–cell embeddings of graphs is a fairly long–standing conjecture in topological graph theory, so–called "strong embedding conjecture".

In this paper, we solve the "cycle double cover conjecture" confirmatively, which is a weak version of the "strong embedding conjecture" ([H],[J],[LR]).

Strong embedding conjecture: Every 2–connected graph has a closed 2–cell embedding into a surface.

Cycle double cover conjecture: Every bridgeless graph has a cycle double cover.

Here cycle double cover means a family of cycles of G such that for each edge of G, there are exactly two cycles containing it.

2. Main results.

Note that if every block of a graph G has a cycle double cover, so does G. Thus in order to solve the cycle double cover conjecture, it suffices to show that every 2–connected graph has a cycle double cover.

We start with the definition of pseudo–surfaces.

Definition([W]). Let A denote a set of $\sum_{i=1}^{t} n_i m_i \geq 0$ distinct points of a surface S_k of genus k, with $1 < m_1 < m_2 < \cdots < m_t$. Partition A into

n_i sets of m_i points for each $i = 1, 2, \cdots, t$. For each set of the partition, identify all the points in the set. The resulting topological space is called a *pseudo-surface*, and is denoted by $S(k; n_1(m_1), n_2(m_2), \cdots, n_t(m_t))$. The points in the pseudo-surface resulting from an identification of m_i points of S_k are called *singular points.* If a graph G is embedded in a pseudo-surface as a [closed] 2-cell embedding, we may assume that each singular point is occupied by a vertex of G and such a vertex is called a *singular vertex.*

Theorem 1. *Let G be a 2-connected graph. Then G has a cycle double cover if and only if it has a closed 2-cell embedding into a pseudo-surface.*
Proof. The necessary condition is clear because the set of boundary cycles of its regions is a cycle double cover. For the proof of the sufficiency, let C be a cycle double cover of G. To each cycle of C, corresponds a 2-cell whose boundary walk is the cycle. Since each edge occurs on exactly two regions as a part of their boundaries, by identifying the corresponding edges, we can obtain a topological space satisfying the property that every point on the interior of each 2-cell or on an edge has a neighborhood homeomorphic to the unit disk in R^2. We call such a neighborhood a *disklike neighborhood.* Since there are only finitely many points without disklike neighborhoods, corresponding to some vertices, the resulting topological space is a pseudo-surface.

Now, we establish an algorithm to check whether or not a vertex v has a disklike neighborhood.

Surface checking algorithm. Consider all edges $vv_1, vv_2, \cdots, vv_n$ meeting v and all cycles $C_1, C_2, \cdots, C_n$ containing v. Then each C_i contains exactly two edges of $vv_1, vv_2, \cdots, vv_n$. Assume that C_1 contains vv_1 and vv_2. Then there is a cycle C_2' containing vv_2 and there is an edge vv_3' contained in C_2'. If $v_3' = v_1$ and $n \neq 2$, then v has no disklike neighborhood. If $v_3' \neq v_1$, then there is a cycle C_3' containing vv_3' and there is an edge vv_4' contained in C_3'. If $v_4' = v_1$ and $n \neq 3$, then v has no disklike neighborhood. If $v_4' \neq v_1$, then there is a cycle C_4' containing vv_4' and there is an edge vv_5' contained in C_4' and so on. If one can complete this process through all vv_i's, then v has a disklike neighborhood.

Corollary 2. *For a 3-regular 2-connected graph, the cycle double cover conjecture is equivalent to the strong embedding conjecture.*

Proof. Clearly the strong embedding conjecture implies the cycle double cover conjecture.
Let C be a cycle double cover of G and let v be a vertex of G. Since G is 3–regular, there are exactly three edges vv_1, vv_2, vv_3 which meet v ,and hence there are also three cycles C_1, C_2, C_3 containing v. Note that each C_i contains two of vv_1, vv_2, vv_3. Assume that C_1 contains vv_1, vv_2. If C_2 contains also vv_1, vv_2, then C_3 must contain the edge vv_3 twice, which is impossible. Hence C_2 contains either vv_1, vv_3 or vv_2, vv_3. Thus either C_3 contains vv_2, vv_3 if C_2 contains vv_1, vv_3, or C_3 contains vv_1, vv_3 if C_2 contains vv_2, vv_3. Then by the surface checking algorithm, we can easily check that the resulting space is a surface.

Consider a standard ladder with n lungs. If we identify the four ends of its two poles as one vertex v, we obtain a CW–complex. A graph G is called a *standard ladder graph* if it is homeomorphic to such a CW–complex. See Figure 1(a).

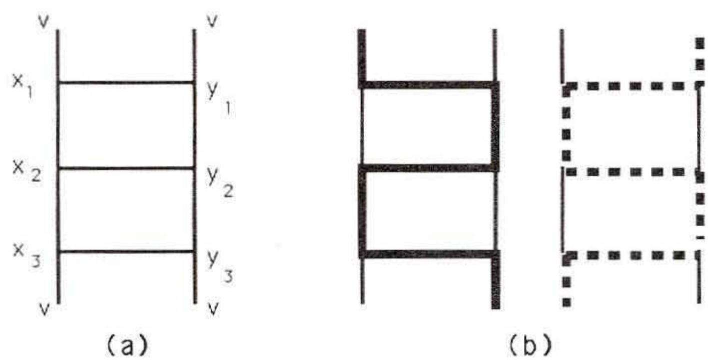

Figure 1.

Now, we introduce the ladder graph chasing.

Ladder graph chasing: Let G be a standard ladder graph and let the vertices on the first pole be $x_1, x_2, \cdots, x_n$ and those on the second pole $y_1, y_2, \cdots, y_n$. To get two cycles simultaneously, start at the top vertex v and go to the vertex $x_1[y_1]$ and through the lung $x_1y_1[y_1x_1]$, go to the vertex $y_1[x_1]$. And through the pole, go to the next vertex $y_2[x_2]$ and then through the lung $y_2x_2[x_2y_2]$, go to $x_2[y_2]$. By repeating this process until the bottom vertex v occurs, we get two cycles $(vx_1y_1y_2x_2 \cdots v)$ and $(vy_1x_1x_2y_2 \cdots v)$, completing the ladder graph chasing.

Note that on these two cycles obtained by the ladder graph chasing, each edge on the lungs is contained in both cycles, and each edge on the poles is contained in only one cycle.

Figure 1(a) shows that a ladder with 3 lungs and in Figure 1(b), the solid lines and the dotted lines show the corresponding ladder chasing.

The following is a key theorem to solve the cycle double cover conjeture.

Theorem 3. *Let G be a 2-connected graph and let $e = uv \in E(G)$. Assume that $G - \{e\}$ has a cycle double cover. Then G has a cycle double cover.*

Proof. Assume that $G-\{e\}$ has a cycle double cover. Then, by Theorem 1, there is a closed 2-cell embedding $i : G-\{e\} \to F$ of $G-\{e\}$ into a pseudo-surface F. Choose an arc $\gamma : I \to F$ from u to v such that for every arc $\gamma' : I \to F$ from u to v, $| \gamma(I) \cap i(G) | \leq | \gamma'(I) \cap i(G) |$. Let $e_1, e_2, \cdots, e_n$ be edges of $G - \{e\}$ meeting γ and let $v_1, v_2, \cdots, v_k$ be vertices of G meeting γ and $R_1, R_2, \cdots, R_m$ regions of the embedding meeting γ, numbered along γ. Denote each e_i by $e_i = x_i y_i$. Then the embedding near γ can be depicted as in Figure 2. Notice that since the embedding $i : G - \{e\} \to F$ is closed 2-cell, the boundary walk of each region of the embedding forms a cycle. In order to construct a cycle double cover of G, as a part of a cycle double cover we take the boundary cycles of regions of the embedding which do not meet γ . In other words, we take the boundary cycles of all regions but $R_1, R_2, \cdots, R_m$ as members of our cycle double cover. Note that an edge on the boundary cycle of a region R_i may be occured on the boundary cycle of another region R_j. Now to get a cycle double cover of G together with those boundary cycles, we have to construct a family of cycles so that the edges $e = uv$, e_i $(i = 1, 2, \cdots, n)$ and the edges which lie on two of the boundary cycles of the regions $R_1, R_2, \cdots, R_m$, occur twice and the edges which lie only on one of the boundary cycles of the regions $R_1, R_2, \cdots, R_m$, occur just once. Consider two closed paths $C_1 = (ux_1v_1x_2 \cdots x_nv_kveu)$ and $C_2 = (uy_1v_1y_2 \cdots y_nv_kveu)$ depicted in Figure 2 as the upper path and the lower path respectively. By regarding C_1 and C_2 as two poles and the edges $e_1, e_2, \cdots, e_n$ as lungs, we get a standard ladder graph $\overline{G}$ if we disregard the multiple occurance of vertices or edges of G on the boundary cycles of the regions $R_1, R_2, \cdots, R_m$. The ladder graph chasing of the ladder graph $\overline{G}$ gives us two long closed walks to get a family of cycles. We can easily check that these cycles together with those boundary cycles of regions which do not meet γ form a cycle double cover of G.

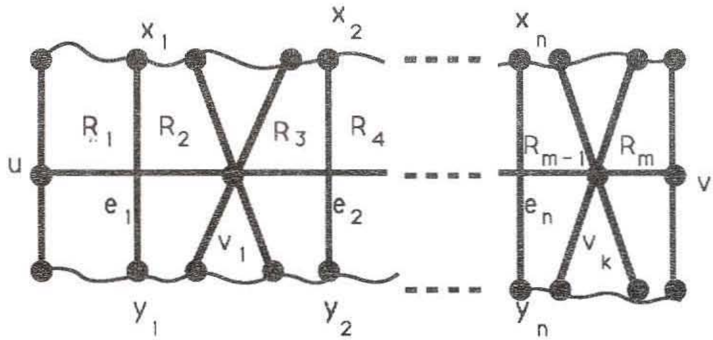

Figure 2.

Now we are ready to prove the cycle double cover conjecture.

Theorem 4 [Cycle Double Cover Theorem]. *Every bridgeless graph has a cycle double cover.*

Proof. Recall that every planar 2–connected graph admits a closed 2–cell embedding. Hence every planar 2–connected graph has a cycle double cover. Assume that G is a non–planar 2–connected graph. Let H be a maximal planar subgraph of G. Note that $V(G) = V(H)$ and $E(G) - E(H) \neq \emptyset$. Let $E(G) - E(H) = \{e_1, e_2, \cdots, e_n\}$. Since H is planar, it has a cycle double cover. By applying Theorem 3, we have that $H \cup \{e_1\}$ has a cycle double cover. Inductively, we can obtain a cycle double cover of G.

References

[GF] J.L. Gross & M. Furst, *Hierachy for imbedding–distribution invariants of a graph*, J. Graph Theory 11 (1987), 205–220.

[GRT] J.L. Gross, D.P. Robbins & T.W. Tucker, *Genus distributions for bouquets of circles*, J. Combinatorial Theory, Series B 47 (1989), 292–306.

[GT] J.L. Gross & T.W. Tucker, *Topological Graph Theory*, John Wiley & Sons, New York, 1987.

[H] G. Hagard, *Edmonds characterization of disk embeddings*, Proc. Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Math. (1977), 291–301.

[J] F. Jaeger, *Nowhere-zero flow problems*, Selected topics in Graph Theory 3 (1988), 71–95.

[L] J. Lee, *Enumerations of graph coverings and bundles*, Ph.D. thesis, Kyungpook National Univ. 1990.

[LR] C.H.C. Little & R.D. Ringeigen, *On the strong graph embedding conjecture*, Proc. Ninth Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Math. (1978), 479–487.

[W] A.T. White, *Graphs, groups and surfaces*, North–Holland, 1984.

Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu, 702–701, KOREA